On the equation $(Du)^t\, H\, Du = G$

S. Bandyopadhyay, B. Dacorogna, V. Matveev and M. Troyanov

S.B.: IISER Kolkata, India; saugata.bandyopadhyay@iiserkol.ac.in

B.D. and M.T.: Section de Mathématiques EPFL, 1015 Lausanne, Switzerland;
bernard.dacorogna@epfl.ch and marc.troyanov@epfl.ch

V.M.: Institut für Mathematatik, Friedrich-Schiller-Universität, 07737 Jena, Germany;
vladimir.matveev@uni-jena.de

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Abstract

In this article, we want to find a map $u:\overline{\Omega}\to\mathbb{R}^n$ solving, in Ω, the equation

$$u^*(H) = G \quad i.e. \quad (Du)^t\, H(u)\, Du = G$$

and coupled, on $\partial\Omega$, either with the Dirichlet-Neumann problem

$$u = \varphi \quad \text{and} \quad Du = D\varphi$$

or the purely Dirichlet problem

$$u = \varphi$$

where $\Omega\subset\mathbb{R}^n$ is a bounded open set, $G, H:\mathbb{R}^n\to\mathbb{R}^{n\times n}$ and $\varphi:\overline{\Omega}\to\mathbb{R}^n$ are given. We discuss the case where G and H are not necessarily symmetric or skew-symmetric, but have invertible symmetric parts.

1 Introduction

1.1 Statement of the problem

Given $\Omega\subset\mathbb{R}^n$ a bounded open set, $G, H:\mathbb{R}^n\to\mathbb{R}^{n\times n}$ and $\varphi:\overline{\Omega}\to\mathbb{R}^n$, we wish to find a map $u:\overline{\Omega}\to\mathbb{R}^n$ solving, in Ω, the equation

$$u^*(H) = G \quad i.e. \quad (Du)^t\, H(u)\, Du = G$$

and coupled, on $\partial\Omega$, either with the Dirichlet-Neumann problem

$$u = \varphi \quad \text{and} \quad Du = D\varphi$$

or the purely Dirichlet problem

$$u = \varphi.$$

We establish existence, uniqueness and regularity of solutions of both problems; see Theorem 15 and Corollary 17 for the first one and Theorem 18 and Corollaries 20 and 22 for the second one.

1.2 Some consequences

An important feature of our results is that we do not assume that G and H are either symmetric or skew-symmetric, but, however, they have invertible symmetric parts. An immediate observation is that the differential equation decouples into

$$(Du)^t H_s\, Du = G_s \quad \text{and} \quad (Du)^t H_a\, Du = G_a \tag{1}$$

where the indices s and a denote the symmetric and skew-symmetric parts of a general matrix. The above observation can be formulated differently and in a more striking way. Given symmetric matrices $H_s\,, G_s$ and skew-symmetric ones $H_a\,, G_a$, we will find, under appropriate conditions, u solving simultaneously the two equations in (1).

Another interesting observation is that we will be able to find a diffeomorphism $u : \overline{\Omega} \to \overline{\Omega}$ solving the equation $(Du)^t H\, Du = G$ and satisfying the very strong boundary condition

$$u = \mathrm{id} \text{ and } Du = I_n \quad i.e. \quad u = \mathrm{id} \text{ and } \partial_\nu u = \nu$$

i.e. solving simultaneously the Dirichlet and the Neumann problems. Such a map will be called a *buckling diffeomorphism*; note that they form a subgroup of the group of diffeomorphisms.

1.3 Some motivations

A natural problem in differential geometry is to determine under which conditions a given tensor field G is equivalent, under a diffeomorphism, to a *constant* tensor field H. The tensor field G is understood here as a covariant $2-$tensor, that is the bilinear form

$$G \sim \sum_{i,j=1}^{n} g_{ij}\,(x)\, dx^i \otimes dx^j .$$

The pullback equation $u^*\,(H) = G$ reads in coordinates as

$$\sum_{k,l=1}^{n} h_{kl} du^k du^l = \sum_{i,j=1}^{n} g_{ij}\,(x)\, dx^i dx^j$$

or, in matrix form,

$$(Du)^t H\, Du = G.$$

Two main cases have received considerable attention.

- G and H (essentially $H = I_n$, the identity matrix) are symmetric. This problem is of fundamental importance in Riemannian geometry, where one wants to determine if a given metric (g_{ij}) is globally isometric to the standard Euclidean metric. The boundary condition $u = \mathrm{id}$ means that the given metric coincides with the Euclidean one on $\partial\Omega$. A particular case of this problem can be reformulated in terms of elasticity; there G is the so called Cauchy-Green tensor. The geometrical problem finds its origins in the work of Riemann.

- G and H are skew-symmetric; in geometry they represent differential $2-$forms. If the forms are non-degenerate and closed, they are then called symplectic forms. The equivalence of symplectic forms is of fundamental importance in symplectic geometry and its study finds its origins in the work of Darboux.

An important difference, from the point of view of partial differential equations, between the two cases is that the first one is elliptic and not the second one (see Proposition 32; a way to remedy to the absence of ellipticity, in the skew-symmetric case, can be found in [10]). This leads to uniqueness and straightforward regularity results in the symmetric case. However when the matrices are skew-symmetric, the regularity is much more involved and there is strong non-uniqueness.

1.4 The linear problem

We conclude this introduction by briefly discussing the linearized problem. It has also been much studied; see, for example, [5] or [6, Theorem 6.18 when $H = I_n$]. It reads as

$$H\, Du + (Du)^t\, H = G.$$

Upon setting $v = H\, u$, the linearized equation becomes when H and G are symmetric, respectively skew-symmetric

$$Dv + (Dv)^t = G \quad \text{respectively} \quad Dv - (Dv)^t = G$$

which behave very differently from the point of view of necessary conditions, uniqueness and regularity, the first one is again elliptic contrary to the second one, which is nothing else than Poincaré lemma for $1-$forms.

2 Notations and preliminaries

2.1 Notations

We use the following notations in this article.

(i) Let $A \in \mathbb{R}^{n\times n}$.

- For every $i, j = 1, \cdots, n$, A_{ij} denotes the (i, j)-th element of A. Furthermore, we write $A_{i,*}$ and $A_{*,j}$ to denote the i-th row and j-th column of A respectively.

- We denote the symmetric and skew-symmetric parts of A by A_s and A_a respectively, namely

$$A_s = \frac{1}{2}\left(A + A^t\right) \quad \text{and} \quad A_a = \frac{1}{2}\left(A - A^t\right).$$

(ii) $\{e_1, \cdots, e_n\}$ denotes the standard orthonormal basis of $\mathbb{R}^n$. For $a, b \in \mathbb{R}^n$, we denote the scalar product by $\langle a; b\rangle$.

(iii) Let $a, b \in \mathbb{R}^n$. The tensor product of a and b is denoted by $a \otimes b$. Note that $(b \otimes a) = (a \otimes b)^t$. Furthermore, for every $A \in \mathbb{R}^{n\times n}$ and $a, b, c \in \mathbb{R}^n$, the following relations are easy to verify

$$(a \otimes b)\, c = a\, \langle b; c\rangle\,, \quad A\,(b \otimes c) = Ab \otimes c, \quad (b \otimes c)\, A = b \otimes A^t c.$$

2.2 Preliminaries

We begin with the definition of Christoffel symbols and recall some of their basic properties. In the present section $\Omega \subset \mathbb{R}^n$ stands for a given open set.

Notation 1 *Let $G = \left((g_{ij})_{1\leq i,j\leq n}\right) \in C\left(\Omega; \mathbb{R}^{n\times n}\right)$ be symmetric and non-degenerate. We write*

$$[G\,(x)]^{-1} = \left(\left(g^{ij}\,(x)\right)^{1\leq i,j\leq n}\right) \quad \textit{for every } x \in \Omega.$$

Definition 2 (Christoffel symbols) *Let $G = \left((g_{ij})_{1\leq i,j\leq n}\right) \in C^1\left(\Omega; \mathbb{R}^{n\times n}\right)$ be symmetric and non-degenerate. We define*

(i) Christoffel symbols (*of the second kind): for every $i, j, k = 1, \cdots, n$,*

$$\Gamma^j_{ik} = \frac{1}{2}\sum_{p=1}^{n} g^{jp}\left(\partial_i g_{kp} + \partial_k g_{ip} - \partial_p g_{ik}\right) \quad \textit{in } \Omega.$$

(ii) Christoffel matrices: *for every* $i = 1, \cdots, n$,

$$(\Gamma_i)_{jk} = \Gamma_{ik}^j \quad \textit{in } \Omega, \ \textit{for every } j, k = 1, \cdots, n.$$

Remark 3 *(i)* It is very convenient to see the set of Christoffel matrices $\{\Gamma_1, \cdots, \Gamma_n\}$ as a $1-$form over the set of matrices, i.e.

$$\Gamma = \sum_{i=1}^{n} \Gamma_i \, dx^i \in \Lambda^1 \left(\Omega; \mathbb{R}^{n \times n}\right).$$

This form is called the *Levi-Civita connection* of G. In particular, if $\Gamma, \Delta \in \Lambda^1 (\Omega; \mathbb{R}^{n \times n})$, then

$$d\Gamma = \sum_{1 \leq i < j \leq n} (\partial_i \Gamma_j - \partial_j \Gamma_i) \, dx^i \wedge dx^j \in \Lambda^2 \left(\Omega; \mathbb{R}^{n \times n}\right)$$

$$\Delta \wedge \Gamma = \sum_{1 \leq i < j \leq n} (\Delta_i \Gamma_j - \Delta_j \Gamma_i) \, dx^i \wedge dx^j \in \Lambda^2 \left(\Omega; \mathbb{R}^{n \times n}\right).$$

(ii) The matrix valued $2-$form

$$\mathcal{R}(G) = 2 \left(d\Gamma + \Gamma \wedge \Gamma\right)$$

is called the *Riemann-Christoffel curvature tensor* associated with G.

