Integrable Hamiltonian Systems. Topological structure of saturated neighborhood of non-degenerate singular points.

A. V. Bolsinov, V. S. Matveev

1 Introduction

Let (M^4, ω) be a symplectic manifold. A Hamiltonian system

$$\dot{x}^i = (\mathrm{sgrad} H)^i = \omega^{ij} \frac{\partial H}{\partial x^j}$$

is called *integrable* , if there exists a function f which is constant on the solutions of the system and functionally independent with H (i. e., the differentials df and dH are independent almost everywhere). The function f is called *an additional integral.*

The mapping $F = (H, f) : M^4 \to \mathbb{R}^2$ is called *the momentum mapping* of the integrable Hamiltonian system. The foliation of the manifold M^4 into connected components of level surfaces of the momentum mapping F is called *a Liouville foliation.* We will assume that for any a and b the common level surface $H^{-1}(a) \cap f^{-1}(b)$ is compact. According to the well-known Liouville theorem [1], any non-singular fiber of this foliation is a two-dimensional torus (*Liouville torus*). The solutions of our equation on each of these tori are periodic or conditionally periodic winding. If the Hamiltonian system is non-resonant then the Liouville foliation does not depend on the choice of the additional integral f.

Definition 1.1. A set $U \subset M^4$ is called *saturated*, if it consists of the whole fibers, that is, for any $x \in U$ the fiber of the Liouville foliation through x belongs to U.

Definition 1.2. A point $x_0 \in M^4$ is called *critical* , if $\lambda df + \mu dH = 0$ for some $\lambda, \mu \in \mathbb{R}$, $\lambda^2 + \mu^2 \neq 0$. The set of critical points is denoted by K. The image $F(K)$ in plane $\mathbb{R}^2(H, f)$ is called *the bifurcation diagram.*

Definition 1.3. Let $L \subset M^4$ be a Liouville fibre. A simple curve $C \subset L$ is called *critical* , if it consist of only critical points and there are no other critical points in some neighborhood $U_C \subset L$ of it.

Definition 1.4 A close critical curve $C \subset M^4$ is called *critical circle.* A nonclose critical curve is called *critical line.*

If $\mathrm{sgrad} H \neq 0$ on C we supply C with the orientation induced by $\mathrm{sgrad} H$. Otherwise we leave C nonoriented.

Definition 1.5. Two saturated sets V and $\tilde{V}$ are called *topologically equivalent* , if there exists a homeomorphism $\tau : V \to \tilde{V}$ which preserves the Liouville foliation structure (i.e., taking fibers to fibers) and takes critical circles to critical circles, preserving their orientations.

Definition 1.6. A critical point $x_0 \in M^4$ is called *singular* if $dH(x_0) = df(x_0) = 0$.

The aim of this paper is to describe the topological structure of integrable Hamiltonian systems in saturated neighborhoods of singular points of the momentum map. A similar problem was formulated by S.Smale in his well-known paper [2]. The papers [3],[4] by L.M.Lerman, Ya.L.Umanskii are also devoted to this problem. Many effects that appear at singular points of a momentum map were discovered by M.P.Kharlamov in his analysis of concrete integrable problems in rigid body dynamics [5], see also [6],[7],[8],[24]. Besides, many papers have been devoted to the investigation of the local structure of momentum map singularities [9], [10], [11], [22], [23], [25], [26], [27], [28].

Recently Zung in [28] announced some describtion of the topological structure of multidimehtional non-degenerate momentum mapping singularities.

Moreover, the analysis of singular points is useful because of the following problem. Suppose h is a regular value of H and the restriction of f to the isoenergy surface $Q_h = \{H = h\}$ is a Bott function. A complete topological invariant of such objects was constructed in [12], [13], [14], [15], [16]. According to [16], the topology of an integrable Hamiltonian system on Q_h can be coded in the form of the so-called marked molecule $W^*(Q^3, H)$.

The calculation of marked molecules for concrete examples of integrable Hamiltonian systems is not an easy problem, and the analysis of saturated neighborhoods of singular points can be very helpful here. Actually, the problem of describing the topological structure is close to the following question.

Let $Q_h = \{H = h\}$ be an isoenergy surface of an integrable Hamiltonian system and $W^*(Q_h, H)$ the corresponding molecule describing the topology of the Liouville foliation on Q_h. The question is : what is happening to the molecule $W^*(Q_h, H)$ when the energy level h changes? Usually $h_0 \in \mathbb{R}$ can be a "bifurcation value" (in other words, $W^*(Q_{h_0}, H)$ changes when the energy level passes through h_0) in the two following cases: 1) the level $\{H = h_0\}$ contains a critical point x_0 at which $dF = 0$, 2) the additional integral $f : Q_{h_0} \to \mathbb{R}$ is not a Bott function, in other words, in Q_{h_0} there is a degenerate critical manifold of f. Thus, to understand how the molecule changes we should first find out what happen near critical points and degenerate critical manifolds.

2 Non-degenerate singular points.

Definition 2.1. Let $H, \hat{H}, f, \hat{f}$ be smooth functions on M^4. Pair (H, f) is called *equivalent* to the pair $(\hat{H}, \hat{f})$ iff there exists autodiffeomorphism $\xi : \mathbb{R}^2 \to \mathbb{R}^2$ such that for any $x \in M^4$ $\xi(H(x), f(x)) = (\hat{H}(x), \hat{f}(x))$.

It is evident that equivalent pairs of functions give the same foliation.

Definition 2.2. Singular point $x_0 \in M^4$ is called *non-degenerate point* if in some neighborhood of x_0 there are exist such coordinate system (p_1, q_1, p_2, q_2) that the pair (H, f) is equivalent to one of the following :

(1) the *center-center* type: $H = p_1^2 + q_1^2$, $f = p_2^2 + q_2^2$;

(2) the *saddle-center* type: $H = p_1 q_1$, $f = p_2^2 + q_2^2$;

(3) the *saddle-saddle* type: $H = p_1 q_1$, $f = p_2 q_2$;

(4) the *focus-focus* type: $H = p_1 q_1 + p_2 q_2$, $f = p_1 q_2 - p_2 q_1$.

Remark Usually one use another definition of a non-degenerate singular point. Consider the linearization A_H and A_f of the vector fields $sgradH$ and $sgradf$ at the singular point x_0. It is easily seen that they generate a commutative subalgebra $K_{H,f}$ in the symplectic Lie algebra $sp(2, \mathbb{R})$. The point x_0 is called non-degenerate iff $K_{H,f}$ is a Cartan subalgebra. In analytical case the two definitions are equivalent, [26], [27], [28], [29].

Proposition 2.1 ([3]) *Let $x_0 \in M^4$ be a non-degenerate critical point of the momentum mapping F of saddle-saddle, saddle-center or center-center type, K be the set of critical points of F. Then there exists a neighbourhood $U(x_0)$ of the point x_0 such that $K \cap U(x_0) = P_1 \cup P_2$, where P_1 and P_2 are two-dimensional symplectic submanifolds which intersect transversally at the point x_0.*

Denote by γ a circle of a small radius on $\mathbb{R}^2$ with center in the image $F(x_0)$ of x_0. Topological structure of preimage $F^{-1}(\gamma)$ can be described by the molecule $W^*(F^{-1}(\gamma), H)$. The molecule $W^*(F^{-1}(\gamma), H)$ is called *the circle molecule* of the point x_0.

Now we are able to formulate the main question we are interested in. Our aim is to describe the topological structure of the Liouville foliation on some thin saturated neighborhood of non-degenerate singular points and to show how the circle molecule can be constructed.

3 Saddle-saddle case

The aim of the following two sections is to classify topologically integrable Hamiltonian systems in saturated regular neighborhoods of singular fibers containing one or several focus-focus or saddle-saddle points.

Definition 3.1. Let a saturated open set $U^4 \subset M^4$ satisfy the following conditions.

(1) The bifurcation diagram of U^4 consists of a pair of tranversally intersecting curves (the point of intersection being denoted by A); each of

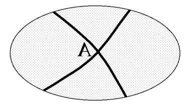

Figure 1: *Bifurcation diagramm of saddle neighborhood*

this curves on the plane $\mathbb{R}^2(H,f)$ is the graph of some smooth function $\alpha=\alpha(H)$, see Fig.1.

(2) The preimage $F^{-1}(A)$ of A is connected.

(3) Any singular point of U^4 is non-degenerate, $F^{-1}(A)$ contains one or several saddle-saddle points.

(4) U^4 is a regular neighborhood of $F^{-1}(A)$.

Such sets are called *saddle neighborhoods.*

Lemma 3.1 *The singular fiber $F^{-1}(A)$ is a 2-dimensional CW-complex whose 0-cells are singular saddle-saddle points, 1-cells are critical lines and 2-cells contain no critical points.*

This lemma was proved, for example, in [20], [21].

