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Towards a Theory of Organizations

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Abstract

In this paper we develop an algebra to describe organizations. Its application is demonstrated with five examples. We start from definitions given by Fontana (1992) of an organization as a closed and self-maintaining set of interacting objects. We develop a formal framework to describe the inner structure of an organization and a relationship between different organizations. The definitions of intersection and union of organizations are developed. Those definitions naturally give rise to a lattice (an algebraic structure over a partially ordered set) which provides a precise basis to study the hierarchical nature of organizations. Some fundamental properties are described and the usefulness of the mathematical concepts demonstrated by application.

1 Introduction

The term *organization* is widely used in science, starting from social sciences and economy to physics and computer science. In nearly all these areas *organization* has a specific meaning, which is sufficient for qualitative statements on a system embedded in a specific context. But once quantitative measurements are sought, an exact definition of organization is required.

This paper tries to define the term organization using precise mathematical and algebraic statements. It intends to give a means for describing organization in systems with a maximum of accuracy, independent of their constituting parts, be they molecules, stars of a galaxy, or departments of a company. These exact statements shall be applied to five examples of systems, stemming from the field af artificial chemistry (AC). Artificial chemistries are able to generate organizations with different characteristics. The concept of an artificial chemistry is an elegant means for dealing with structures that are able to change or maintain themselves, and especially with systems that are able to create new components.

Varela, Maturana, and Uribe (1974) investigate the basic organization of living systems with simple autopoietic models. Their approach can be seen as one of the early works in artificial chemistry that in the 1980s contributed to the formation of the field *Artificial Life* (AL), an interdisciplinary area of research that deals with the abstract foundations of living systems. Many AL researcher investigated the theory of organizations of artificial living systems (e.g. (Kampis 1991; Fontana and Buss 1994; Szathmary 1995)). It soon emerged that properties like *self-maintenance, self-creation, seclusion, and openness*, observable in different artificial (and natural) chemical systems, are crucial to understand the structure and dynamics of organizations, independent from their instantiating structure. In order to investigate these phenomena artificial chemistries have been proven to be important constructive and analytical tools. McCaskill (1988), Banzhaf (1993b), Ikegami and Hashimoto (1995), Ehricht, Ellinger, and McCascill (1997), Dittrich and Banzhaf (1998), Breyer, Ackermann, and McCaskill (1999), and Ono and Ikegami (1999) for instance, studied different artificial chemistries, ranging from abstract automata to rewriting systems; whereas Hjelmfelt, Weinberger, and Ross (1991), Adleman (1994), Rambidi and Maximychev (1997), Segré, Lancet, Kedem, and Pilpel (1998) analyzed natural

chemical systems, like DNA or enzyme-subtrate solutions, in order to utilize this knowledge to achieve a new way of information processing. Therefore the natural self-organizing properties of chemical-like systems can also been utilized to perform beneficial tasks.

2 Algebra of Organizations

In this contribution we will only concentrate on the static structure of organizations and will neglect their dynamics. For this reason we introduce the concept of an *algebraic chemistry*. An algebraic chemistry is an artificial chemistry without any dynamics.

Definition 2.1. *(algebraic chemistry)*
Let S be a set of elements, R a reaction rule of arity n such that $R : S^n \to S \bigcup \{\emptyset\}$. We call $\mathbb{S} = \langle S, R \rangle$ an **algebraic chemistry**.

Usually, an artificial chemistry is simulated by setting up a population of molecules taken from S. In the most simple case – the well-stirred tank reactor – the population is a multi-set P with elements from S. For a simulation step the population P is updated by using the reaction rule[1] R. Normally what a simulation step does to P is to extract some of its elements (usually two, say s_1 and s_2), apply one to the other, and add the result to the multi-set. Some elements are then eliminated to keep the multiset from becoming too large (called *dilution flow*). The specific "reactor" algorithm can be different from one work to the other. Sometimes the extracted elements (s_1 and s_2) are eliminated from the multi-set after they are extracted (Suzuki and Tanaka 1997), sometimes not (Fontana and Buss 1994). Sometimes the number of molecules in the population is kept constant (Stadler, Fontana, and Miller 1993; Banzhaf 1993a), sometimes it can vary (Speroni di Fenizio 1999). Even the operation sometimes does not produce only one result but a whole multi-set (Hofstadter 1979; Speroni di Fenizio 2000). All those differences, which surely add to the beauty of artificial chemistries, make them more difficult to define. For this reason we try to keep things as simple as possible here and concentrate on the operation R neglecting the dynamics. Thus the formal framework presented now allows a *static* description and analysis of an artificial chemistry.