We state few classical elementary properties of Christoffel symbols; see pages 213 and 186-187 in [17].

Lemma 4 (Ricci lemma) *Let* $G = \left((g_{ij})_{1 \leq i,j \leq n}\right) \in C^1 (\Omega; \mathbb{R}^{n \times n})$ *be symmetric and non-degenerate. Then, for every* $i, j, k = 1, \cdots, n$ *and in* Ω,

(i) $\Gamma_{ij}^k = \Gamma_{ji}^k$

(ii) $dG = \Gamma^t G + G \, \Gamma$ *i.e.* $\partial_k G = (\Gamma_k)^t G + G \, \Gamma_k$.

Proposition 5 *Let* $H \in \mathbb{R}^{n \times n}$ *be constant, symmetric and invertible. Let* $u \in C^2 (\Omega; \mathbb{R}^n)$ *be such that* $\det Du(x) \neq 0$, *for every* $x \in \Omega$, *and* $G \in C^1 (\Omega; \mathbb{R}^{n \times n})$ *be defined, in* Ω, *as*

$$G = F^t H \, F \quad \textit{with } F = Du.$$

The Christoffel matrices $\{\Gamma_1, \cdots, \Gamma_n\}$ *(i.e.* Γ *is the Levi-Civita connection of* G*), in addition to the properties of Lemma 4, satisfy the following two conclusions.*

(i) $dF = F \, \Gamma$, *i.e.*

$$\partial_{ij} u = \sum_{r=1}^{n} \Gamma_{ij}^r \partial_r u, \quad \forall \, i, j = 1, \cdots, n$$

or equivalently

$$\Gamma_{ij}^k = \left\langle \left((Du)^{-1}\right)_{k,*} ; \partial_{ij} u \right\rangle = \Gamma_{ji}^k, \quad \forall \, i, j, k = 1, \cdots, n.$$

(iii) $d\Gamma + \Gamma \wedge \Gamma = 0$, *i.e.*

$$\partial_i \Gamma_j - \partial_j \Gamma_i + \Gamma_i \Gamma_j - \Gamma_j \Gamma_i = 0, \quad \forall \, i, j = 1, \cdots, n.$$

3 Global Frobenius theorem

3.1 Cauchy problem for Pfaff system

In the sequel we write $x = (x', x_n) = (x_1 \cdots, x_{n-1}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and, for $p = (p', p_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$,

$$C_{r,\epsilon}(p) = B_r(p') \times (p_n - \epsilon, p_n + \epsilon) = \left\{ (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x' - p'| < r \text{ and } |x_n - p_n| < \epsilon \right\}.$$

We start by defining the meaning of sets with Lipschitz boundary.

Definition 6 *A bounded open set $\Omega \subset \mathbb{R}^n$ is said to have* Lipschitz boundary, *if for every $p = (p', p_n) \in \partial\Omega$, there exist $r, \epsilon > 0$ and a Lipschitz function $\varphi : B_r(p') \subset \mathbb{R}^{n-1} \to (p_n - \epsilon, p_n + \epsilon)$ such that, upon rotation and relabeling of coordinate axes if necessary,*

$$\Omega \cap C_{r,\epsilon}(p) = \{x \in C_{r,\epsilon}(p) : x_n < \varphi(x')\} \quad \text{and} \quad \partial\Omega \cap C_{r,\epsilon}(p) = \{x \in C_{r,\epsilon}(p) : x_n = \varphi(x')\}$$

i.e. $\partial\Omega \cap C_{r,\epsilon}(p) = \{(x', \varphi(x')) : x' \in B_r(p')\}$.

Remark 7 A direct consequence of the definition is (cf., for example, Lemma 10.4 in [2]) that a Lipschitz domain has the following property (in geometry, sometimes, such a domain Ω is said to be quasi-convex or to have the geodesic property). There exists $C_1 = C_1(\Omega) > 0$ such that, for every $x, y \in \Omega$, there exists $\alpha_{xy} \in C^{0,1}([0,1]; \Omega)$, satisfying

$$\alpha_{xy}(0) = x, \quad \alpha_{xy}(1) = y \quad \text{and} \quad L(\alpha_{xy}) := \int_0^1 \left|\alpha'_{xy}(t)\right| dt \leqslant C_1 |x - y|.$$

The following theorem extends classical results by proving existence, uniqueness and regularity (with estimates) up to the boundary. The theory was initiated by Pfaff and further developed by Jacobi, Clebsch, Frobenius, Darboux and E. Cartan. We refer to [15] for a history of the subject. The sharper regularity result, (i.e. by considering continuous Γ) is due to Hartman and Wintner [14] and [13] (for a more recent presentation see [6] or [7]).

Theorem 8 *Let $r \geq 0$ be an integer, $\Omega \subset \mathbb{R}^n$ be open, bounded, simply connected with Lipschitz boundary and $x_0 \in \overline{\Omega}$, $F^0 \in \mathbb{R}^{n \times n}$. Let $\Gamma_1, \cdots, \Gamma_n \in C^r\left(\overline{\Omega}; \mathbb{R}^{n \times n}\right)$ satisfy in Ω*

$$d\Gamma + \Gamma \wedge \Gamma = 0 \quad \textit{i.e.} \quad \partial_i \Gamma_j - \partial_j \Gamma_i + \Gamma_i \Gamma_j - \Gamma_j \Gamma_i = 0, \quad \forall\, i, j = 1, \cdots, n. \tag{2}$$

There exists a unique $F \in C^{r+1}\left(\overline{\Omega}, \mathbb{R}^{n \times n}\right)$ such that $F(x_0) = F^0$ and in Ω

$$dF = F\,\Gamma \quad \textit{i.e.} \quad \partial_i F = F\,\Gamma_i \quad \textit{for every } i = 1, \cdots, n. \tag{3}$$

Furthermore the following properties hold.
(i) *The rank of F is constant, i.e.*

$$\operatorname{rank} F(x) = \operatorname{rank} F^0, \quad \textit{for every } x \in \overline{\Omega}.$$

(ii) *If $F^0 \in GL_n(\mathbb{R})$, then (2) is also necessary.*
(iii) *For every integer $r \geq 0$, there exist constants c_r, depending only on Ω, such that*

$$\left\|F - F^0\right\|_{C^0} \leq c_0 \left|F^0\right| \|\Gamma\|_{C^0} \exp\{c_0 \|\Gamma\|_{C^0}\}$$

$$\left\|F - F^0\right\|_{C^{r+1}} \leq c_{r+1} \left|F^0\right| (1 + \|\Gamma\|_{C^r}^r) \|\Gamma\|_{C^r} \exp\{c_0 \|\Gamma\|_{C^0}\}.$$

Remark 9 When $r=0$, the condition (2) has to be understood in the weak sense, i.e., for every $\psi \in C_0^1(\Omega;\mathbb{R}^{n\times n})$ and for every $1 \leqslant i < j \leqslant n$, the following holds

$$\int_{\Omega} [(-\partial_i \psi\, \Gamma_j + \partial_j \psi\, \Gamma_i) + \psi\, (\Gamma_i \Gamma_j - \Gamma_j \Gamma_i)] = 0. \tag{4}$$

Note that (4) is equivalent to

$$\int_{a_j}^{b_j} [\Gamma_j]_{x_i = a_i, b_i}\, dx_j - \int_{a_i}^{b_i} [\Gamma_i]_{x_j = a_j, b_j}\, dx_i + \int_{a_i}^{b_i} \int_{a_j}^{b_j} (\Gamma_i \Gamma_j - \Gamma_j \Gamma_i)\, dx_i dx_j = 0,$$

for every $1 \leqslant i < j \leqslant n$ and every $x \in \Omega$ with $a_i < x_i < b_i$, $a_j < x_j < b_j$ and $\prod_{i=1}^n [a_i, b_i] \subset \Omega$, where

$$[\Gamma_j]_{x_i = a_i, b_i} = \Gamma_j(x_1, \cdots, x_{i-1}, b_i, x_{i+1} \cdots, x_n) - \Gamma_j(x_1, \cdots, x_{i-1}, a_i, x_{i+1} \cdots, x_n).$$

Another way of writing the above condition is

$$\int_{\partial R} \Gamma + \iint_R \Gamma \wedge \Gamma = 0$$

for any oriented two dimensional rectangle R with sides parallel to the coordinate axis.

Proof The existence and uniqueness part, in the interior of the domain Ω, is in Corollaries 6.1 and 6.2 of Chapter VI of Hartman [13].

Step 1 (existence and regularity). Let us prove the existence of the solution with regularity up to the boundary. We consider two cases.

Case 1: $x_0 \in \Omega$. Using the result of [13], we find a unique $F \in C^{r+1}(\Omega;\mathbb{R}^{n\times n})$ satisfying

$$dF = F\,\Gamma, \text{ in } \Omega \quad \text{and} \quad F(x_0) = F^0. \tag{5}$$

We show that $F \in C^{r+1}(\overline{\Omega};\mathbb{R}^{n\times n})$. As Ω is Lipschitz, there exists $C_1 = C_1(\Omega) > 0$ such that, for every $x, y \in \Omega$, there exists $\alpha_{xy} \in C^{0,1}([0,1];\Omega)$, satisfying

$$\alpha_{xy}(0) = x, \quad \alpha_{xy}(1) = y \quad \text{and} \quad L(\alpha_{xy}) := \int_0^1 \left|\alpha_{xy}'(t)\right| dt \leqslant C_1 |x - y| \tag{6}$$

We establish the regularity of F in two sub-steps.