Note that the vector field sgrad$H$ has no singularities on edges of C. Hence we can naturally orient the edges of C by sgrad$H$. Thus if we have a saddle neighborhood we can naturally construct CW-complex C with an oriented 1-skeleton.

Definition 3.2. CW-complex C with the orientation on its 1-skeleton is called *the saddle complex* of the saddle neighborhood.

Let us recall the definition of an atom introduced in [16].

Definition 3.3. *An atom* is a pair (B^2,G) where B^2 is an oriented two-dimensional surface, G is a graph, embedded in B^2 such that

(1) Each vertex of G has degree 0,2 or 4.

(2) The subtraction $B^2\backslash G$ is a collection of rings $S^1\times$I.

(3) There exists a decomposition of rings of $B^2\backslash G$ into two classes (positive and negative) such that any edge of G attached to one positive and one negative ring.

(4) G is a connected set on each connected component of B^2.

Usually vertices of G with degree 2 are called and denoted by *stars* .

Unlike the original definition we do not require here the connectedness of the surface B^2 in the definition of the atom and we assume that the decomposition of $B^2\backslash G$ into positive and negative rings is fixed. In section 8 we deal with connected atoms.

Lemma 3.2 $K\cap U^4$ *is a pair of transversally intersecting 2-dimensional smooth surfaces, each of which has a natural structure of atoms* (B_w,G_w)

and (B_b, G_b), *where* $G_w = B_w \cap F^{-1}(A), G_b = B_b \cap F^{-1}(A)$.

This lemma becomes evident if we consider the local structure of the set K.

Definition 3.4. The pair of atoms $(B_w, G_w), (B_b, G_b)$ is called *the L-type* of the saddle neighborhood.

It is clear that the L-type and saddle complex C are both topological invariants of saddle neighborhoods. But there are examples of non-equivalent saddle neighborhoods with the identical L-types and different complexes, or, conversely, with the identical complexes and different L-types.

Definition 3.5. The triple $[C, (B_w, G_w), (B_b, G_b)]$, where C is the saddle complex and the pair $((B_w, G_w), (B_b, G_b))$ is the L-type of the saddle neighborhood is called *the CL-type* of the saddle neighborhood.

Note that $G_w \cap G_b$ consists of saddle-saddle points.

Let $[C, (B_w, G_w), (B_b, G_b)]$ and $[\tilde{C}, (\tilde{B}_w, \tilde{G}_w), (\tilde{B}_b, \tilde{G}_b)]$ be CL-types of saddle neighborhoods U^4 and $\tilde{U}^4$ respectively.

Definition 3.6. CL-types $[C, (B_w, G_w), (B_b, G_b)]$ and $[\tilde{C}, (\tilde{B}_w, \tilde{G}_w), (\tilde{B}_b, \tilde{G}_b)]$ are called *equivalent* if there exists a homeomorphism $X : C \cup B_w \cup B_b \to \tilde{C} \cup \tilde{B}_w \cup \tilde{B}_b$ such that $X(C) = \tilde{C}$ and X preserves the orientation of edges of C.

Theorem 3.1 *CL-type is a complete topological invariant of saddle neighborhoods (i. e., two saddle neighborhoods are topologically equivalent if and only if their CL-types are equivalent).*

Plan of the proof. Let CL-types $[C, (B_w, G_w), (B_b, G_b)]$ and $[\tilde{C}, (\tilde{B}_w, \tilde{G}_w), (\tilde{B}_b, \tilde{G}_b)]$ be equivalent. We can extend the homeomorphism X to that between U^4 and $\tilde{U}^4$ so that the new homeomorphism $\tilde{X}$ we obtain preserves the fibers and the orientation of critical circles by the vector field $\mathrm{sgrad} H$.

First we extend X to the homeomorphism between some spherical neighborhoods of saddle-saddle points. We can do it in such a way that this homeomorphism preserves fibers. Indeed, each saddle neighborhood has the same topological structure near saddle- saddle points. Denote the result of such extension by X_1.

Then we extend X_1 to a homeomorphism between a regular neighborhood of edges (denote it by X_2) and finally we extend X_2 to a homeomorphism between a neighborhood of the 2-cells, see [21] for details.

4 Focus-focus case

Definition 4.1. A saturated open set $U^4 \subset M^4$ is called *a focus neighborhood* if the following conditions hold.

(1) The bifurcational diagram of U^4 consists of a single point (denoted by A).

(2) The preimage $F^{-1}(A)$ is connected.

(3) Any singular point of U^4 is non-degenerate.

(4) U^4 is a regular neighborhood of $F^{-1}(A)$.

It is evident that all critical points of U^4 have a focus-focus type.

Definition 4.2. The number of critical points in the focus neighborhood is called *the weight* of the focus neighborhood.

Theorem 4.1 *The weight is a complete topological invariant of a focus neighborhood (i.e., two focus neighborhoods are topologically equivalent if and only if their weights are equal).*

Plan of the proof. First, we describe the topological structure of $F^{-1}(A)$.

Let $S_1, S_2, \ldots, S_n$ be 2-dimensional spheres. It is useful to suppose that index belongs to $\mathbb{Z}_n$, so $(n-1)+1=0$. Choose two points A_i and B_i on each sphere S_i and glue A_i and B_{i+1} for any i. As a result we obtain some topological space V.

Definition 4.3. V is called *the n-chain.*

Lemma 4.1 [21] *Let the weight of the focus neighborhood be equal to n. Then the preimage $F^{-1}(A)$ is homeomorphic to the n-chain.*

Then, if U_1^4 and U_2^4 are focus neighborhoods with equal weights then their singular fibers are homeomorphic. If we stepwise extend this homeomorphism to a homeomorphism between some sufficiently small spherical neighborhood of vertex, next to a homeomorphism between some neighborhoods of the union of this vertex and their neighboring "rings" $R^1$and R^2, then to some neighborhood of the union $R^1 \cup R^2$ and vertices that touched R^1 and R^2 and so on we obtain homeomorphism between U_1^4 and U_2^4. It is evident that on each step we may choose an extension in such a way that it preserves the Liouville fibers.

5 Admissible CL-type

Let $[C,(B_w,G_w),(B_b,G_b)]$ be the CL-type of some saddle neighborhood. Suppose that G_w is colored in white, G_b is colored in black.

Definition 5.1. Let $[C,(B_w,G_w),(B_b,G_b)]$ be the CL-type of some saddle neighborhood. We say that a 2-cell $W \subset C$ is *a square* if the following conditions hold.

(1) W is a quadrangle.

(2) The opposite sides of W are colored in the same colors: two in white, two in black.

(3) The opposite sides of W have parallel orientation (see Fig. 2).

Two of its sides are called repellent, another two sides are called attractive (see Fig. 2).

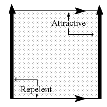

Figure 2: *Square*

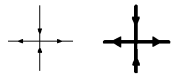

Figure 3: *Orientation of rays of crosses*

Sometimes we will assume that every square has the same canonical linear structure as the square $\{0 \leq x \leq 1, 0 \leq y \leq 1\}$ has.

Property 1. Each 2-cell of the saddle complex C (with colored 1-skeleton C_1) is a square.

Let P be a vertex of C.

Property 2. Some neighborhood of P is homeomorphic to the direct product of two pairs of intersecting segments (i. .e, of two crosses). If we assume that one of these pairs is black, another is white and the rays of these crosses are oriented as in Fig. 3, we can choose this homeomorphism in such a way that it will preserve color and orientation.

This property is evident.

Corollary *Each edge of C is a result of attaching together of 4 sides of the squares.*

Let G be the critical graph of the atom (B^2, G) (B^2 may be nonconnected, the decompositions of $B^2 \backslash G$ into positive and negative rings is fixed).

Let e_1, e_2 be the edges of G with the common vertex Q.

Definition 5.2. The angle $(e_1, e_2)_Q$ is called *positive* if there exist a positive ring R^2 such that $\overline{(R^2 \cap U_Q)} \supset [(e_1 \cup e_2) \cap U_Q]$ for any neighborhood U_Q of Q.

Let a square $xay*$ belong to C. We will say that the edge x corresponds to the edge y along the edge a.

Property 3. Suppose that the edge x corresponds to x_1 along a, the edge y corresponds to y_1 along a, and A_1 and A_2 are the vertices of the edge a. Then the angles $(x, y)_A$ and $(x_1, y_1)_{A_1}$ are simultaneously positive or nonpositive.

Definition 5.3. The triple $[C,(B_w,G_w),(B_b,G_b)]$, where C is a finite CW-complex with oriented 1-skeleton, (B_w,G_w) and (B_b,G_b) are atoms, $C\cap B_w=G_w, C\cap B_b=G_b$, is called *an admissible CL-type* if it satisfies the properties 1, 2 and 3.