The following definitions like *organization*, *closure*, and *self-maintenance* are formulated following Fontana (1992) and Fontana and Buss (1994).

Definition 2.2. *(organization)*
Let O be a set of elements, subset of S, ($O \subseteq S$). Then $\mathbb{O} = \langle O, R \rangle$ is an **organization** *iff $\mathbb{O}$ is closed and self-maintaining. That is iff $\forall a_1, \ldots, a_n \in O, R(a_1, \ldots, a_n) = b, b \in O$ (property of* **closure***) and $\forall a \in O, \exists b_1, \ldots, b_n \in O$ such that $R(b_1, \ldots, b_n) = a$ (property of* **self-maintenance***).*

To simplify matters the properties "closed" and "self-maintaining" are also used to characterize sets if it is clear what kind of reaction rule R is meant. So, we can also call a *set* O an organization with respect to a reaction rule R, if the properties of closure and self-maintenance of Def. (2.2) hold. Given a set of elements and a reaction rule we can, of course, consider the set of all possible organizations on this set. We will indicate this set as $\mathfrak{O}$. $\mathfrak{O} \equiv \{\mathbb{O} \subseteq \mathbb{S} : \mathbb{O}$ is closed and self-maintaining$\}$.

[1] Note that sometimes the term "reaction rule" refers to rules where each rule is defined only for a small subset of S. In this case R is defined by a collection of such "reaction rules".

Definition 2.3. *(sub-organization of an organization)*
Let $\mathbb{A} = \langle A, R\rangle, \mathbb{B} = \langle B, R\rangle \in \mathfrak{O}$ *be two organizations* ($\mathbb{A}, \mathbb{B} \in \mathfrak{O}$)*, such that* $A \subset B$ ($A \subseteq B$)*, than* $\mathbb{A}$ *is a* ***sub-organization*** *of* $\mathbb{B}$*. We will indicate this with* $\mathbb{A} \lhd \mathbb{B}$ ($\mathbb{A} \trianglelefteq \mathbb{B}$)*.*

Definition 2.4. *(maximal sub-organization)*
Let $\mathbb{A} = \langle A, R\rangle, \mathbb{B} = \langle B, R\rangle \in \mathfrak{O}$ *be two organizations* ($\mathbb{A}, \mathbb{B} \in \mathfrak{O}$)*, with* $\mathbb{A} \lhd \mathbb{B}$*, and such that there is no* $\mathbb{C}$*, sub-organization, with* $\mathbb{A} \lhd \mathbb{C} \lhd \mathbb{B}$*. Then* $\mathbb{A}$ *is called a* ***maximal sub-organization*** *of* $\mathbb{B}$*.*

We will now list some trivial properties of closed and self-maintaining sets.

Proposition 1. *The intersection of two closed sets is still a closed set.*

Proposition 2. *The union of two self-maintaining sets is still a self-maintaining set.*

This is true not only if we deal with a finite number of sets but also if we deal with an infinite number of sets. Thus given any set $T \subseteq S$ we can always define C the **smallest closed set that contains** T and M the **biggest self-maintaining set that is contained in** T. C will be defined as the intersection of all the closed sets that contain T. $C \equiv \{x \in S : \forall D \subseteq S, T \subseteq D,$ and D closed, such that $x \in D\}$. M will be defined as the union of all the self-maintaining sets contained in T. $M \equiv \{x \in S : \exists D \subseteq T,$ with D self-maintaining, such that $x \in D\}$

2.1 Organization Generated by a Self-Maintaining Set

Let A be a self-maintaining set. A uniquely defines an organization O. Note that from here on we use O with the understanding that the reaction rules R are implicit and $\mathbb{O}$ if we need them explicitly. Let A^1 be the set whose elements are directly generated by A. $A^1 \equiv \{x \in S :$ $\exists a_1, \ldots, a_n \in A,$ such that $R(a_1, \ldots, a_n) = x\}$ Since A is self-maintaining $A^1 \supseteq A$. But then, since $A^1 \supseteq A$, A^1 is self-maintaining too. Thus, if $A^1 \neq A$, we can generate A^2, etc. In general $A^n \supseteq A^{(n-1)}$ with

$$A^n \equiv \{x \in S : \exists a_1, \ldots, a_n \in A^{n-1} \text{such that} R(a_1, \ldots, a_n) = x\}. \quad (1)$$

If there exists n such that $A^n = A^{(n-1)}$ then $O \equiv A^n$ is closed and an organization. If there does not exists n such that $A^n = A^{(n-1)}$ then we can define O as the union of all the A^i. $O \equiv \bigcup_{i \in \mathcal{N}} A^i$. O is well defined, closed and self-maintaining, thus O is an organization. We will call $\mathbb{O} =< O, R >$ the **organization generated by the self-maintaining set** A. O is also the smallest closed set that contains A. This definition is equivalent to Fontana's definition of closure.