We first prove that F is bounded. More precisely, for some $C_2 = C_2(\Omega) > 0$,

$$|F(x)| \leqslant C_2, \quad \text{for every } x \in \Omega. \tag{7}$$

Indeed, for every $x \in \Omega$ and $t \in [0,1]$, using (5), we have

$$\begin{aligned} F(\alpha_{x_0 x}(t)) &= F^0 + \int_0^t \frac{d}{d\tau}[F(\alpha_{x_0 x}(\tau))]\, d\tau = F^0 + \sum_{k=1}^n \int_0^t \partial_k F(\alpha_{x_0 x}(\tau))\, [\alpha_{x_0 x}]_k'(\tau)\, d\tau \\ &= F^0 + \sum_{k=1}^n \int_0^t F(\alpha_{x_0 x}(\tau))\, \Gamma_k(\alpha_{x_0 x}(\tau))\, [\alpha_{x_0 x}]_k'(\tau)\, d\tau. \end{aligned}$$

Therefore, for every $x \in \Omega$ and $t \in [0,1]$,

$$\begin{aligned} |F(\alpha_{x_0 x}(t))| &\leqslant \left|F^0\right| + \sum_{k=1}^n \int_0^t |F(\alpha_{x_0 x}(\tau))|\, \|\Gamma_k\|_{C^0} \left|[\alpha_{x_0 x}]_k'(\tau)\right| d\tau \\ &\leqslant \left|F^0\right| + M\sqrt{n} \int_0^t |F(\alpha_{x_0 x}(\tau))| \left|\alpha_{x_0 x}'(\tau)\right| d\tau, \end{aligned}$$

where $M = \max_{1\leqslant k\leqslant n}\left[\|\Gamma_k\|_{C^0}\right]$. Using Grönwall inequality and (6), we get, for every $x \in \Omega$,

$$\begin{aligned}|F(x)| = |F(\alpha_{x_0x}(1))| &\leqslant \left|F^0\right| \exp\left\{M\sqrt{n}\int_0^1 \left|\alpha'_{x_0x}(\tau)\right| d\tau\right\} = \left|F^0\right| \exp\left\{M\sqrt{n}\, L(\alpha_{x_0x})\right\} \\ &\leqslant \left|F^0\right| \exp\left\{M\sqrt{n}\, C_1 |x_0 - x|\right\} \leqslant \left|F^0\right| \exp\left\{M\sqrt{n}\, C_1 \operatorname{diam}\Omega\right\} := C_2(\Omega) = C_2\end{aligned}$$

and thus

$$|F(x)| \leqslant C_2 := \left|F^0\right| \exp\left\{M\sqrt{n}\, C_1 \operatorname{diam}\Omega\right\} \tag{8}$$

which proves (7). Hence, F is bounded.

We next show that F is Lipschitz, i.e. for some $C_3 = C_3(\Omega) > 0$,

$$|F(x) - F(x)| \leqslant C_3 |x - y|, \quad \text{for every } x, y \in \Omega. \tag{9}$$

Indeed, for every $x, y \in \Omega$, using (5),

$$\begin{aligned}F(y) - F(x) &= \int_0^1 \frac{d}{d\tau}\left[F(\alpha_{xy}(\tau))\right] d\tau = \sum_{k=1}^{n} \int_0^1 \partial_k F(\alpha_{xy}(\tau)) \left[\alpha_{xy}\right]'_k(\tau)\, d\tau \\ &= \sum_{k=1}^{n} \int_0^1 F(\alpha_{xy}(\tau))\, \Gamma_k(\alpha_{xy}(\tau)) \left[\alpha_{xy}\right]'_k(\tau)\, d\tau.\end{aligned}$$

Hence, as F is bounded, it follows from (6) and (7) that, for every $x, y \in \Omega$,

$$|F(y) - F(x)| \leqslant C_2 M \sqrt{n}\, L(\alpha_{xy}) \leqslant M\, C_1 C_2 \sqrt{n}\, |x - y| = C_3 |x - y| \tag{10}$$

where $C_3 = C_3(\Omega) = M\, C_1 C_2 \sqrt{n}$. This proves (9). Hence, F is uniformly continuous. Therefore, $F \in C\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$. Using (5), we see that $F \in C^1\left(\overline{\Omega}, \mathbb{R}^{n\times n}\right)$. Bootstrapping, it follows that $F \in C^{r+1}\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$ which settles the first case.

Case 2: $x_0 \in \partial\Omega$. Let $(x^p)_{p\in\mathbb{N}}$ be a sequence in Ω such that $\lim_{p\to\infty}[x^p] = x_0$. Using Case 1, for each $p \in \mathbb{N}$, we find a unique $F^p \in C^{r+1}\left(\overline{\Omega}, \mathbb{R}^{n\times n}\right)$ such that

$$\begin{cases} \partial_i F^p = F^p\, \Gamma_i & \text{in } \Omega \text{ and } i = 1, \cdots, n \\ F^p(x^p) = F^0. \end{cases} \tag{11}$$

Since, thanks to (7) and (9), for every $p \in \mathbb{N}$ and every $x, y \in \Omega$,

$$|F^p(x)| \leqslant C_2 \quad \text{and} \quad |F^p(x) - F^p(y)| \leqslant C_3 |x - y|,$$

we invoke Ascoli-Arzela theorem to find a subsequence $(F^{p_k})_{k\in\mathbb{N}}$ of $(F^p)_{p\in\mathbb{N}}$ and $F \in C\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$ such that $(F^{p_k})_{k\in\mathbb{N}}$ converges to F in $C\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$. We claim that

$$dF = F\,\Gamma, \text{ in } \Omega \quad \text{and} \quad F(x_0) = F^0. \tag{12}$$

Indeed, as $(F^{p_k})_{k\in\mathbb{N}}$ converges to some F, and $(\partial_i F^{p_k})_{k\in\mathbb{N}}$ converges to $F\,\Gamma_i$ in $C\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$ for every $i = 1, \cdots, n$, it follows that

$$\partial F = F\,\Gamma, \quad \text{in } \Omega.$$

Furthermore, using (11),

$$F(x_0) = \lim_{k\to\infty}\left[F^{p_k}(x^{p_k})\right] = F^0$$

which proves (12). This, in turn, implies that $F \in C^1\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$. Bootstrapping, it follows that $F \in C^{r+1}\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$. This settles the second case, and completes the proof of existence and uniqueness.

Step 2 (constant rank). We now prove that F has constant rank. Let $x^1 \in \overline{\Omega}$ where the rank is minimal, i.e.

$$m := \operatorname{rank} F\left(x^1\right) \leq \operatorname{rank} F\left(x\right), \quad \text{for every } x \in \overline{\Omega}.$$

We can therefore find $A \in \mathbb{R}^{n\times n}$, with $\operatorname{rank} A = (n-m)$ such that $A\,F\left(x^1\right) = 0$. Define $G \in C^1\left(\overline{\Omega};\mathbb{R}^{n\times n}\right)$ as

$$G\left(x\right) = A\,F\left(x\right), \quad \text{for every } x \in \overline{\Omega}.$$

Then, G satisfies, for every $i = 1,\cdots,n$,

$$dG = G\,\Gamma, \text{ in } \Omega \quad \text{and} \quad G\left(x^1\right) = 0.$$

It therefore follows, by uniqueness and continuity, that $G\left(x\right) = A\,F\left(x\right) = 0$ for every $x \in \overline{\Omega}$. Since A is a constant matrix with $\operatorname{rank} A = (n-m)$, we deduce that

$$m := \operatorname{rank} F\left(x^1\right) \leq \operatorname{rank} F\left(x\right) \leq m, \quad \text{for every } x \in \overline{\Omega}$$

and thus the claim.

Step 3 (necessity). We now prove that (2), under its weak form (4), holds. Let $\psi \in C_0^1\left(\Omega;\mathbb{R}^{n\times n}\right)$ be arbitrary and define $\varphi \in C_0^1\left(\Omega;\mathbb{R}^{n\times n}\right)$ by $\varphi = \psi\,F^{-1}$ (this is well defined since, by Step 2, $F\left(x\right) \in GL_n\left(\mathbb{R}\right)$ for every $x \in \Omega$). Call

$$A = \int_\Omega \left[\left(-\partial_i\psi\,\Gamma_j + \partial_j\psi\,\Gamma_i\right) + \psi\left(\Gamma_i\Gamma_j - \Gamma_j\Gamma_i\right)\right].$$

We have to show that $A = 0$. We find, since $\psi = \varphi\,F$,

$$A = \int_\Omega \varphi\left[-\partial_i F\,\Gamma_j + \partial_j F\,\Gamma_i + F\left(\Gamma_i\Gamma_j - \Gamma_j\Gamma_i\right)\right] + \int_\Omega \left[-\partial_i\varphi\,F\,\Gamma_j + \partial_j\varphi\,F\,\Gamma_i\right].$$

Using the fact that $\partial_i F = F\,\Gamma_i$, we obtain

$$A = \int_\Omega \left[-\partial_i\varphi\,\partial_j F + \partial_j\varphi\,\partial_i F\right].$$

If φ were $C_0^2\left(\Omega;\mathbb{R}^{n\times n}\right)$, the divergence theorem immediately gives that $A = 0$. If φ is only $C_0^1\left(\Omega;\mathbb{R}^{n\times n}\right)$, the result follows by a straightforward argument of density.