6 Realization theorems for saddle and focus neighborhoods.

Theorem 6.1 *For any natural number n there exists a focus neighborhood with the weight n.*

Theorem 6.2 *For any admissible CL-type $[C,(B_w,K_w),(B_b,K_b)]$ there exists a saddle neighborhood with CL-type $[C,(B_w,K_w),(B_b,K_b)]$.*

Proofs. Let us suppose that quadruple (A^4,ω,H,f) sets an integrable Hamiltonian system (here (A^4,ω) is a symplectic manifold, H is a Hamiltonian and f is an additional integral), we do not require the compactness of fibers. Suppose that a discrete group G acts on the manifold A and preserves the symplectic form ω, the hamiltonian H and the additional integral f. It is clear that $(A^4/G,\omega,H,f)$ sets an integrable Hamiltonian system.

To prove the theorems 1, 2 we will construct the integrable Hamiltonian systems (F,ω,H,f) and (S,ω,H,F) and two systems of discrete groups $FG=\{G_i\}(i=1,2,\ldots)$ and $SG=\{G_\alpha\}$ (α runs over the set of all admissible CL-types) such that

$(F/G_i,\omega,H,f)$ is a focus neighborhood with the weight i,

$(S/G_\alpha,\omega,H,F)$ is a saddle neighborhood with the Cl-type α.

The construction of the integrable Hamiltonian system (F,ω,H,f) and the system of groups FG.

It is known that there exists a focus neighborhood $(\tilde{F},\tilde{\omega},\tilde{H},\tilde{f})$ with weight 1. For example, the focus-focus point appearing in the famous Lagrange case in rigid body dynamics has such a neighborhood. It is easy to see that the fundamental group $\pi_1(\tilde{F})$ is $\mathbb{Z}$. Indeed, the singular fiber of this system is the deformation retract of $\tilde{F}$. The singular fiber is a 1-chain, so its fundamental group is $\mathbb{Z}$.

Let (F,p) be the universal covering of $\tilde{F}$ and $f=p^*(\tilde{f}), H=p^*(\tilde{H}), \omega=p^*(\tilde{\omega})$.

Note that the group $\mathbb{Z}$ acts on F as the fundamental group of the base. This action preserves H,f,ω. Let G_i be the subgroup of $\mathbb{Z}$, generated by the number i. By definition, put $FG=\{G_i\}(i=1,2,\ldots)$.

To complete the proof of the theorem 1, it is sufficient to note that $(F/G_i,\omega,H,f)$ is a focus neighborhood with the weight i.

The construction of the integrable Hamiltonian system (S,ω,H,f) and the system of groups FS

Let (P^2, K) be an atom (possibly nonconnected). Mark (for example, by arcs) all the positive angles of the graph K. One can see that the graph K and marked positive angles define the atom (P^2, K) uniquely. Therefore, CL-type is defined uniquely by the saddle complex C and marked positive angles.

Let $[C, (B_w, K_w), (B_b, K_b)]$ be an admissible CL-type.

Definition 6.1. The saddle complex C together with coloring of the edges of its 1-skeleton (as above, see the previous section) and marking positive angles is called *the marked complex* of the CL-type $[C, (B_w, K_w), (B_b, K_b)]$.

So, if we prove that for any marked complex C^* there exists a saddle neighborhood with the marked complex C^* we have proved theorem 2.

Let C_1^* and C_2^* be marked complexes (possibly, infinite).

Definition 6.2. A cell mapping $\vartheta : C_1^* \to C_2^*$ is called *a right mapping* if the following conditions hold.

(1) ϑ preserves color and orientation.

(2) If an angle $(e_1, e_2)_A$ is positive, then the angle $(\vartheta(e_1), \vartheta(e_2))_{\vartheta(A)}$ is positive too.

(3) ϑ is a linear mapping on the cells of C_1^*.

Lemma 6.1 *Let C^* be a simply connected marked complex, C_1^* an arbitrary marked complex, $\alpha \in C^*, \alpha_1 \in C_1^*$ their 2-cells. Then there exists a unique right mapping $\kappa : C^* \to C_1^*$ such that $\kappa(\alpha) = \alpha_1$.*

Proof. Let α' be an arbitrary 2-cell of C. Choose a path γ on C^* such that

(1) γ goes from the center of α to the center of α'.

(2) γ transversally intersects 1-skeleton of C^* at the interior points of edges.

Let $e_1, e_2, .., e_n$ be the edges of C^*, sequentially intersected by γ. Let γ go sequentially through 2-cells $\beta_1 = \alpha, \beta_2, \beta_3, .., \beta_n = \alpha'$. We can associate the sequence of quadruples (t_i, β_i, s_i, n_i) with any path, where

t_i is a coordinate of the point of intersection of e_i and γ,

β_i=1 if e$_i$ is a black edge, $\beta_i = -1$ if e_i is a white edge,

$s_i = 1$ if the edge e_i is the repellent edge with respect to $\beta_i, s_i = -1$ in another case,

$n_i = 0$ if e_{i-1} and e_i are both either repellent or attractive with respect to $\beta_i, n_i = -1$ if β_{i-1} and β_i form a positive angle and $n = -1$ in another case.

It is easy to see that there is a path γ_1 on C^* such that it begins in the center of α' and the sequence of quadruples of γ_1 is equal to the sequence of quadruples of γ.

Let the end of γ_1 belong to the cell α_1' . Construct the linear mapping $\kappa_{\alpha_1} : \alpha_1 \to \alpha'^1$ preserving color and orientation. Since the complex C^* is simply connected the definition of κ_{α_1} is unambigious. (It does not depend

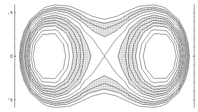

Figure 4: *Level curves of* $\tilde{H}$

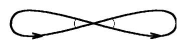

Figure 5: *Figure eight curve with oriented edges and marked positive angles*

on the path.) It is evident that if two cells α_1 and α_2 have a common edge or vertex then $\kappa_{\alpha_1}|_{\overline{\alpha}_1\cap\overline{\alpha}_2} = \kappa_{\alpha_2}|_{\overline{\alpha}_1\cap\overline{\alpha}_2}$. The set of mappings $\{\kappa_\alpha\}$ yields the mapping κ we need.

Corollary *Let C_1^*, C_2^* be simply connected. Then there exist a right homeomorphism between C_1^* and C_2^*.*

Consider the symplectic space $\mathbb{R}^4$ with coordinates p_1, q_1, p_2, q_2 and the form $dp_1 \wedge dq_1 + dp_2 \wedge dq_2$. Functions $\tilde{H} = ((p_1-1)^2+q_1^2)((p_1+1)^2+q_1^2)$ and $\tilde{f} = ((p_2-1)^2+q_2^2)((p_2+1)^2+q_2^2)+((p_1-1)^2+q_1^2)((p_1+1)^2+q_1^2)$ commute, so $(\mathbb{R}^4, \tilde{\omega}, \tilde{H}, \tilde{f})$ sets an integrable Hamiltonian system. Level curves of $\tilde{H}$ are shown in Fig. 4. The set $\tilde{S} = \{0 < \tilde{H} < 2, 0 < \tilde{f} - \tilde{H} < 2\}$ is a saddle neighborhood. The marked complex $\tilde{C}^*$ of this neighborhood is the direct product of white and black figures eight curves by assuming that the orientation of edges and marking positive angles (by arcs) on figures eight curves is shown in Fig. 5.

Let (S, p) be the universal covering of $\tilde{S}$. Consider $H = p^*(\tilde{H}), f = p^*(\tilde{f}), \omega = p^*(\tilde{\omega})$. (S, ω, H, f) sets an integrable Hamiltonian system (with noncompact fibers). Consider the marked complex of this system. It is the direct product of two (black and white) infinite 4-degree trees.

Let G be the group of right automorphisms of C^*. One can note that G is the direct product of groups of right automorphisms of the white and the black tree. Each of these two groups is generated by two elements (a, s, for example) with one relation: $s^2 = 1$. Actually, such marked and oriented tree is the universal covering of the figure eight curve from Fig. 5. So the group of automorphisms includes the fundamental group of figure eight curve (elements a and b) and a central symmetry (s). And it is clear that any automorphism can be obtained as a composition of some elements above.

Let $\tilde{G}$ be the group of automorphisms of (S, ω, H, f). There exist natural homomorphism $\vartheta : G \to \tilde{G}$. Actually, the subgroup of $\tilde{G}$, generated by the fundamental group of base and the symmetries relatively the planes (p_1, q_1) and (p_2, q_2) is exactly the group G.