2.2 Organization Generated by a Closed Set

Proposition 3. *Let* A *be a closed set, let* $O \subseteq A$ *be the biggest self-maintaining set inside* A*. Then* $\mathbb{O} = \langle O, R\rangle$ *is an organization.*

Proof: O is a self-maintaining set by definition. We need to prove that O is also closed. Let $a \in A \setminus O$ if $\exists o_1, \ldots, o_n \in O : R(o_1, \ldots, o_n) = a$ then $O \cup \{a\}$ would be a self-maintaining set in A, bigger than O, against the hypothesis. Thus there is no $a \in A \setminus O$ generated by the elements in O, thus O is closed. ■

We will call $\mathbb{O}$ the **organization generated by the closed set** A.

2.3 Organization Generated by a Set

Let A be a set, B its largest self-maintaining set in A, and C its smallest closed set containing A. Thus $B \subseteq A \subseteq C$. C is a closed set, thus it uniquely generates an organization. **Let D be the organization generated by** C. In general $B \subseteq D \subseteq C$. We will define D as the **organization generated by the set** A. Of course if A is an organization itself then B, C and D are all equal to A. In the most general case $A \nsubseteq D$ nor $D \nsubseteq A$.

2.4 Union and Intersection of two Organizations

Definition 2.5. *(union of organizations)*
Let $\mathbb{A} = \langle A, R\rangle, \mathbb{B} = \langle B, R\rangle \in \mathfrak{O}$ be two organizations. We will define the **union of organizations** *as the organization generated by the union of the two sets A and B $(A \cup B)$. We will indicate this organization as $\mathbb{A} \sqcup \mathbb{B}$.*

Definition 2.6. *(intersection of organizations)*
Let $\mathbb{A} = \langle A, R\rangle, \mathbb{B} = \langle B, R\rangle \in \mathfrak{O}$ be two organizations. We will define the **intersection of organizations** *as the organization generated by the intersection of the two sets A and B $(A \cap B)$. We will indicate this organization as $\mathbb{A} \sqcap \mathbb{B}$.*

Note that the term "generated" is important here and should be understood as defined in Sec. 2.1-2.3. In general, given two organization $\mathbb{A} = \langle A, R\rangle, \mathbb{B} = \langle B, R\rangle \in \mathfrak{O}$.
$\mathbb{A} \sqcap \mathbb{B} \subseteq A \cap B \subseteq A, B \subseteq A \cup B \subseteq \mathbb{A} \sqcup \mathbb{B}$. And $\mathbb{A} \sqcap \mathbb{B} \trianglelefteq \mathbb{A}, \mathbb{B} \trianglelefteq \mathbb{A} \sqcup \mathbb{B}$.

2.5 Lattice of Organizations

Organizations in an algebraic chemistry nicely relate one with the other generating a well known algebraic structure called lattice. What follows is the standard definition of a lattice (Rutherford 1966), as well as the definition of a lattice of organizations of an artificial chemistry. Later we shall demonstrate this concept which was already recognized in (Fontana and Buss 1996) on a number of examples and show that it is of importance in understanding the properties of an organization.

Definition 2.7. *(lattice)*
Let L be a set of elements, where two binary operations are defined by $\cup$ and $\cap$. Let those operations be such that $\forall x, y, z \in L$ the following properties are valid:

$$x \cup y = y \cup x \quad \textit{(commutative law on } \cup\textit{)}, \tag{2}$$
$$x \cap y = y \cap x \quad \textit{(commutative law on } \cap\textit{)}, \tag{3}$$
$$x \cup (y \cup z) = (x \cup y) \cup z \quad \textit{(associative law on } \cup\textit{)}, \tag{4}$$
$$x \cap (y \cap z) = (x \cap y) \cap z \quad \textit{(associative law on } \cap\textit{)}, \tag{5}$$
$$x \cap (x \cup y) = x \quad \textit{(absorptive law)}, \tag{6}$$
$$x \cup (x \cap y) = x \quad \textit{(absorptive laws)}. \tag{7}$$
$$\tag{8}$$

Then $< L, \cup, \cap >$ is a **lattice**.