Step 4 (estimates). *(i)* The C^0 estimate follows at once from (8) and (10); choosing $c_0 = \sqrt{n}\,C_1 \operatorname{diam}\Omega$. Note that, from (8), we have

$$\|F\|_{C^0} \leq \left|F^0\right| \exp\left\{c_0\,\|\Gamma\|_{C^0}\right\}.$$

(ii) Before starting with the estimates of higher order, we recall that

$$\|F\,\Gamma\|_{C^r} \leq a_r\,\|F\|_{C^r}\,\|\Gamma\|_{C^r}\ .$$

In fact, using Theorem 16.28 in [8], one can refine the estimate to

$$\|F\,\Gamma\|_{C^r} \leq a_r\left(\|F\|_{C^r}\,\|\Gamma\|_{C^0} + \|F\|_{C^0}\,\|\Gamma\|_{C^r}\right).$$

Using this more refined inequality, we can improve the estimates of the present step in a natural way, but, for the sake of simplicity, we will not do it.

(iii) We now prove the C^{r+1} estimates by induction. Note first that

$$\begin{aligned}\left\|F - F^0\right\|_{C^{r+1}} &= \left\|F - F^0\right\|_{C^0} + \|dF\|_{C^r} = \left\|F - F^0\right\|_{C^0} + \|F\,\Gamma\|_{C^r}\\ &\leq \left\|F - F^0\right\|_{C^0} + a_r\,\|F\|_{C^r}\,\|\Gamma\|_{C^r}\ .\end{aligned}$$

We can now proceed with the induction proof and consider first the case $r = 0$. We have

$$\left\| F - F^0 \right\|_{C^1} \leq \left\| F - F^0 \right\|_{C^0} + a_0 \left\| F \right\|_{C^0} \left\| \Gamma \right\|_{C^0} \leq \left| F^0 \right| (c_0 + a_0) \left\| \Gamma \right\|_{C^0} \exp\{c_0 \left\| \Gamma \right\|_{C^0}\}$$

as wished. We next discuss the case $r \geq 1$. Assume that the result has already been proved for r and let us prove it for $(r+1)$. We have

$$\begin{aligned}\left\| F - F^0 \right\|_{C^{r+1}} &\leq \left\| F - F^0 \right\|_{C^0} + a_r \left\| F \right\|_{C^r} \left\| \Gamma \right\|_{C^r} \leq \left\| F - F^0 \right\|_{C^0} + a_r \left[\left\| F - F^0 \right\|_{C^r} + \left| F^0 \right| \right] \left\| \Gamma \right\|_{C^r} \\ &\leq \left| F^0 \right| \left[c_0 \left\| \Gamma \right\|_{C^0} + a_r c_r \left(1 + \left\| \Gamma \right\|_{C^{r-1}}^{r-1} \right) \left\| \Gamma \right\|_{C^{r-1}} \left\| \Gamma \right\|_{C^r} + a_r \left\| \Gamma \right\|_{C^r} \right] \exp\{c_0 \left\| \Gamma \right\|_{C^0}\}\end{aligned}$$

and the claim follows. ■

3.2 Dirichlet problem for Pfaff system

We start with an elementary proposition.

Proposition 10 *Let $\Omega \subset \mathbb{R}^n$ be open with connected Lipschitz boundary and outward unit normal ν. Let $f \in C^1\left(\overline{\Omega}\right)$. Then,*

$$f = 0 \quad \text{on } \partial\Omega$$

if and only if

$$\nu \wedge Df = 0 \quad \mathcal{H}^{n-1} \text{ a.e. on } \partial\Omega \quad \text{and} \quad f(p) = 0, \text{ for some } p \in \partial\Omega. \tag{13}$$

Proof We first prove that (13) implies $f = 0$ on $\partial\Omega$. Fix $x \in \partial\Omega$ and invoke Proposition 30 to find a Lipschitz curve $\gamma : [0,1] \to \partial\Omega$ such that $\gamma(0) = p$, $\gamma(1) = x$ and

$$\langle \nu(\gamma(t)); \gamma'(t) \rangle = 0 \quad \text{for } \mathcal{H}^1 - \text{a.e. } t \in (0,1).$$

It follows from (13) that

$$\langle Df(\gamma(t)); \gamma'(t) \rangle = 0 \quad \text{for } \mathcal{H}^1 - \text{a.e. } t \in (0,1)$$

and thus

$$f(x) = f(x) - f(p) = \int_0^1 \frac{d}{dt}\left[f(\gamma(t)) \right] dt = \int_0^1 \langle Df(\gamma(t)); \gamma'(t) \rangle \, dt = 0.$$

The reverse implication being immediate, we have indeed established the proposition. ■
The main result of the present section is the following.

Theorem 11 *Let $r \geq 0$ be an integer and $\Omega \subset \mathbb{R}^n$ be open, bounded, simply connected, with connected Lipschitz boundary and outward unit normal ν. Let $\Phi \in C^{r+1}\left(\partial\Omega; \mathbb{R}^{n\times n}\right)$ with* $\det \Phi \neq 0$ *on $\partial\Omega$ and $\Gamma_1, \cdots, \Gamma_n \in C^r\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$. There exists $F \in C^{r+1}\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$ satisfying*

$$\begin{cases} dF = F\,\Gamma \text{ i.e. } \partial_i F = F\,\Gamma_i\,,\ i = 1, \cdots, n & \text{in } \Omega \\ \qquad F = \Phi & \text{on } \partial\Omega \end{cases} \tag{14}$$

if and only if in Ω

$$d\Gamma + \Gamma \wedge \Gamma = 0 \quad \text{i.e.} \quad \partial_i \Gamma_j - \partial_j \Gamma_i + \Gamma_i \Gamma_j - \Gamma_j \Gamma_i = 0, \text{ for every } i, j = 1, \cdots, n \tag{15}$$

and $\mathcal{H}^{n-1}$ a.e. on $\partial\Omega$

$$\nu \wedge (d\Phi - \Phi\,\Gamma) = 0 \quad \text{i.e.} \quad \nu^i\left(\partial_j \Phi - \Phi\,\Gamma_j\right) - \nu^j\left(\partial_i \Phi - \Phi\,\Gamma_i\right) = 0, \text{ for every } i, j = 1, \cdots, n. \tag{16}$$

Furthermore, if a solution of (14) exists, then it is unique and, for every $x_0 \in \partial\Omega$, there exist constants c_r , depending only on Ω, such that

$$\|F - \Phi(x_0)\|_{C^0} \leq c_0 |\Phi(x_0)| \|\Gamma\|_{C^0} \exp\{c_0 \|\Gamma\|_{C^0}\}$$

$$\|F - \Phi(x_0)\|_{C^{r+1}} \leq c_{r+1} |\Phi(x_0)| (1 + \|\Gamma\|_{C^r}^r) \|\Gamma\|_{C^r} \exp\{c_0 \|\Gamma\|_{C^0}\}.$$

Remark 12 Note that if Φ is constant (and invertible), then (16) reduces to

$$\nu \wedge \Gamma = 0 \quad i.e. \quad \nu^i(\Gamma_j) - \nu^j(\Gamma_i) = 0, \text{ for every } i, j = 1, \cdots, n.$$

Proof *Step 1 (necessity).* We assume that $F \in C^1(\overline{\Omega}; \mathbb{R}^n)$ satisfy (14), then (15) follows from Theorem 8. To establish (16), we note, cf. Proposition 10, that $\nu \wedge DF_{pq} = \nu \wedge D\Phi_{pq}$ $\mathcal{H}^{n-1}$ a.e. on $\partial\Omega$, for every $p, q = 1, \cdots, n$, i.e.

$$\left[\nu^i \partial_j \Phi - \nu^j \partial_i \Phi\right]_{pq} = \left[\nu^i \partial_j F - \nu^j \partial_i F\right]_{pq}$$

Observe next that, since $F = \Phi$ and $dF = F\,\Gamma$ on $\partial\Omega$, then, for every $i, j, p, q = 1, \cdots, n$ and $\mathcal{H}^{n-1}$ a.e. on $\partial\Omega$,

$$\left[\nu^i (\partial_j \Phi - \Phi\,\Gamma_j) - \nu^j (\partial_i \Phi - \Phi\,\Gamma_i)\right]_{pq} = \left[\nu^i (\partial_j F - F\,\Gamma_j) - \nu^j (\partial_i F - F\,\Gamma_i)\right]_{pq}$$

from where (16) follows.

Step 2 (sufficiency). Conversely, let us suppose that (15), (16) are satisfied. Let $x_0 \in \partial\Omega$ be fixed. Using Theorem 8, we find $F \in C^{r+1}(\overline{\Omega}; \mathbb{R}^{n \times n})$ satisfying

$$dF = F\,\Gamma, \text{ in } \Omega \quad \text{and} \quad F(x_0) = \Phi(x_0).$$

It remains to show that $F(x) = \Phi(x)$ for every $x \in \partial\Omega$. Invoking Proposition 30, we find $\alpha \in C^{0,1}([0,1]; \partial\Omega)$ be such that $\alpha(0) = x_0$ and $\alpha(1) = x$; note that

$$\langle \alpha'(t); \nu(\alpha(t)) \rangle = 0 \quad \text{a.e. } t \in (0,1). \tag{17}$$

Call

$$X(t) = \Phi(\alpha(t)) \quad \text{and} \quad A(t) = \sum_{j=1}^{n} \Gamma_j(\alpha(t))\,\alpha_j'(t)$$

and observe that, a.e. $t \in (0,1)$ and for every $i = 1, \cdots, n$,

$$\begin{aligned} \nu^i(\alpha(t))\,[X'(t) - X(t)\,A(t)] &= \nu^i(\alpha) \left\{ \left[\sum_{j=1}^{n} \partial_j \Phi(\alpha) - \Phi\,\Gamma_j(\alpha) \right] \alpha_j' \right\} \\ &= \sum_{j=1}^{n} \left[\nu^i(\alpha)\,(\partial_j \Phi(\alpha) - \Phi\,\Gamma_j(\alpha))\right] \alpha_j'. \end{aligned}$$

Invoking (16) and then (17) we find, for every $i = 1, \cdots, n$,

$$\nu^i(\alpha(t))\,[X'(t) - X(t)\,A(t)] = (\partial_i \Phi(\alpha) - \Phi\,\Gamma_i(\alpha)) \sum_{j=1}^{n} \nu^j \alpha_j' = 0.$$

It therefore follows that $X = \Phi \circ \alpha$ satisfies

$$\begin{cases} X'(t) = X(t)\,A(t) & \text{a.e. } t \in (0,1) \\ X(0) = \Phi(x_0) \end{cases}$$

where $A \in L^\infty((0,1); \mathbb{R}^{n \times n})$. Since $F \circ \alpha$ satisfies the same equation, it follows from Grönwall lemma that $\Phi \circ \alpha = F \circ \alpha$. As $\partial\Omega$ is connected, we have that $F = \Phi$ on the boundary, as wished.