The last step. Let C_1^* be an arbitrary marked complex. Consider its universal covering (C^*, p'). (It follows the above corollary that the simply connected covering space is S). Consider the subgroup $G_{C_1^*}$ such that $C^*/G_{C_1^*} = C_1^*$. Let $\vartheta(G_{C_1^*}) = \tilde{G}_{C_1^*}$.

It is easy to see that $S/\tilde{G}_{C_1^*}$ is a saddle neighborhood with the marked complex C_1^*, so $SG = \{\tilde{G}_{C_\alpha^*}\}$ and S are the objects we need.

7 How many

Saddle neighborhoods can be algorithmically listed. This has been done for saddle neighborhoods with 1,2 and 3 singular points.

Theorem 7.1 *Let n be the number of the saddle-saddle points in a saddle neighborhood.*

- *If n=1 then there exist exactly 4 non-equivalent saddle neighborhoods.*
- *If n=2 then there exist exactly 39 non-equivalent saddle neighborhoods.*
- *If n=3 then there exist exactly 256 non-equivalent saddle neighborhoods.*

This theorem was proved by L. Lerman and Ya. Umanskii ($n = 1$), A. Bolsinov ($n = 2$) and N. Maximova ($n = 3$). The list of all possible admissible CL-types with one and two vertices is given in Table 1. In this table the positive rings on non-connected atoms are shaded, the horizontal sides of the squares are directed from left to right, the vertical ones are directed upward. The simplest atoms A, B, C, D have been described in [16] and we omit their description here. Some new atoms appearing within the framework of this theory are described in Table 2.

8 Center-center and center-saddle cases

The aim of this section is to classify topologically integrable Hamiltonian systems in the saturated regular neighborhood of a singular fiber containing center-center or center-saddle points.

Definition 8.1. A saturated open set $U^4 \subset M^4$ is called *center-center neighborhood* if the following conditions hold:

(1) The bifurcation diagramm of U^4 is an embedding of the pair of half-open intervals $I_1 = \{p = 0, 1 > q \geq 0\}$, $I_2 = \{q = 0, 1 > p \geq 0\}$ with the common origin (Denote the imbedding of origin by A). Embedding of each interval on the plain $\mathbb{R}^2(H, f)$ is the graph of some smooth function $\alpha = \alpha(H)$.

(2) The preimage $F^{-1}(A)$ is connected.

(3) Any singular point of U^4 is non-degenerate.

(4) U^4 is homeomorphic to 4-ball.

Theorem 8.1 [4,20] *Each center-center neighborhood is topoloically equivalent to center-center neighborhood* (U^4, ω, H, f), *where* $U^4 = \{p_1^2 + q_1^2 < 1, p_2^2 + q_2^2 < 1\}$, $H = p_1^2 + q_1^2$, $f = p_2^2 + q_2^2 + H$, $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$.

Definition 8.2. A saturated open set $U^4 \subset M^4$ is called *a center-saddle neighborhood* if the following conditions hold.

(1) Exactly one fiber of U^4 includes zero rank points and those points have the center-saddle type (denote this fiber by L).

(2) U^4 is a regular neighborhood of L.

(3) Any curve on the bifurcation diagram of U^4 can be represented as a graph of some smooth function of H.

The following lemma follows from the local structure in a small neighborhood of non-degenerate points.

Lemma 8.1 *L is a connected graph whose vertices are all of degree 4.*

Consider the set of critical points K. Let us recall that proposition 2.1 guarantees the existence of two critical two-dimensional manifolds P_1 and P_2 in some neighborhood of L. It is easy to see that one of them (for example, P_2) contains L.

Lemma 8.2 (P_2, L) *is a connected atom without stars.*

Since the vector field sgrad$H$ is not equal to 0 on the edges of L then we can naturally orient the edges of L.

Definition 8.3. The atom (P^2, L) (together with the natural orientation on its edges) is called *the L-type* of the center-saddle neighborhood.

Theorem 8.2 *L-type is a complete topological invariant of center-saddle neighborhoods (i. e., two center-saddle neighborhoods are topologically equivalent if and only if their L-types are identical).*

Definition 8.4. Let (P^2, L) be an atom. An orientation of a 4-degree graph L is called admissible if it is induced by some orientation of boundaries of rings of decomposition $P^2 \backslash L$.

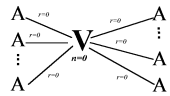

Figure 6: *Circle molecule of center-saddle neighborhood with L-type $V = (P^2, G)$. Number of letters A in right part equals to number of positive rings, number of letters A in left part equals to number of negative rings of letter V*

Theorem 8.3 *Let (P^2, L) be a connected atom without stars and the orientation of L be admissible. Then there exists a center-saddle neighborhood such that its L-type is equal to (P^2, L) with the orientation.*

Proof. It is clear that there exists a function $g : P^2 \to \mathbb{R}$ such that its single critical level coincides L. Consider an area form $\tilde{\omega}$ on P^2. It is evident that either $\mathrm{sgrad}(g)$ or $\mathrm{sgrad}(-g)$ orients the graph L as it is necessary. Consider the disc

$$D^2 = \{p_1 + q_1 < 1\},$$

and put $V^4 = D^2 \times P^2$, $f = p_1^2 + q_1^2$, $H = \pm g + f$, $\omega = dp_1 \wedge dq_1 + \tilde{\omega}$. It is evident that $\{V^4, \omega, H, f\}$ is an integrable Hamiltonian system with the desired L-type.

9 Circle molecules.

Theorem 9.1 The circle molecule of a center-center point is

$$A \to A$$

Theorem 9.2 *The circle molecule of the center-saddle neighborhood with L-type $V = (P^2, G)$ is shown in Fig. 6.*

These theorems are evident.

Theorem 9.3 *There exists an algorithm of construction of circle molecule by the CL-type.*

This algorithm has been proposed by A. Bolsinov.

The idea of the algorithm. Consider the saddle neighborhood $\tilde{S}$ from the proof of the theorem 2. Cut $\tilde{S}$ along the sets $\{\mid p_1 \mid < 1,\ q_1 = 0\}$, $\{p_1 = 0\}$, $\{\mid p_2 \mid < 1,\ q_2 = 0\}$, $\{p_2 = 0\}$. Each level becomes decomposed into the collection of squares. The structure of the squares is induced by the decomposition of the saddle complex. Since the subgroup $\vartheta(G)$ preserves such squares, we see that the fibers on each saddle neighborhood are decomposed in squares, and reglueing of squares comply with some rules of reglueing of squares of $\tilde{S}$ case. Bolsinov's algorithm is exactly the formalization of such rules.

The list of circle molecules of saddle neighborhoods with 1 and 2 vertices is given in Table 1.

Consider finally the focus-focus case. Let U be a focus neighborhood. The circle molecule of U is not a molecule in the usual sense because it has no atoms. Indeed, the preimage of a small circle on the bifurcation diagram is a fiber bundle over the circle whose fibers are 2-dimensional tori. Every such bundle can be given by a conjugacy class of 2×2 matrices.

Theorem 9.4 *For a focus neighborhood with the weight n this fiber bundle is determined by the conjugacy class of the matrix* $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

This theorem can be proved by standard methods of covering mapping theory.