We also present (from the same cited work (Rutherford 1966)) the concept of a *chain* which will be used in our future definition of depth. A chain is a lattice where given any two elements of the chain, one is always contained in the other.

Definition 2.8. *(chain)*
A lattice $\mathbb{L} = \langle L, \cup, \cap \rangle$ *is a* **chain** *if* $\forall a, b \in L, a \cup b = a$ *or* $a \cup b = b$.

Finally we come to a definition central to this paper, the lattice of organizations. In the application part, Sec. 3, demonstrates by visualizing the lattice of organizations that it provides valuable insights into the organizational structure of an artificial chemistry.

Definition 2.9. *(lattice of organizations of an algebraic chemistry)*
Let $\mathfrak{O}$ *be the set of all organizations of an algebraic chemistry* $\mathbb{S}$. *Let* $\sqcup$ *and* $\sqcap$ *be the union and intersection of organizations, then* $\langle \mathfrak{O}, \sqcup, \sqcap \rangle$ *will be defined as the* **lattice of organizations** *of the algebraic chemistry* $\mathbb{S}$.

Some obvious definitions follow:

Definition 2.10. *(finite and infinite organizations)*
An organization is called **finite** *if it holds a finite number of elements. It is, instead, called* **infinite** *if it holds an infinite number of elements.*

Definition 2.11. *(replicator and self-replicator)*
Let $\mathfrak{O} = \langle O, R \rangle$ *be an organization and* c *be an element in* O, *then* c *is a* **replicator** *in* O *if there exist* d *in* O *such that* $R(d, c) = c$ *or* $R(c, d) = c$. *An element* c *is called a* **self-replicator** *if* $R(c, c) = c$.

2.6 Reformulating Earlier Definitions

We now reformulate some of the definitions given in (Fontana and Buss 1994; Fontana and Buss 1996) using our previous terms.

Definition 2.12. *(L0 organization)*
An organization $\mathfrak{O} = \langle O, R \rangle$ *is called an* **L0 organization** *if it is finite and if every element* $c \in O$ *is a replicator in* O. *That is, if* $\forall c \in O$ *there is* $d \in O$ *such that* $R(c, d) = c$ *or* $R(d, c) = c$

Definition 2.13. *(L1 organization)*
An organization $\mathfrak{O} = \langle O, R \rangle$ *is called* **L1 organization** *if it is an infinite organization and if no element of* O *is a replicator in* O.

Thus, no element of an L1 organization is a self-replicator. In (Fontana and Buss 1994; Fontana and Buss 1996) there is only an informal definition of an L2 organization. We try to formulate what seems to be the most closely related definition:

Definition 2.14. *(L2 organization)*
An L1 organization O *is also a* **L2 organization** *iff it contains two L1 organizations* T_1, T_2 *such that* T_1 *is not in* T_2, T_2 *is not in* T_1 *and* $T_1 \sqcup T_2 = O$.

We can now present an important extension to the concept of organization. The concept of depth. This concept recalls the concept of level of an organization, yet we present this new concept in a formal mathemathical way, thus permitting extensions and future work.

Definition 2.15. *(depth of an organization)*
Let $\mathbb{O} \in \mathfrak{O}$ be an organization. Let $\mathbb{C} \subseteq \langle O, \sqcup, \sqcap \rangle$ be the longest chain contained in its lattice of sub-organizations. Then we define the **depth of an organization** *as the number of elements in $\mathbb{C}$.*

So the depth of an organization is the length of the longest chain contained. Note that an organization with no suborganization has depth 1, while only the null organization (the organization with no elements inside) has depth 0. Note also that a finite organization has always a finite depth, while an infinite organization can have either a finite or an infinite depth.

2.7 Definition of Glue and Seed

Some interesting other concepts defined or described in the previously mentioned work are the definition of *glue*, *seed* and of *center*.

Definition 2.16. *(glue with respect to two sub-organizations)*
Let $A, B, O \in \mathfrak{O}$, let O be a L2 organization and $A, B \subseteq O$ the two L1 organizations that generate O. Then the set $G_{A,B} \equiv (A \sqcup B) \setminus (A \cup B)$ will be the glue of O with respect to A and B.