Step 3 (uniqueness and estimates). The uniqueness and the estimates are already at the level of the Cauchy problem, we have thus completed the proof. ■

4 Pullback equation

In this section, we study the following nonlinear problem

$$u^*(H) = G \quad i.e. \quad (Du)^t \, H \, Du = G \quad \text{in } \Omega \tag{18}$$

when G and H have invertible symmetric parts. We discuss the unconstrained, the Dirichlet-Neumann and the Dirichlet problems, namely

$$\begin{cases} (Du)^t \, H \, Du = G & \text{in } \Omega \\ u = \varphi \quad \text{and} \quad Du = D\varphi & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} (Du)^t \, H \, Du = G & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Our result includes the purely symmetric case (i.e. G and H are symmetric and non-degenerate). The purely symmetric case has received considerable attention, since the work of Riemann; it is related to the problem of equivalence of Riemannian metrics. The first results were concerned with the local problem (see [17] where several proofs are provided). The global case for the unconstrained problem was first established by Cartan. The Dirichlet-Neumann problem presented here is new, even in the purely symmetric case.

Our analysis does not include the purely skew-symmetric case, which also received considerable attention, since the time of Darboux (see [1], [18] or any book on symplectic geometry for more modern developments). It is more involved, both from the point of view of uniqueness and regularity. The optimal regularity, for the local problem, was obtained in [4], in the framework of Hölder spaces. The Dirichlet problem has been treated in [4] and slightly improved in [8] and [11].

4.1 Unconstrained problem

The unconstrained and Cauchy problems are intimately related and under mild conditions the second can be deduced from the first one. Indeed let $\Omega \subset \mathbb{R}^n$ be open, $H \in \mathbb{R}^{n \times n}$, $(x_0, c_0, C_0) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n \times n}$ and $G \in C(\Omega; \mathbb{R}^{n \times n})$ be such that $(C_0)^t \, H \, C_0 = G(x_0)$. Let $v \in C^1(\Omega; \mathbb{R}^n)$ verify, in Ω,

$$(Dv)^t \, H \, Dv = G \quad \text{and} \quad \det Dv(x_0) \neq 0.$$

Setting $u(x) = C_0 \left[Dv(x_0)\right]^{-1} \left[v(x) - v(x_0)\right] + c_0$, we obtain that

$$\begin{cases} (Du)^t \, H \, Du = G, & \text{in } \Omega \\ u(x_0) = c_0 \quad \text{and} \quad Du(x_0) = C_0\,. \end{cases}$$

Note that this construction is independent of the symmetry or the rank of G and H. Moreover if v is locally invertible and C_0 is invertible, then so is u.

We recall the following notations. For a matrix G we denote by G_s and G_a its symmetric and skew-symmetric parts respectively. Below we write $\{\Gamma_1, \cdots, \Gamma_n\}$ to denote the Christoffel matrices of G_s (i.e. Γ is the Levi-Civita connection of G_s).

The main theorem of the present section is the following (in the symmetric case it is a standard theorem in differential geometry).

Theorem 13 (Unconstrained case) *Let $r \geq 1$ be an integer and $\Omega \subset \mathbb{R}^n$ be a simply connected open set. Let $H \in \mathbb{R}^{n \times n}$ with H_s invertible and $G \in C^r(\Omega; \mathbb{R}^{n \times n})$ with G_s non-degenerate. There exists $u \in C^{r+1}(\Omega; \mathbb{R}^n)$ satisfying, in Ω,*

$$(Du)^t \, H \, Du = G$$

if and only if

(i) *there exists* $C_0 \in GL_n(\mathbb{R})$ *such that* $(C_0)^t H\, C_0 = G(x_0)$, *for some* $x_0 \in \Omega$,

(ii) $d\Gamma + \Gamma \wedge \Gamma = 0$ *(i.e.* $\partial_i \Gamma_j - \partial_j \Gamma_i + \Gamma_i \Gamma_j - \Gamma_j \Gamma_i = 0$, *for every* $i, j = 1, \cdots, n$*)*,

(iii) $dG_a = \Gamma^t G_a + G_a \Gamma$ *i.e.* $\partial_k G_a = (\Gamma_k)^t G_a + G_a \Gamma_k$ *in* Ω, *for every* $k = 1, \cdots, n$.

Furthermore, the solution, if it exists, is unique up to an affine transformation; more precisely if v *and* w *are two solutions, there exist* $a \in \mathbb{R}$ *and* $A \in GL_n(\mathbb{R})$, *with* $A^t H\, A = H$, *such that* $w = A\, v + a$.

Remark 14 *(i)* The theorem includes the purely symmetric case where G and H are symmetric; the condition (iii) in the theorem being then trivially true.

(ii) There are some implicit conditions on the symmetric part; for example H_s and G_s should have the same signature (i.e. H_s and G_s have the same number of positive eigenvalues). In particular if $H = I_n$, then G should be positive definite.

(iii) There are also several hidden necessary conditions on the skew-symmetric part.

- Since $\operatorname{rank}[G_a(x)] = \operatorname{rank}[H_a]\ \forall\, x \in \Omega$, then $G_a(x_0) = 0$ implies $G_a(x) = 0\ \forall\, x \in \Omega$.

- Note that G_a is uniquely determined by

$$\begin{cases} \partial_k G_a = (\Gamma_k)^t G_a + G_a \Gamma_k & \text{in } \Omega \text{ and } k = 1, \cdots, n \\ G_a(x_0) = (C_0)^t H_a\, C_0 \,. \end{cases}$$

- Since H is constant, looking at G_a as a $2-$from, we deduce that G_a is closed (i.e. $dG_a = 0$).

(iii) In terms of differential geometry, looking at G_a as a $2-$tensor (or a $2-$form), the condition $\partial_k G_a = (\Gamma_k)^t G_a + G_a \Gamma_k$, says that the covariant derivative of G_a vanishes.

Proof (Theorem 13) We have to show that, for every $(x_0, c_0, C_0) \in \Omega \times \mathbb{R}^n \times GL_n(\mathbb{R})$, there exists $u \in C^{r+1}(\Omega; \mathbb{R}^n)$ satisfying

$$\begin{cases} (Du)^t H\, Du = G \quad \text{and} \quad \det Du \neq 0 \quad \text{in } \Omega \\ u(x_0) = c_0 \quad \text{and} \quad Du(x_0) = C_0 \end{cases} \tag{19}$$

if and only if (i), (ii) and (iii) hold.

Step 1 (preliminaries). Observe that any solution of (19) satisfies

$$\begin{cases} (Du)^t H_s\, Du = G_s \quad \text{and} \quad (Du)^t H_a\, Du = G_a \quad \text{in } \Omega \\ u(x_0) = c_0 \quad \text{and} \quad Du(x_0) = C_0 \,. \end{cases}$$

- The strategy is to solve first

$$\begin{cases} (Du)^t H_s\, Du = G_s \quad \text{and} \quad \det Du \neq 0 \quad \text{in } \Omega \\ u(x_0) = c_0 \quad \text{and} \quad Du(x_0) = C_0 \end{cases} \tag{20}$$

showing that $(C_0)^t H_s\, C_0 = G_s(x_0)$ and (ii) are necessary (Step 2) and sufficient (Step 3) conditions to achieve this goal. We prove as well the uniqueness result (Step 4).

- In Step 5, we prove that any solution of (20) solves

$$(Du)^t H_a\, Du = G_a$$

if and only if $(C_0)^t H_a\, C_0 = G_a(x_0)$ and (iii) are verified.

Step 2 (necessity). Let $u \in C^{r+1}(\Omega;\mathbb{R}^n)$ satisfy (20); the conclusion $(C_0)^t H_s C_0 = G_s(x_0)$ is trivial, while the condition $d\Gamma + \Gamma \wedge \Gamma = 0$ follows from Proposition 5.