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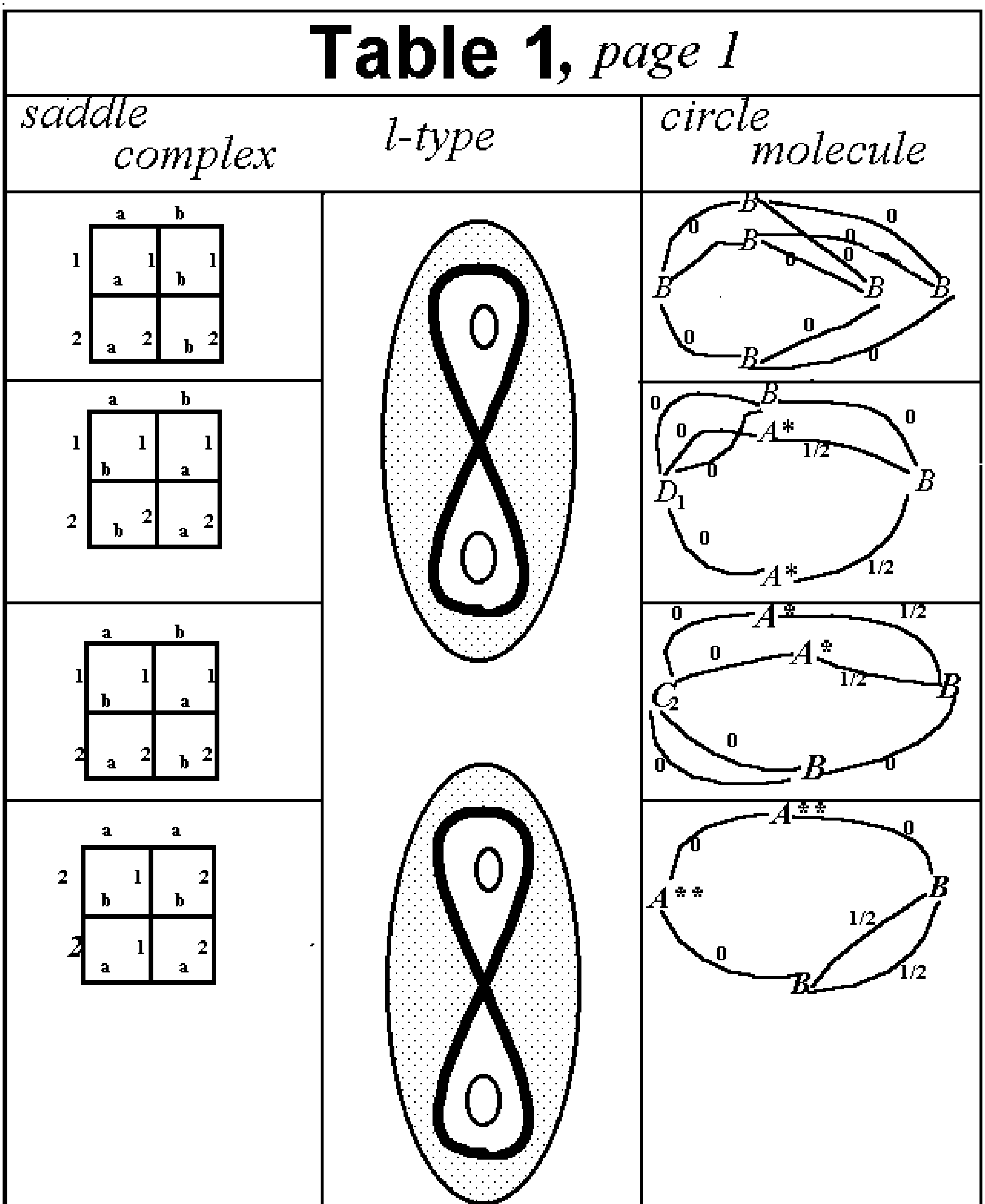


Figure 7: Trajectories of the vector field $\lambda\vec{B}$ for the Goryachev-Chaplygin case

Table 1, *page 2*

saddle complex	*l-type*	*circle molecule*

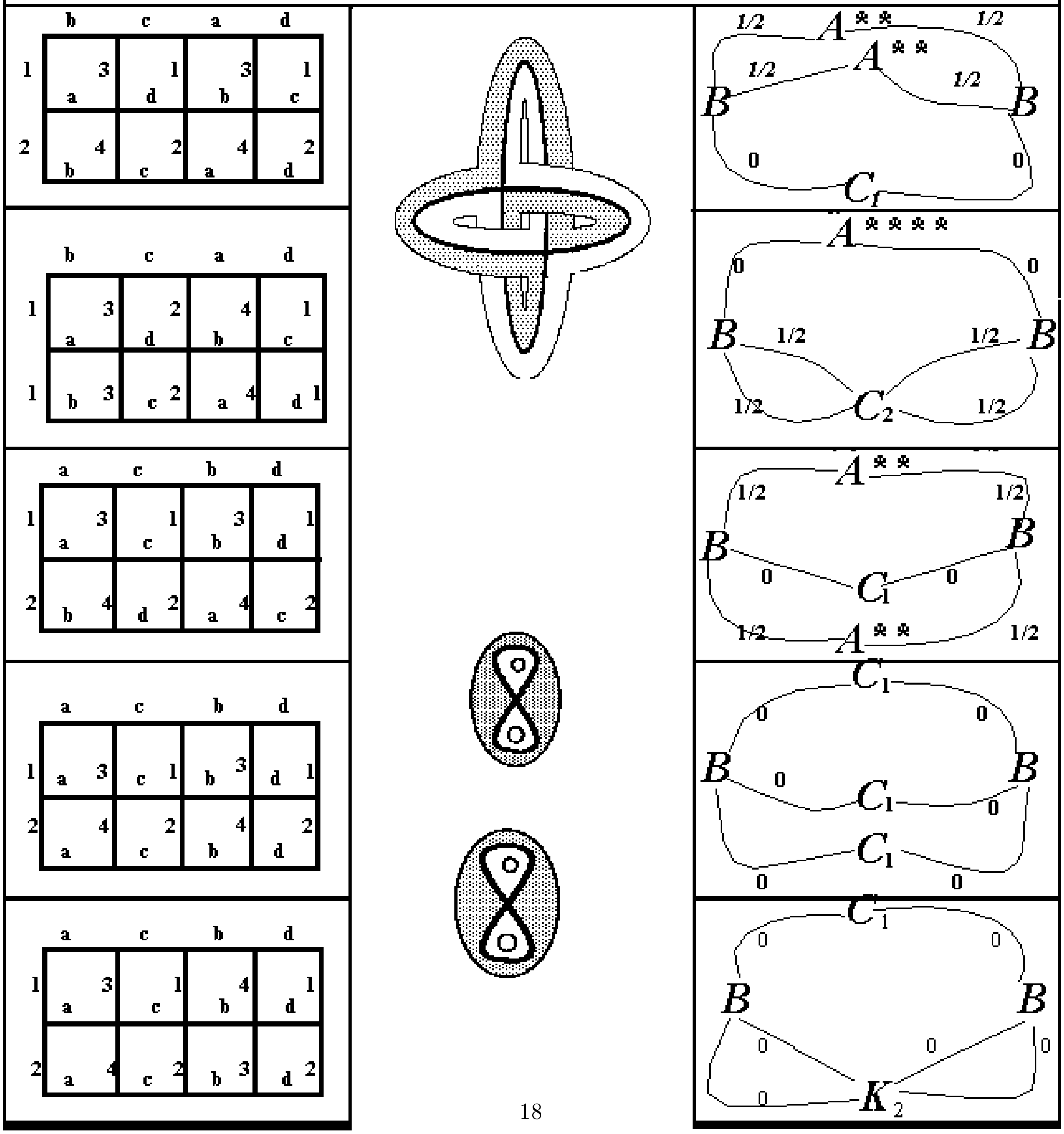


Figure 8: Trajectories of the vector field $\lambda\vec{B}$ for the Goryachev-Chaplygin case