This definition mirrors the definition by Fontana and Buss (1994), adding the terms "with respect to ..." to show how this definition depends upon the choice of the two L1 organizations. We define a more general concept of glue which does not depend upon the number of maximal sub-organizations, nor upon the number of elements in the organization or in its sub-structures.

Definition 2.17. *(glue)*
Let O be an organization. Let $A_1, \ldots, A_i, \ldots$ be the maximal sub-organizations in O, then the set $G \equiv O \setminus (\cup_{A_i})$ will be the **glue of** *O.*

Let O be an organization, let A be a set, subset of O ($A \subseteq O$). In (Fontana and Buss 1994) A is defined to be a seed if A generates O. We would like to present here a different definition of seed. We start with the more general concept of a *generator set*.

Definition 2.18. *(generator)*
et O be an organization, G a set. We define G to be a **generator** *for an organization O iff G generates O.*

Note that we do not require in any way that the set G is contained in O. G could be totally disjointed from O. We then define the smallest generators of an organization as seed.

Definition 2.19. *(seed)*
Let G be a generator for O such that there does not exists a generator T of O contained in G, $T \subset G$, then G is a **seed** *for O.*

Again we have not required in any way that a seed is contained in the organization. If it is necessary we can further define a seed G as being an **internal seed** (**external seed**) if G is in O (G is disjointed from O). The definitions of *internal generator* and *external generator* will be given equivalently. With those tools we can now proceed in reformulating the definition of center:

Definition 2.20. *(center)*
Let O be an organization, let S be a generator of O. S is said to be a **center** *for O iff S is self-maintaining, and $\nexists B$, $B \subseteq S$, self-maintaining and a generator of O.*

Since S in a self-maintaining set, $S \subseteq O$. It may seem that our definition of internal seed just recalls the definition of center. But the two definitions are different. In fact, it is possible to have an internal seed which is not a self-maintaining set. For a set to be self-maintaining, it needs to build all the elements by immediate interaction, while an internal seed may need to generate some elements of the organization before being able to regenerate itself. Similarly, a biological seed needs to build an entire tree before the seed is regenerated.

3 Application

In this Section we apply the mathematical concepts described above in order to demonstrate their feasibility. Five different artificial chemistries are investigated by looking at their reaction network and displaying the lattice of organizations. As we will see each chemistry possesses a characteristic lattice.

3.1 Modulo Chemistry

Given two prime numbers p_1, p_2 and their product $m = p_1 p_2$ the **modulo chemistry** is defined as follows.

Molecules: Molecules are natural numbers smaller than m, including zero. $S = \{0, 1, ..., m-1\}$.

Reaction: The reaction product s_3 is the sum of s_1 and s_2 module m: $s_3 = s_1 + s_2 \bmod m$.

Figure 1 shows the reaction table and the lattice of organizations. The lattice structure is simple. There are only two non-trivial sub-organizations with more than one element. This stems from the fact that the divisor m is the product of two prime numbers.

3.2 4-Bit Matrix-Multiplication Chemistry

The matrix-multiplication chemistry has been introduced by Banzhaf (1993a) in order to abstract fundamental principles of an RNA world.

Molecules: Binary strings of length 4 bit. $S = \{0, 1\}^4$.

Reaction: The reaction product s_3 is created from s_1 and s_2 in the following way: (1) s_1 is folded to a 2x2 matrix. (2) s_2 is divided into two sub-strings of length 2 bit. (3) The matrix is multiplied with each sub-string. (4) The resulting two sub-strings are concatenated in order to form the product string s_3. Table 1 shows the reaction table of a 4-bit matrix reaction using the so called non-topological horizontal folding. For more details see (Banzhaf 1993b; Banzhaf 1993a).

The resulting lattice of organizations (Fig. 2) contains a surprisingly large number of organizations and is relatively complex compared to the number of molecules constituting the artificial chemistry.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	2	3	4	5	6	7	8	9	10	11	12	13	14	0
2	3	4	5	6	7	8	9	10	11	12	13	14	0	1
3	4	5	6	7	8	9	10	11	12	13	14	0	1	2
4	5	6	7	8	9	10	11	12	13	14	0	1	2	3
5	6	7	8	9	10	11	12	13	14	0	1	2	3	4
6	7	8	9	10	11	12	13	14	0	1	2	3	4	5
7	8	9	10	11	12	13	14	0	1	2	3	4	5	6
8	9	10	11	12	13	14	0	1	2	3	4	5	6	7
9	10	11	12	13	14	0	1	2	3	4	5	6	7	8
10	11	12	13	14	0	1	2	3	4	5	6	7	8	9
11	12	13	14	0	1	2	3	4	5	6	7	8	9	10
12	13	14	0	1	2	3	4	5	6	7	8	9	10	11
13	14	0	1	2	3	4	5	6	7	8	9	10	11	12
14	0	1	2	3	4	5	6	7	8	9	10	11	12	13