Step 3 (sufficiency). Let $x_0 \in \Omega$ and $C_0 \in GL_n(\mathbb{R})$ be such that $(C_0)^t H_s C_0 = G_s(x_0)$. Theorem 8 implies that we can find $F \in C^r(\Omega;\mathbb{R}^{n\times n})$ satisfying

$$dF = F\,\Gamma, \text{ in } \Omega \quad \text{and} \quad F(x_0) = C_0\,.$$

Using Lemma 4, we have, for every $i,j,k = 1,\cdots,n$ and in Ω,

$$\partial_k F_{ij} = (F\,\Gamma_k)_{ij} = \sum_{p=1}^{n} F_{ip}(\Gamma_k)_{pj} = \sum_{p=1}^{n} F_{ip}\Gamma_{kj}^{p} = \sum_{p=1}^{n} F_{ip}\Gamma_{jk}^{p} = \sum_{p=1}^{n} F_{ip}(\Gamma_j)_{pk} = \partial_j F_{ik}$$

which implies that

$$\operatorname{curl}(F_{i,*}) = 0, \quad \text{in } \Omega \text{ and } i = 1,\cdots,n.$$

Since Ω is simply connected, we find $u \in C^{r+1}(\Omega;\mathbb{R}^n)$ satisfying, for $c_0 \in \mathbb{R}^n$,

$$Du = F \text{ in } \Omega \quad \text{and} \quad u(x_0) = c_0\,.$$

Note that, for every $k = 1,\cdots,n$ and in Ω,

$$\partial_k\left(F^t H_s\,F\right) = (\partial_k F)^t H_s\,F + F^t H_s(\partial_k F) = (\Gamma_k)^t\left(F^t H_s\,F\right) + \left(F^t H_s\,F\right)\Gamma_k\,.$$

Hence, both $F^t H_s\,F$ and G_s (invoking Lemma 4 (ii)) satisfy the following system of equations

$$\begin{cases} \partial_k X = (\Gamma_k)^t X + X\,\Gamma_k & \text{in } \Omega \text{ and } k = 1,\cdots,n \\ X(x_0) = G_s(x_0)\,. \end{cases} \tag{21}$$

The uniqueness of solutions of (21) implies that $F^t H_s\,F = G_s$ in Ω; i.e.

$$\begin{cases} (Du)^t H_s\,Du = G_s & \text{in } \Omega \\ u(x_0) = c_0 \quad Du(x_0) = C_0\,. \end{cases} \tag{22}$$

This proves Step 3.

Step 4 (uniqueness). (i) We have to prove that the solution of (22) is unique; so let $u, v \in C^{r+1}(\Omega;\mathbb{R}^n)$ satisfy (22). Then, using Proposition 5, we see that Du, Dv satisfy

$$dF = F\,\Gamma, \text{ in } \Omega \quad \text{and} \quad F(x_0) = C_0\,.$$

Hence, it follows from Theorem 8 that $Du = Dv$ in Ω, which implies that $u = v$ in Ω as $u(x_0) = v(x_0) = c_0$. This establishes the uniqueness of solutions of (22).

(ii) From the above argument we deduce immediately the uniqueness stated in the theorem. Indeed let v and w satisfy the equation $(Du)^t H\,Du = G$. Fix a point $x_0 \in \Omega$, then, because of the uniqueness in (i) above, we have, setting

$$A = (Dw(x_0))\,(Dv(x_0))^{-1} \quad \text{and} \quad a = w(x_0) - A\,v(x_0)$$

that $w = A\,v + a$, as claimed.

Step 5 (the skew-symmetric equation). Let $u \in C^{r+1}(\Omega;\mathbb{R}^n)$ be a solution of (20).

- Assume that u also satisfies the equation $(Du)^t H_a Du = G_a$ and let us prove that condition (iii) of the theorem is verified. Indeed we get from Proposition 5 that, for every $k = 1, \cdots, n$,

$$\begin{aligned}
\partial_k G_a &= \partial_k \left((Du)^t H_a Du \right) = (\partial_k (Du))^t H_a Du + (Du)^t H_a \partial_k (Du) \\
&= ((Du) \Gamma_k)^t H_a Du + (Du)^t H_a (Du) \Gamma_k = (\Gamma_k)^t (Du)^t H_a Du + (Du)^t H_a (Du) \Gamma_k \\
&= (\Gamma_k)^t G_a + G_a \Gamma_k .
\end{aligned}$$

- Conversely, assume that (iii) is verified and let us show that $(Du)^t H_a Du = G_a$. To this end, we use Proposition 5 to note that, for every $k = 1, \cdots, n$,

$$\begin{aligned}
\partial_k \left((Du)^t H_a Du \right) &= (\partial_k (Du))^t H_a Du + (Du)^t H_a \; \partial_k (Du) = ((Du) \Gamma_k)^t H_a Du + (Du)^t H_a (Du) \Gamma_k \\
&= (\Gamma_k)^t (Du)^t H_a Du + (Du)^t H_a (Du) \Gamma_k .
\end{aligned}$$

Therefore, both G_a and $(Du)^t H_a Du$ satisfy the following equation

$$\begin{cases} \partial_k X = (\Gamma_k)^t X + X \Gamma_k & \text{in } \Omega \text{ and } k = 1, \cdots, n \\ X(x_0) = G_a(x_0) = (C_0)^t H_a C_0 . \end{cases} \tag{23}$$

Hence, it follows from the uniqueness of solutions of (23) that $(Du)^t H_a Du = G_a$ in Ω, as wished. ■

4.2 Dirichlet-Neumann problem

In the present section we let $r \geq 0$ be an integer and $\Omega \subset \mathbb{R}^n$ be open, bounded, simply connected, with connected Lipschitz boundary and outward unit normal ν.

For $G \in C^{r+1} \left(\overline{\Omega}; \mathbb{R}^{n \times n} \right)$ with G_s non-degenerate, we let $\{\Gamma_1, \cdots, \Gamma_n\}$ be the Christoffel matrices of G_s (i.e. Γ is the Levi-Civita connection of G_s). We recall that $d\Gamma + \Gamma \wedge \Gamma = 0$ means

$$\partial_i \Gamma_j - \partial_j \Gamma_i + \Gamma_i \Gamma_j - \Gamma_j \Gamma_i = 0, \text{ for every } i, j = 1, \cdots, n$$

i.e.

$$\partial_i (\Gamma_j)_{kl} - \partial_j (\Gamma_i)_{kl} + (\Gamma_i \Gamma_j - \Gamma_j \Gamma_i)_{kl} = 0, \quad \text{for every } i, j, k, l = 1, \cdots, n$$

while $dG_a = \Gamma^t G_a + G_a \Gamma$ stands for

$$\partial_k G_a = (\Gamma_k)^t G_a + G_a \Gamma_k , \quad \text{for every } k = 1, \cdots, n.$$

We now give the main theorem.

Theorem 15 (Dirichlet-Neumann problem) *Let $\varphi \in C^{r+2} \left(\overline{\Omega}; \mathbb{R}^n \right)$. Let $H \in \mathbb{R}^{n \times n}$ with H_s invertible, $G \in C^{r+1} \left(\overline{\Omega}; \mathbb{R}^{n \times n} \right)$ with G_s non-degenerate. The following two statements are equivalent.*

(i) *The four following conditions hold*

$$d\Gamma + \Gamma \wedge \Gamma = 0, \quad \textit{in } \Omega \tag{24}$$

$$dG_a = \Gamma^t G_a + G_a \Gamma, \quad \textit{in } \Omega \tag{25}$$

$$(D\varphi(x_0))^t H \, D\varphi(x_0) = G(x_0), \quad \textit{for some } x_0 \in \partial\Omega \tag{26}$$

$$\nu \wedge (d(D\varphi) - (D\varphi) \Gamma) = 0, \quad \mathcal{H}^{n-1} \; \textit{a.e. on } \partial\Omega. \tag{27}$$

(ii) *There exists* $u \in C^{r+2}\left(\overline{\Omega};\mathbb{R}^n\right)$ *satisfying the Dirichlet-Neumann problem*

$$\begin{cases} u^*(H) = G \quad i.e. \quad (Du)^t\, H\, Du = G & in\ \Omega \\ \qquad u = \varphi \quad and \quad Du = D\varphi & on\ \partial\Omega. \end{cases} \tag{28}$$

Moreover, if the solution of (28) exists, then it is unique and if $\varphi \in \operatorname{Diff}^{r+2}\left(\overline{\Omega};\varphi\left(\overline{\Omega}\right)\right)$, *then* $u \in \operatorname{Diff}^{r+2}\left(\overline{\Omega};\varphi\left(\overline{\Omega}\right)\right)$. *Furthermore, for every* $x_0 \in \partial\Omega$, *there exist constants* c_r, *depending only on* Ω, *such that*

$$\|Du - D\varphi(x_0)\|_{C^0} \leq c_0 \|D\varphi\|_{C^0} \|\Gamma\|_{C^0} \exp\{c_0 \|\Gamma\|_{C^0}\}$$

$$\|Du - D\varphi(x_0)\|_{C^{r+1}} \leq c_{r+1} \|D\varphi\|_{C^0} (1 + \|\Gamma\|_{C^r}^r) \|\Gamma\|_{C^r} \exp\{c_0 \|\Gamma\|_{C^0}\}.$$

In particular, if φ *is affine (i.e.* $D\varphi$ *is constant), then there exist constants* $\widetilde{c}_r$, *depending only on* Ω, *such that*

$$\|u - \varphi\|_{C^{r+2}} \leq \widetilde{c}_r \|D\varphi\|_{C^0} (1 + \|\Gamma\|_{C^r}^r) \|\Gamma\|_{C^r} \exp\{c_0 \|\Gamma\|_{C^0}\}. \tag{29}$$

Remark 16 *(i)* When we write $u \in \operatorname{Diff}^r\left(\overline{\Omega};\varphi\left(\overline{\Omega}\right)\right)$, we mean that u is a diffeomorphism from $\overline{\Omega}$ onto $\varphi\left(\overline{\Omega}\right)$ with u and u^{-1} belonging to C^r.

(ii) The estimate (29) implies, in particular, that if φ is affine, $r \geq 0$ is an integer and $\left\{G_{(m)}\right\}$ is a sequence converging in the C^{r+1} topology to the constant matrix H, then the solution $\left\{u_{(m)}\right\}$ converges to φ in C^{r+2}. This follows at once from the estimate and the fact that the corresponding sequence $\left\{\Gamma_{(m)}\right\}$ converges to 0 (since H is constant) in the C^r topology.