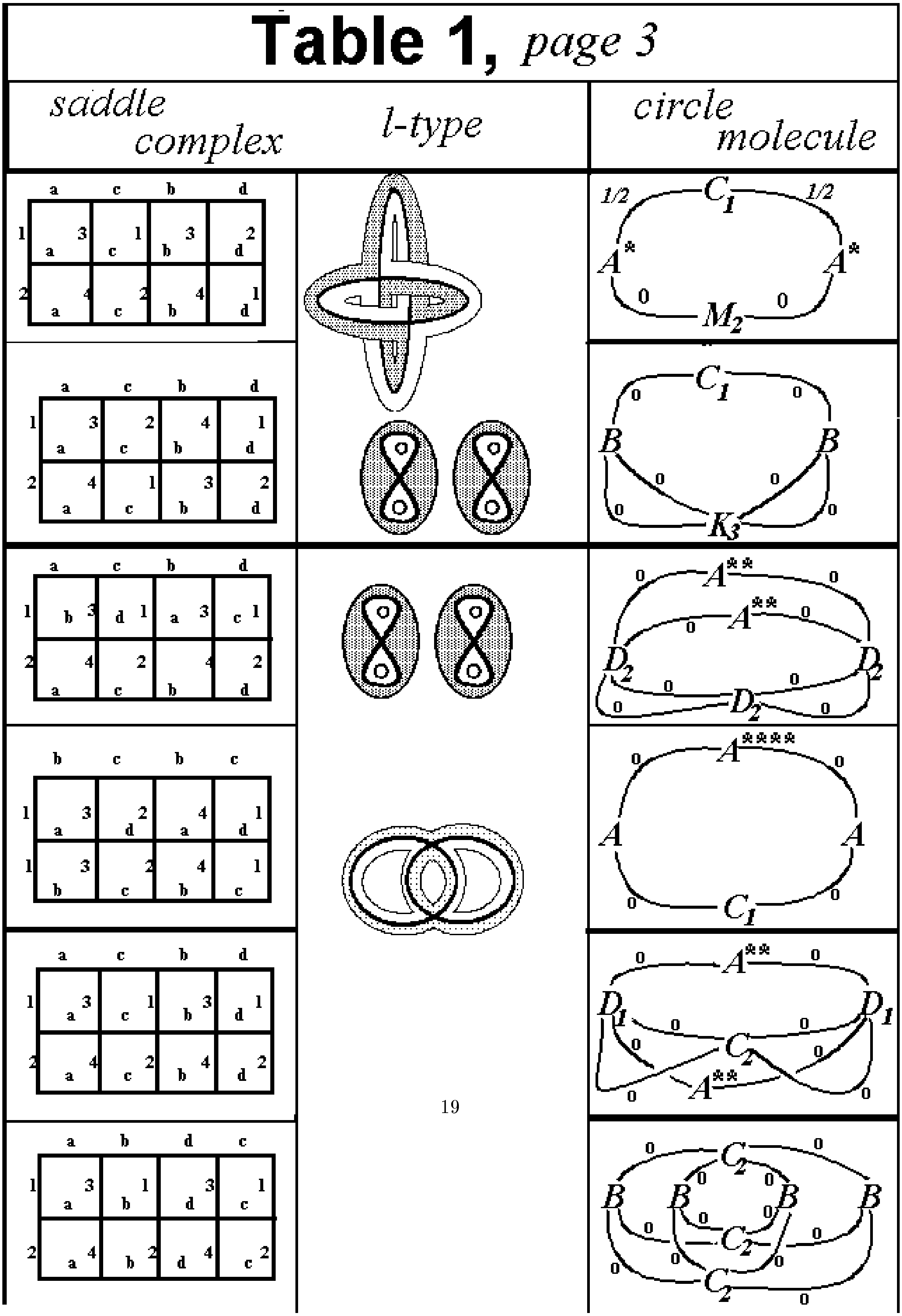


Figure 9: Trajectories of the vector field $\lambda\vec{B}$ for the Goryachev-Chaplygin case

Table 1, *page 4*

saddle complex	*l-type*	*circle molecule*

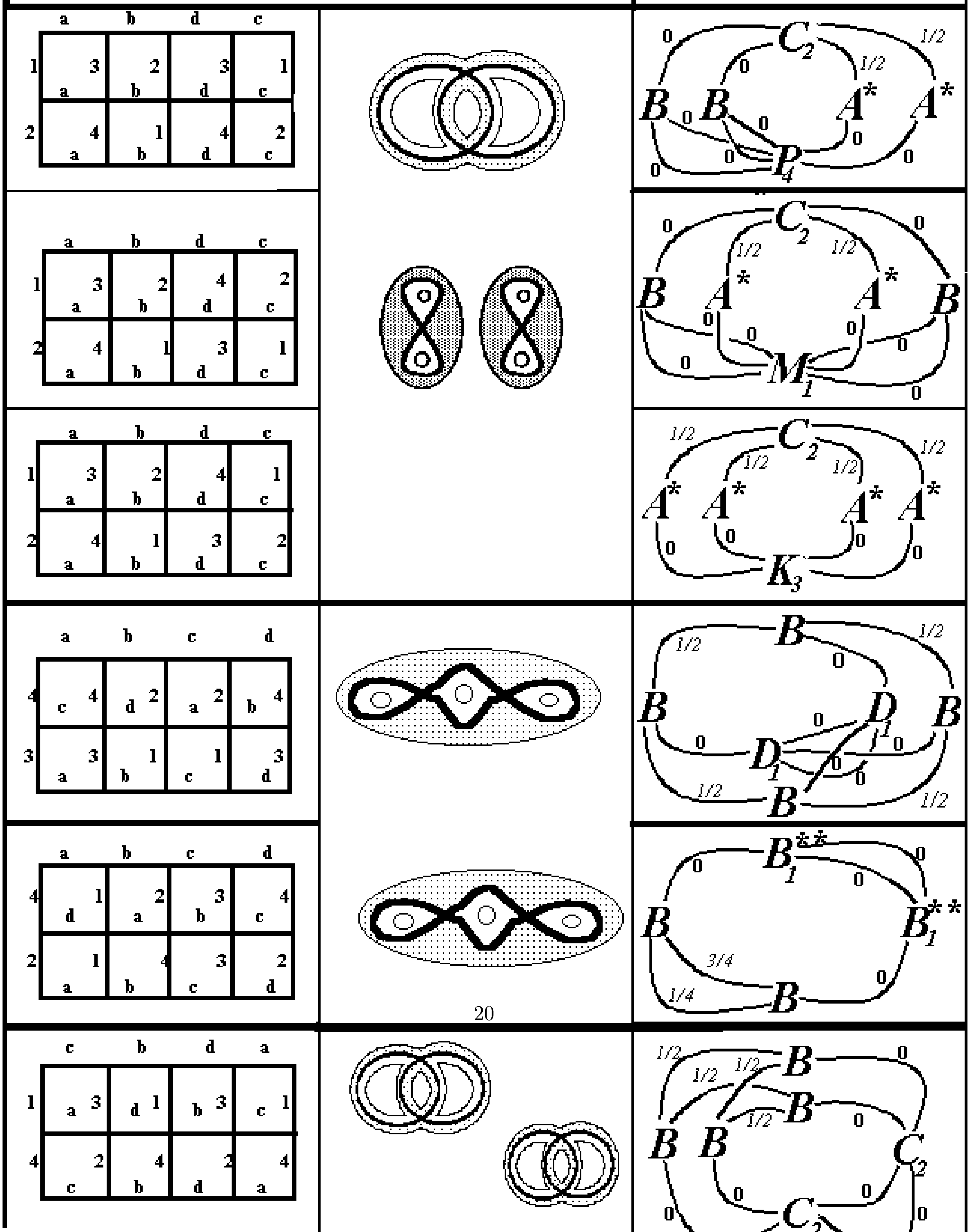


Figure 10: Trajectories of the vector field $\lambda\vec{B}$ for the Goryachev-Chaplygin case

Table 1, *page 5*

saddle complex	*l-type*	*circle molecule*

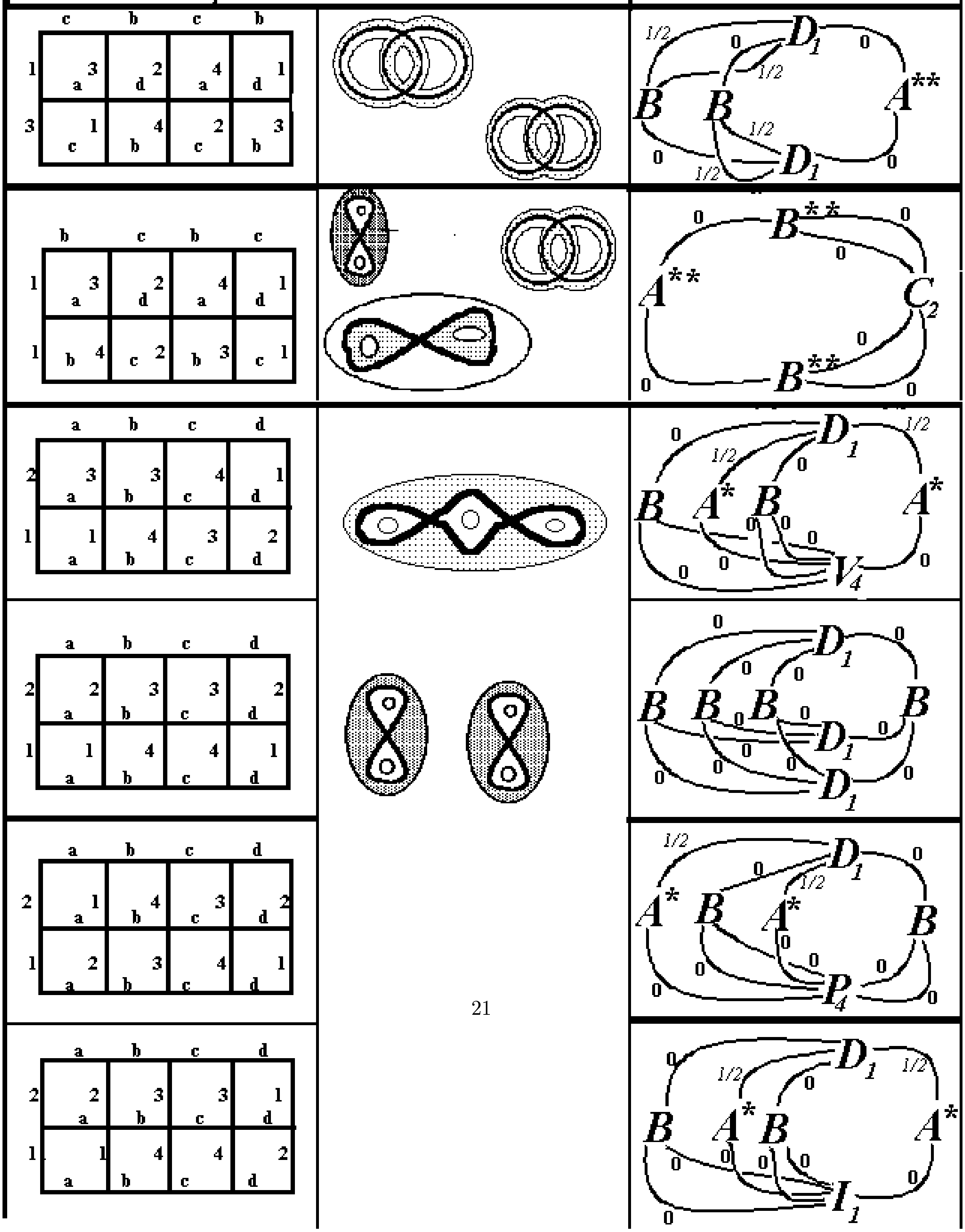


Figure 11: Trajectories of the vector field $\lambda\vec{B}$ for the Goryachev-Chaplygin case