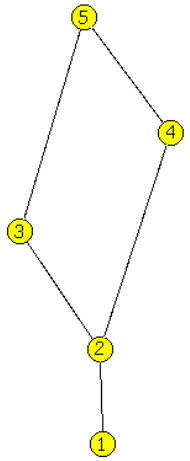

Figure 1: *Modulo Chemistry with* $m = 15$. *Reaction table (left) and lattice of organizations (right). (1) The empty organization:* $O_1 = \{\}$. *(2) Organization containing only the self-replicator:* $O_2 = \{0\}$. *(3) Organization containing all valid numbers which can be divided by 5:* $O_3 = \{0, 5, 10\}$. *(4) Organization of all valid numbers which can be divided by 3:* $O_4 = \{0, 3, 6, 9, 12\}$. *(5) Organization of all molecules:* $O_5 = S$.

operator s_1	operand string s_2															
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	1	4	5	4	5	0	1	0	1	4	5	4	5
2	0	0	1	1	0	0	1	1	4	4	5	5	4	4	5	5
3	0	1	1	1	4	5	5	5	4	5	5	5	4	5	5	5
4	0	2	0	2	8	10	8	10	0	2	0	2	8	10	8	10
5	0	3	0	3	12	15	12	15	0	3	0	3	12	15	12	15
6	0	2	1	3	8	10	9	11	4	6	5	7	12	14	13	15
7	0	3	1	3	12	15	13	15	4	7	5	7	12	15	13	15
8	0	0	2	2	0	0	2	2	8	8	10	10	8	8	10	10
9	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
10	0	0	3	3	0	0	3	3	12	12	15	15	12	12	15	15
11	0	1	3	3	4	5	7	7	12	13	15	15	12	13	15	15
12	0	2	2	2	8	10	10	10	8	10	10	10	8	10	10	10
13	0	3	2	3	12	15	14	15	8	11	10	11	12	15	14	15
14	0	2	3	3	8	10	11	11	12	14	15	15	12	14	15	15
15	0	3	3	3	12	15	15	15	12	15	15	15	12	15	15	15

Table 1: *Reaction table of a 4-bit matrix-multiplication chemistry (non-topological horizontal folding). A "0" indicates an elastic collision. This means that there is no reaction product defined so that in case of a collision the molecuels do not react.*

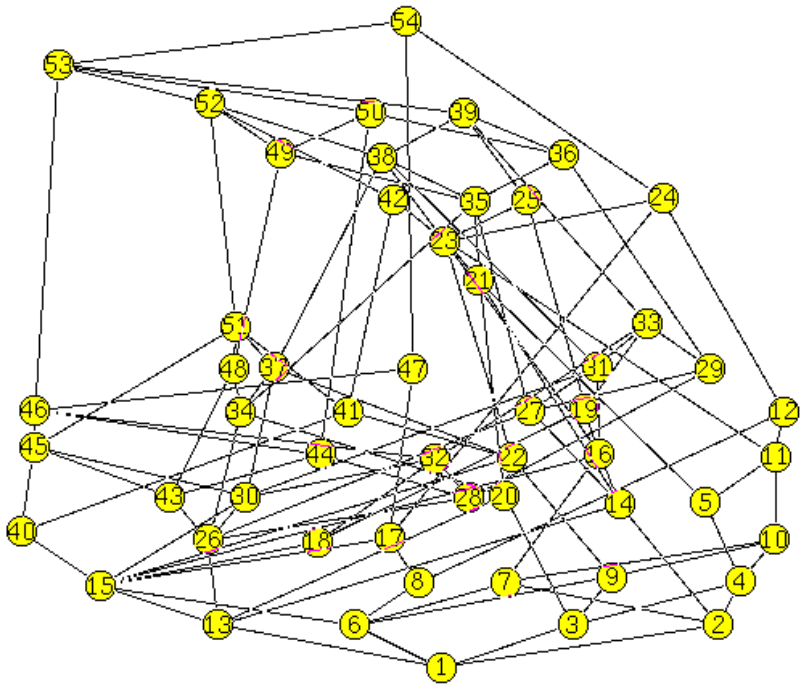

Figure 2: *Lattice of organizations of the 4-bit matrix-multiplication chemistry shown in Tab. 1.*

3.3 Variable-Length Matrix-Multiplication Chemistry

The matrix-multiplication chemistry can be generalized to operate on binary strings of arbitrary length as shown by Banzhaf (1994) and Dittrich (2000). Here we take from one of these chemistries a small finite organization that emerged in one run using the MV1 reaction.