(iii) It turns out that the condition $(D\varphi(x_0))^t\, H\, D\varphi(x_0) = G(x_0)$ for some $x_0 \in \partial\Omega$ is equivalent to $(D\varphi)^t\, H\, D\varphi = G$ everywhere on $\partial\Omega$; this is a direct consequence of the theorem.

(iv) The condition (27) reads, $\mathcal{H}^{n-1}$ a.e. on $\partial\Omega$, as

$$\nu^i\left(\partial_j(D\varphi) - (D\varphi)\Gamma_j\right) = \nu^j\left(\partial_i(D\varphi) - (D\varphi)\Gamma_i\right), \quad \text{for every } i,j = 1,\cdots,n$$

or, in other words, for every $i,j,k,l = 1,\cdots,n$

$$\nu^i\left(\partial_j(\partial_l\varphi_k) - \sum_{p=1}^n (\partial_p\varphi_k)(\Gamma_j)_{pl}\right) = \nu^j\left(\partial_i(\partial_l\varphi_k) - \sum_{p=1}^n (\partial_p\varphi_k)(\Gamma_i)_{pl}\right).$$

(v) Note that if φ is affine, then $D\varphi$ is invertible, in view of (26). Therefore condition (27) reads, in this case, $\nu \wedge \Gamma = 0$.

Proof (Theorem 15) *Preliminary step. (i)* As in Theorem 13, the system decouples into

$$\begin{cases} (Du)^t\, H_s\, Du = G_s \text{ and } (Du)^t\, H_a\, Du = G_a & \text{in } \Omega \\ \qquad u = \varphi \quad \text{and} \quad Du = D\varphi & \text{on } \partial\Omega. \end{cases}$$

(ii) We then solve the problem

$$\begin{cases} (Du)^t\, H_s\, Du = G_s & \text{in } \Omega \\ u = \varphi \quad \text{and} \quad Du = D\varphi & \text{on } \partial\Omega \end{cases} \tag{30}$$

showing that the conditions

$$d\Gamma + \Gamma \wedge \Gamma = 0 \text{ in } \Omega,$$

$$\nu \wedge (d(D\varphi) - (D\varphi)\,\Gamma) = 0 \text{ a.e. on } \partial\Omega \quad \text{and} \quad (D\varphi(x_0))^t\, H_s D\varphi(x_0) = G_s(x_0)$$

are necessary and sufficient to find a unique solution. It remains then, exactly as in Theorem 13, to prove that any solution of (30) satisfy $(Du)^t H_a\, Du = G_a$ in Ω if and only if

$$\partial_k G_a = (\Gamma_k)^t G_a + G_a \Gamma_k \quad \text{and} \quad (D\varphi(x_0))^t H_a D\varphi(x_0) = G_a(x_0)$$

It is therefore enough to prove the theorem under the further assumption that G and H are symmetric and we therefore drop the index s.

Step 1: (i) $\Rightarrow$ (ii)*.* We use Theorem 11 to find $F \in C^{r+1}\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$ satisfying

$$\begin{cases} \partial_k F = F\,\Gamma_k & \text{in } \Omega \text{ and } k = 1, \cdots, n \\ F = D\varphi & \text{on } \partial\Omega. \end{cases}$$

The same argument as in Step 3 of Theorem 13 leads to the existence of u satisfying (30), proving (ii).

Step 2: (ii) $\Rightarrow$ (i)*.* Let $u \in C^{r+2}\left(\overline{\Omega}; \mathbb{R}^n\right)$ satisfy

$$\begin{cases} (Du)^t H\, Du = G & \text{in } \Omega \\ u = \varphi \quad \text{and} \quad Du = D\varphi & \text{on } \partial\Omega. \end{cases}$$

That Γ satisfies (24) has already been proved in Theorem 13. It is evident that $(D\varphi)^t H\, D\varphi = G$ everywhere on $\partial\Omega$; it therefore remains to prove (27). Define $F \in C^{r+1}\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$ as $F = Du$ in $\overline{\Omega}$. Using Proposition 5, we find that F satisfies

$$\begin{cases} \partial_k F = F\,\Gamma_k & \text{in } \Omega \text{ and } k = 1, \cdots, n \\ F = D\varphi & \text{on } \partial\Omega. \end{cases}$$

Applying Theorem 11, we obtain that (27) holds. This proves the equivalence properties of the theorem.

Step 3 (uniqueness). Let $u, v \in C^{r+2}(\Omega; \mathbb{R}^n)$ satisfy (28). Then, using Proposition 5, we see that Du, Dv satisfy

$$\begin{cases} \partial_k F = F\,\Gamma_k & \text{in } \Omega \text{ and } k = 1, \cdots, n \\ F = D\varphi & \text{on } \partial\Omega. \end{cases}$$

Hence, it follows from Theorem 11 that $Du = Dv$ in Ω, which implies that $u = v$ in Ω as $u = v = \varphi$ on $\partial\Omega$. This establishes the uniqueness of solutions of (30). That $u \in \operatorname{Diff}^{r+2}\left(\overline{\Omega}; \varphi\left(\overline{\Omega}\right)\right)$, follows from Theorem 19.12 of [8], see also [16].

Step 4 (estimate). The general estimate follows from the construction of Step 1 and the corresponding estimates in Theorem 11. We now discuss the more specific estimate (29). Observe first that, since $D\varphi$ is constant, we have from Remark 7 and the general estimate that

$$\|u - \varphi\|_{C^0} \le C_1 \operatorname{diam}(\Omega) \|Du - D\varphi\|_{C^0} \le c_0 \|D\varphi\|_{C^0} \|\Gamma\|_{C^0} \exp\{c_0 \|\Gamma\|_{C^0}\}.$$

Noting that

$$\|u - \varphi\|_{C^{r+2}} = \|u - \varphi\|_{C^0} + \|Du - D\varphi\|_{C^{r+1}}$$

we have the desired result and the proof of the theorem is therefore complete. ■

Theorem 15 can be extended to the case where H is not constant.

Corollary 17 *Let $\varphi \in \operatorname{Diff}^{r+2}\left(\overline{\Omega}; \varphi\left(\overline{\Omega}\right)\right)$ and ν_φ be the normal to $\partial\varphi(\Omega)$. Let $G \in C^{r+1}\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$, $H \in C^{r+1}\left(\varphi\left(\overline{\Omega}\right); \mathbb{R}^{n\times n}\right)$ with G_s and H_s non-degenerate. Let Γ and Δ be the Levi-Civita connection of G_s and H_s respectively. If*

$$d\Gamma + \Gamma \wedge \Gamma = 0 \quad \textit{and} \quad dG_a = \Gamma^t G_a + G_a \Gamma, \quad \textit{in } \Omega$$

$$d\Delta + \Delta \wedge \Delta = 0 \quad and \quad dH_a = \Delta^t H_a + H_a \Delta, \quad in\ \varphi(\Omega)$$
$$(D\varphi(x_0))^t H(\varphi(x_0)) D\varphi(x_0) = G(x_0), \quad for\ some\ x_0 \in \partial\Omega$$
$$\nu \wedge (d(D\varphi) - (D\varphi)\Gamma) = 0, \quad \mathcal{H}^{n-1}\ a.e.\ on\ \partial\Omega$$
$$\nu_\varphi \wedge \Delta = 0, \quad \mathcal{H}^{n-1}\ a.e.\ on\ \partial\varphi(\Omega)$$

then, there exists $u \in \operatorname{Diff}^{r+2}\left(\overline{\Omega}; \varphi\left(\overline{\Omega}\right)\right)$ *satisfying*

$$\begin{cases} u^*(H) = G \quad i.e. \quad (Du)^t H(u) Du = G & in\ \Omega \\ u = \varphi \quad and \quad Du = D\varphi & on\ \partial\Omega. \end{cases} \tag{31}$$

Proof Call $A = H(\varphi(x_0))$. Using Theorem 15, we find $v \in \operatorname{Diff}^{r+2}\left(\overline{\Omega}; \varphi\left(\overline{\Omega}\right)\right)$ solving

$$\begin{cases} v^*(A) = G & \text{in } \Omega \\ v = \varphi \quad \text{and} \quad Dv = D\varphi & \text{on } \partial\Omega \end{cases}$$

and $w \in \operatorname{Diff}^{r+2}\left(\varphi\left(\overline{\Omega}\right); \varphi\left(\overline{\Omega}\right)\right)$ satisfying

$$\begin{cases} w^*(A) = H & \text{in } \varphi(\Omega) \\ w = \text{id} \quad \text{and} \quad Dw = I_n & \text{on } \partial\varphi(\Omega). \end{cases}$$

Then, $u \in \operatorname{Diff}^{r+2}\left(\overline{\Omega}; \varphi\left(\overline{\Omega}\right)\right)$ defined as $u = w^{-1} \circ v$ solves (31). Indeed in Ω we have

$$u^*(H) = v^*\left(\left(w^{-1}\right)^*(H)\right) = v^*(A) = G$$

while on $\partial\Omega$

$$u = \varphi \quad \text{and} \quad Du = D\varphi.$$

This achieves the proof of the corollary. ■

4.3 Dirichlet problem

In the present section we let $r \geq 0$ be an integer and $\Omega \subset \mathbb{R}^n$ be open, bounded, simply connected, with connected Lipschitz boundary and outward unit normal ν.

For $G \in C^{r+1}\left(\overline{\Omega}; \mathbb{R}^{n \times n}\right)$ with G_s non-degenerate, we let $\{\Gamma_1, \cdots, \Gamma_n\}$ be the Christoffel matrices of G_s (i.e. Γ is the Levi-Civita connection of G_s). We recall that $d\Gamma + \Gamma \wedge \Gamma = 0$ and $dG_a = \Gamma^t G_a + G_a \Gamma$ mean respectively

$$\partial_i \Gamma_j - \partial_j \Gamma_i + \Gamma_i \Gamma_j - \Gamma_j \Gamma_i = 0, \text{ for every } i, j = 1, \cdots, n$$

$$\partial_k G_a = (\Gamma_k)^t G_a + G_a \Gamma_k, \text{ for every } k = 1, \cdots, n.$$

When dealing with the purely Dirichlet problem, Theorem 15 takes the following abstract form.