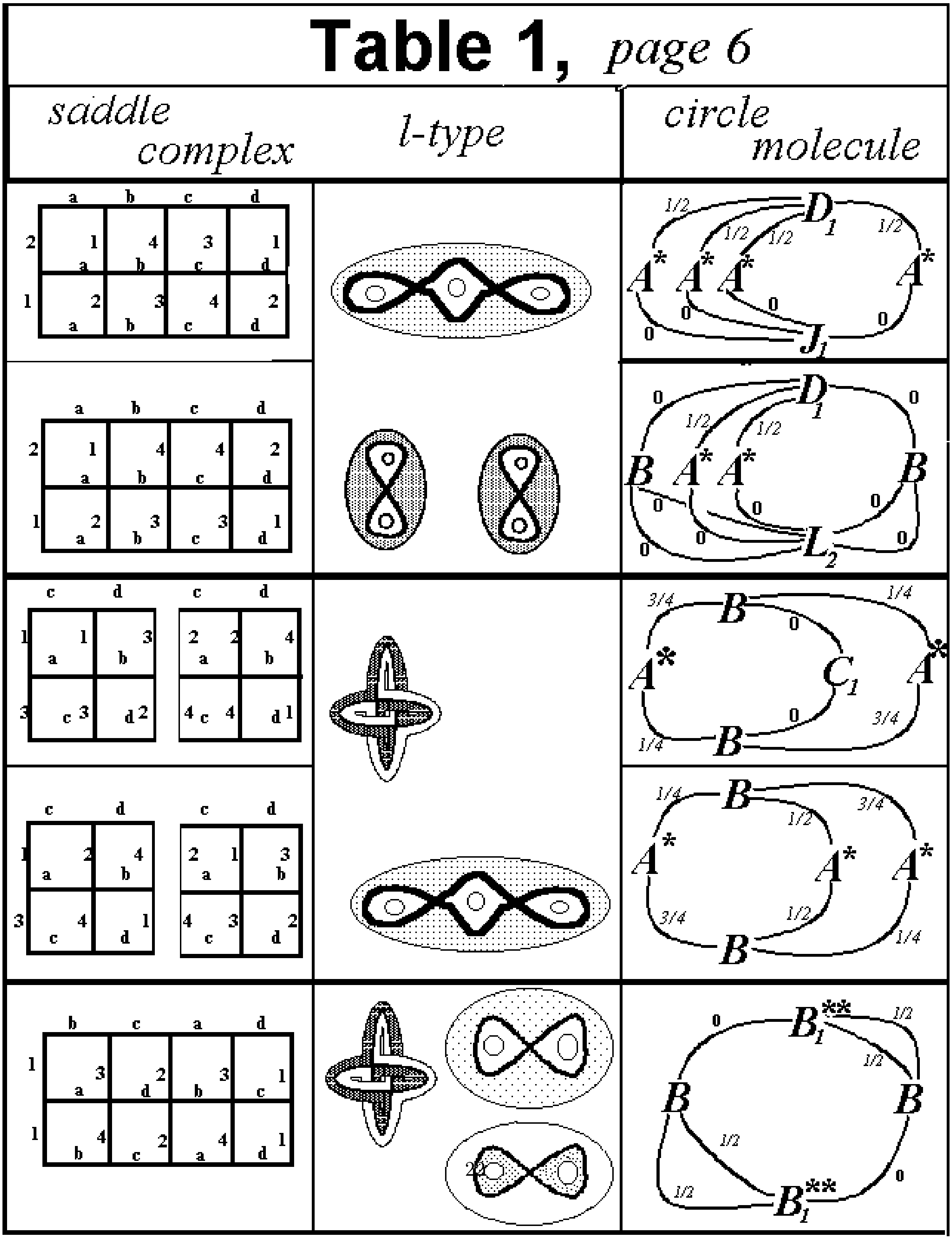


Figure 12: Trajectories of the vector field $\lambda\vec{B}$ for the Goryachev-Chaplygin case

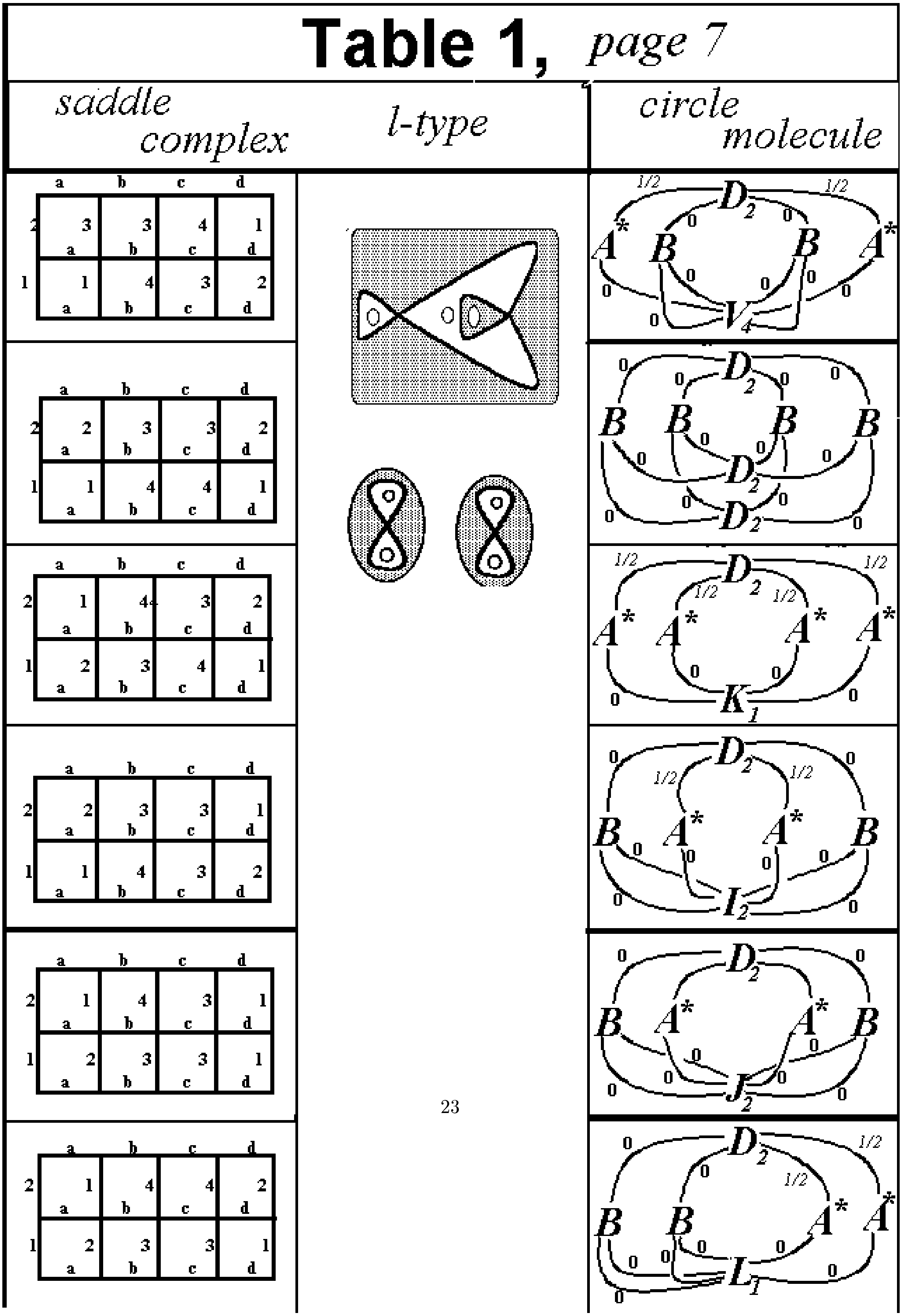


Figure 13: Trajectories of the vector field $\lambda\vec{B}$ for the Goryachev-Chaplygin case