Molecules: Binary strings of arbitrary length $S = \{0,1\}^*$.

Reaction: The reaction product s_3 is created in a similar way as described for the 4-bit matrix-multiplication chemistry in Sec. 3.2. The major difference is that the operator matrix is likely to be non-square. In this case the reaction product s_3 can become larger or smaller than s_2.

The lattice of organizations shown in Fig. 3 has an interesting regularity. The origin of this regularity and whether it is typical for the specific variable-length matrix-multiplication chemistry should be investigated in the future.

3.4 Shuffled 4-Bit Matrix-Multiplication Chemistry

Molecules: Binary strings of length 4 bit.

Reaction: We take the reaction table of the 4-bit matrix-multiplication chemistry and randomly shuffle it. In order to preserve (at least partly) its statistical properties we conserve the number of symmetric reactions rules ($s_1+s_2 = s_2+s_1$) and the number of self-replications ($s_1+s_1 = s_1$).

In all chemistries generated by shuffling the 4-bit matrix-multiplication reaction table the organizational structure became more simple and the number of organization became much smaller as exemplified by Fig. 4.

0	1	0	1	4	0	4	1	0	4	11	11	1
2	3	2	3	6	2	6	3	2	6	10	10	3
0	1	0	1	4	0	4	1	0	4	11	11	1
2	3	2	3	6	2	6	3	2	6	10	10	3
5	7	5	7	9	5	9	7	5	9	13	13	7
0	1	0	1	4	0	4	1	0	4	11	11	1
5	7	5	7	9	5	9	7	5	9	13	13	7
2	3	2	3	6	2	6	3	2	6	10	10	3
0	1	0	1	4	0	4	1	0	4	11	11	1
5	7	5	7	9	5	9	7	5	9	13	13	7
8	12	8	12	14	8	14	12	8	14	15	15	12
8	12	8	12	14	8	14	12	8	14	15	15	12
2	3	2	3	6	2	6	3	2	6	10	10	3

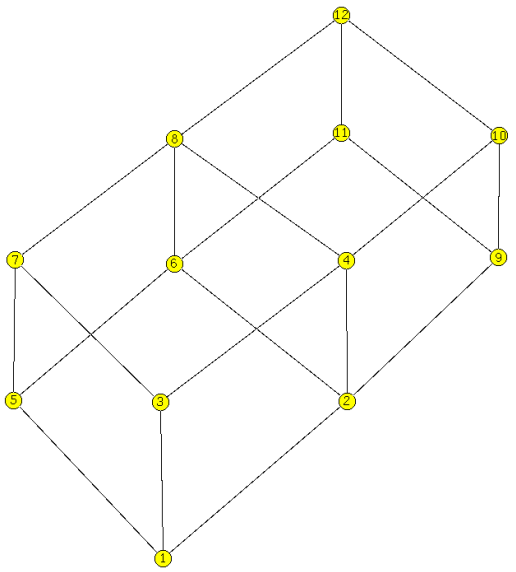

Figure 3: *Variable-length matrix multiplication chemistry. Reaction table (left) and lattice of organizations (right).*

0	12	0	3	4	0	11	12	0	10	15	1	14	15	0	8
12	1	12	14	0	15	0	2	0	0	6	12	10	5	15	15
0	12	0	14	4	2	15	12	2	9	10	15	3	5	3	0
3	14	14	13	0	5	2	15	2	1	5	7	7	12	15	12
4	0	4	0	10	0	3	1	12	10	5	12	5	4	5	10
0	15	2	5	0	8	5	0	3	0	2	2	4	5	11	1
11	0	15	2	3	5	8	8	0	13	1	1	4	15	10	10
12	12	12	15	1	0	8	0	3	8	0	3	0	15	2	12
0	0	2	2	12	3	0	3	8	4	0	11	5	15	15	15
10	0	9	1	10	0	13	8	4	9	8	10	0	4	1	0
15	6	10	5	5	2	1	0	0	8	1	3	4	3	0	8
1	12	15	7	12	2	1	3	11	10	3	15	15	15	15	15
14	10	3	7	5	4	3	0	5	0	4	15	10	0	10	13
15	5	5	12	4	5	15	15	15	4	3	15	0	15	0	7
0	15	3	15	5	11	5	2	15	1	0	15	10	0	3	8
8	15	0	12	10	1	10	12	15	0	8	15	13	7	8	15