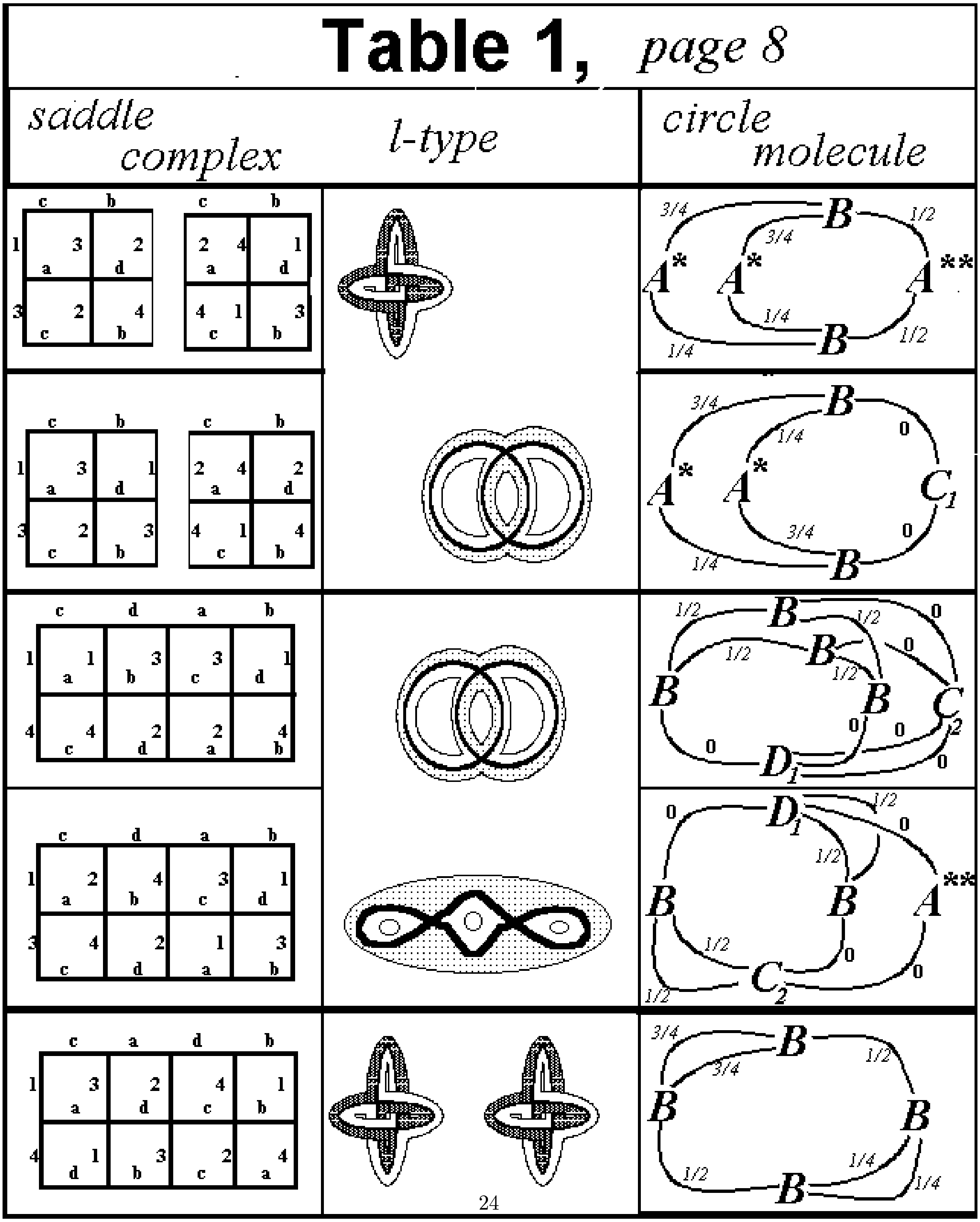


Figure 14: Trajectories of the vector field $\lambda\vec{B}$ for the Goryachev-Chaplygin case

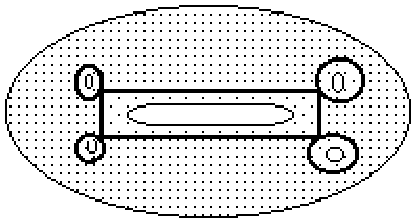
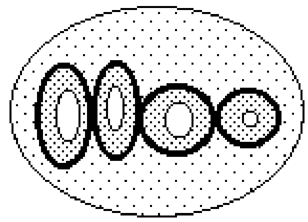
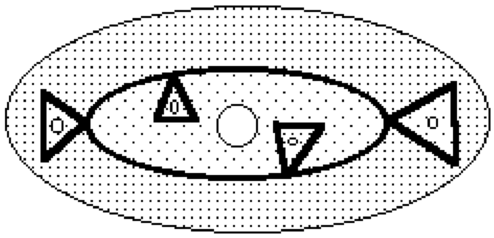
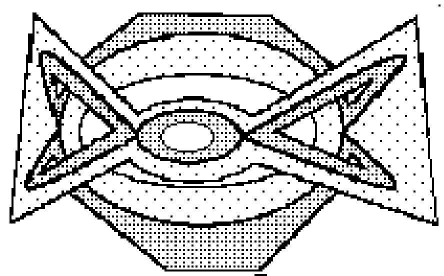
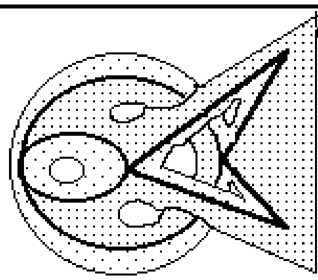
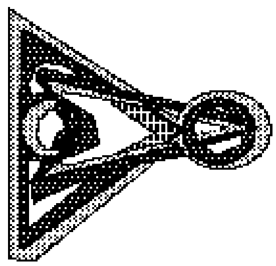
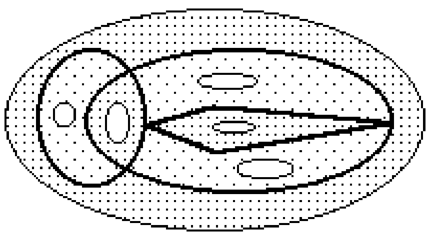
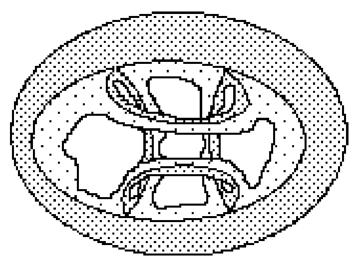
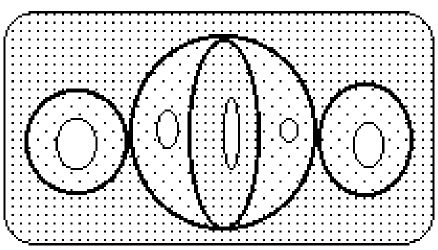
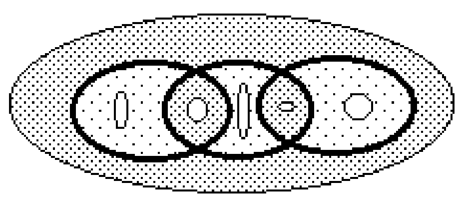
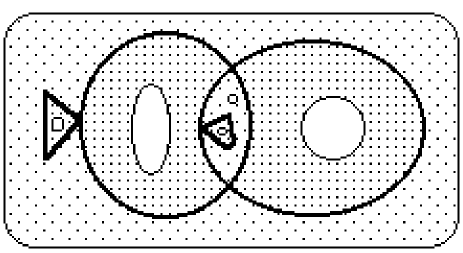
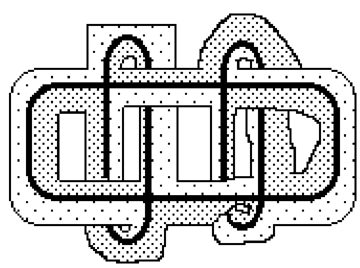
Table 2			
name	atom	*name*	atom
$\mathbf{I}_1$		$\mathbf{V}_4$	
$\mathbf{I}_2$		$\mathbf{K}_1$	
$\mathbf{J}_1$		$\mathbf{K}_2$	
$\mathbf{J}_2$		$\mathbf{K}_3$	
$\mathbf{L}_1$		$\mathbf{M}_1$	
$\mathbf{L}_2$		$\mathbf{M}_2$	

Figure 15: Trajectories of the vector field $\lambda\vec{B}$ for the Goryachev-Chaplygin case