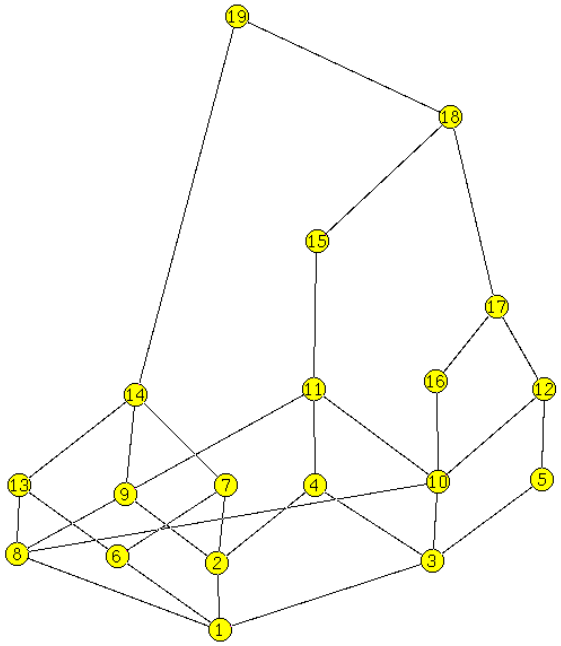

Figure 4: *Typical shuffled 4-bit matrix-multiplication chemistry. Reaction table (left) and lattice of organizations (right). Here, "0" denotes an elastic collision.*

Figure 5: *Typical lattice of organizations of a random reaction table with 15 molecular types. The random chemistry contains three organizations. (1) The empty organization, (2) one self-replicator, and (3) the organization of all molecules.*

3.5 Random Reaction Table

Now we take a look at artificial chemistries which are randomly generated. Randomness does not refer to the interaction process; thus R deterministic.

Molecules: Natural numbers. $S = \{1, 2, \ldots, 15\}$.

Reaction: The reaction table is generated by drawing randomly for each entry a number from S (without removing that number).

Figure 5 shows a typical case where the lattice of organization contains only three organizations. All three organizations are trivial in the sense that they contain either none, only one, or all elements from S. It is also likely that there are only two organizations, the empty organization and the organization containing all elements. A larger number of organizations appears with rapidly decreasing probability.

4 Discussion

The five examples above showed that artificial chemistries with different underlying principles can have a variety of different lattices of organizations. The organizational structure cannot be seen easily by just looking at the reaction table. Thus we conclude that the mathematical concepts discussed here are a useful means to characterize an artificial chemistry, especially if it is given as a (relatively small) finite reaction table.

The definitions given earlier in (Fontana and Buss 1994; Fontana and Buss 1996) had been developed while studying one particular artificial chemistry, *AlChemy*. This model, however, is in some sense special and lacks – in the form it has been studied – the generality to support all possible organizations. In this contribution we have therefore embarked on a more general route, trying to develop a generic and mathematically more rigorous picture of organizations where we have taken into account different artificial chemistries. For instance, we consider finite organizations without replicators and infinite organizations which contain replicators. Other notions have been newly introduced, for instance generators or the depth of an organization.

We have also demonstrated that if the reaction table is shuffled, the number of organizations is reduced on average, even if we conserve properties like number of self-replicators and symmetric reactions. Note that there are "pathological cases" where the number of organizations may on average increase by shuffling. E.g. a random artificial chemistry which does not contain any self-replicator and consists only of two organizations: the empty organization and the organization of all molecules. Shuffling would likely create a self-replicator which would increase the number of organizations.

The last example shows that a random chemistry possesses a very simple organizational structure. Under this light the lattice of the 4-bit matrix-multiplication chemistry is amazingly complex. The lattices of the other artificial chemistries are also significantly different to the lattice of a random chemistry. Thus the lattice of organizations can be applied to characterize the randomness of an organization or an artificial chemistry.

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