Theorem 18 (Dirichlet problem) *Let* $\varphi \in C^{r+2}\left(\overline{\Omega}; \mathbb{R}^n\right)$, *with* $\det D\varphi \neq 0$ *in* $\overline{\Omega}$. *Let* $H \in \mathbb{R}^{n \times n}$ *with* H_s *invertible,* $G \in C^{r+1}\left(\overline{\Omega}; \mathbb{R}^{n \times n}\right)$ *with* G_s *non-degenerate. Then, there exists* $u \in C^{r+2}\left(\overline{\Omega}; \mathbb{R}^n\right)$ *satisfying the Dirichlet problem*

$$\begin{cases} u^*(H) = G \quad i.e. \quad (Du)^t H\, Du = G & in\ \Omega \\ u = \varphi & on\ \partial\Omega. \end{cases} \tag{32}$$

if and only if
(i) *In* Ω
$$d\Gamma + \Gamma \wedge \Gamma = 0 \quad \textit{and} \quad dG_a = \Gamma^t G_a + G_a \Gamma.$$
(ii) *There exists* $\Phi \in C^{r+1}\left(\partial\Omega; \mathbb{R}^{n \times n}\right)$ *such that,* $\mathcal{H}^{n-1}$ *a.e. on* $\partial\Omega$,
$$\Phi^t H \, \Phi = G, \quad \det \Phi \det D\varphi > 0, \quad \nu \wedge \Phi = \nu \wedge D\varphi, \quad \nu \wedge (d\Phi - \Phi \, \Gamma) = 0.$$
Furthermore, the solution of (32), if it exists, is unique and satisfies the Dirichlet-Neumann conditions on $\partial\Omega$, *i.e.*
$$u = \varphi \quad \textit{and} \quad Du = \Phi.$$
Moreover if $\varphi \in \operatorname{Diff}^{r+2}\left(\overline{\Omega}; \varphi\left(\overline{\Omega}\right)\right)$, *then so is* u.

Remark 19 *(i)* If $\varphi = \mathrm{id}$, then Φ is of the form
$$\Phi = I_n + \alpha \otimes \nu$$
for an appropriate α such that $\alpha \otimes \nu \in C^{r+1}\left(\partial\Omega; \mathbb{R}^{n \times n}\right)$. In particular

- if $H = I_n$, then $\left\langle G^{-1}\nu; \nu \right\rangle \neq 0$ and
$$\Phi = I_n + \alpha \otimes \nu \quad \text{where} \quad \alpha = \left[\sqrt{\det G} \, I_n - \frac{G^{-1}}{\langle G^{-1}\nu; \nu \rangle} \right] \nu.$$
Note that in order to have $\alpha \otimes \nu \in C^{r+1}$, one will have, in general, to assume that the domain Ω is at least C^{r+2}.

- if $G = H$ on $\partial\Omega$ (in fact it suffices that $G(x_0) = H(x_0)$ for a certain $x_0 \in \partial\Omega$), then $\Phi = I_n$ (i.e. $\alpha = 0$) and therefore the only condition on the boundary is
$$\nu \wedge \Gamma = 0.$$
Note that in this case we do not need extra regularity of the domain and we are back to Theorem 15.

(ii) We recall that $\nu \wedge (d\Phi - \Phi \, \Gamma) = 0$ means
$$\nu^i \left(\partial_j \Phi - \Phi \, \Gamma_j\right) = \nu^j \left(\partial_i \Phi - \Phi \, \Gamma_i\right), \quad \text{for every } i, j = 1, \cdots, n.$$
It can be proved that the condition $\nu \wedge (d\Phi - \Phi \, \Gamma) = 0$ is equivalent to the second fundamental forms on $\partial\Omega$ of G_s and H_s are equal.

Proof (Theorem 18) The proof of the existence part is exactly the same as that of Theorem 15 and we will not reproduce it. The uniqueness also follows as the corresponding one in Theorem 15, once it has been observed, thanks to Lemma 23, that Φ is unique, since $\nu \wedge \Phi = \nu \wedge D\varphi$ implies $\operatorname{rank}(\Phi - D\varphi) \leq 1$. ■

In both corollaries below we further assume more regularity on $\partial\Omega$, namely C^{r+2}. A direct consequence of Theorems 18 and 27 is the following corollary.

Corollary 20 *Let* $H \in \mathbb{R}^{n \times n}$, $G \in C^{r+1}\left(\overline{\Omega}; \mathbb{R}^{n \times n}\right)$ *with* H_s, G_s *non-degenerate satisfying*
$$\left\langle H_s^{-1}\nu; \nu \right\rangle \left\langle G_s^{-1}\nu; \nu \right\rangle \neq 0 \quad \textit{on } \partial\Omega.$$
Then, there exists $u \in C^{r+2}\left(\overline{\Omega}; \mathbb{R}^n\right)$ *satisfying the Dirichlet problem*
$$\begin{cases} u^*(H) = G \quad \textit{i.e.} \quad (Du)^t \, H \, Du = G & \textit{in } \Omega \\ \qquad\qquad u = \mathrm{id} & \textit{on } \partial\Omega \end{cases} \tag{33}$$

if and only if
(i) $d\Gamma + \Gamma \wedge \Gamma = 0$ *in* Ω
(ii) $\det G_s \det H_s > 0$ *in* $\overline{\Omega}$
(iii) $\langle G_s a; b\rangle = \langle H_s a; b\rangle$ *on* $\partial\Omega$ *and for every* $\langle a; \nu\rangle = \langle b; \nu\rangle = 0$
(iv) Φ *defined as*

$$\Phi = I_n + \left[\sqrt{\frac{\det G_s}{\det H_s}}\left(\frac{H_s^{-1}\nu}{\left\langle H_s^{-1}\nu; \nu\right\rangle}\right) - \frac{G_s^{-1}\nu}{\left\langle G_s^{-1}\nu; \nu\right\rangle}\right] \otimes \nu.$$

satisfies on $\partial\Omega$

$$\nu \wedge (d\Phi - \Phi\,\Gamma) = 0, \quad i.e. \quad \nu_i\left(\partial_j \Phi - \Phi\,\Gamma_j\right) = \nu_j\left(\partial_i \Phi - \Phi\,\Gamma_i\right) \text{ for every } i, j = 1, \cdots, n.$$

(v) $dG_a = \Gamma^t G_a + G_a\,\Gamma$ *i.e.* $\partial_k G_a = (\Gamma_k)^t\, G_a + G_a\,\Gamma_k$, *for every* $k = 1, \cdots, n$
(vi) $G_a \wedge \nu = H_a \wedge \nu$ *on* $\partial\Omega$.
Furthermore, the solution, if it exists, is unique and $u \in \operatorname{Diff}^{r+2}\left(\overline{\Omega}; \overline{\Omega}\right)$.

Remark 21 Note that the hypotheses (i) to (iv) of the corollary guarantee, invoking Theorem 27, only that $\Phi^t H_s \Phi = G_s$. The conditions (v) and (vi) imply then that $\Phi^t H_a \Phi = G_a$.

This result, when $H = I_n$, takes even the simpler form.

Corollary 22 *Let* $G \in C^{r+1}\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$ *be symmetric and non-degenerate. Then, there exists* $u \in C^{r+2}\left(\overline{\Omega}; \overline{\Omega}\right)$ *satisfying the Dirichlet problem*

$$\begin{cases} u^*(I_n) = G \quad i.e. \quad (Du)^t\, Du = G & \text{in } \Omega \\ \qquad\qquad\qquad\qquad u = \mathrm{id} & \text{on } \partial\Omega \end{cases}$$

if and only if
(i) $d\Gamma + \Gamma \wedge \Gamma = 0$ *in* Ω
(ii) $\det G > 0$ *in* $\overline{\Omega}$
(iii) $\langle Ga; b\rangle = \langle a; b\rangle$ *on* $\partial\Omega$ *and for every* $\langle a; \nu\rangle = \langle b; \nu\rangle = 0$
(iv) Φ *defined as*

$$\Phi = I_n + \left[G + \left(\sqrt{\det G} - 1 - \langle G\nu; \nu\rangle\right) I_n\right] \nu \otimes \nu.$$

satisfies on $\partial\Omega$

$$\nu \wedge (d\Phi - \Phi\,\Gamma) = 0.$$

Furthermore, the solution, if it exists, is unique and $u \in \operatorname{Diff}^{r+2}\left(\overline{\Omega}; \overline{\Omega}\right)$.

5 Appendix 1: some algebraic results

5.1 Preliminary results

We start with few elementary results.

Lemma 23 *Let* $A \in GL_n(\mathbb{R})$ *be symmetric and* $X, Y \in GL_n(\mathbb{R})$ *be such that*

$$X^t A\, X = Y^t A\, Y, \quad \det X \det Y > 0 \quad and \quad \operatorname{rank}(X - Y) \leqslant 1.$$

Then, $X = Y$. *In particular if*

$$X^t A\, X = A, \quad \det X = 1 \quad and \quad \operatorname{rank}(X - I_n) \leqslant 1,$$

then, $X = I_n$.

Remark 24 Lemma 23 fails if A is skew-symmetric. To see this, let $n = 2$ and let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = I_2 + e_1 \otimes e_2\,.$$

Then, $X^t A\, X = A$, $\det X = 1$, $\operatorname{rank}(X - I_2) \leqslant 1$, but $X \neq I_2$.

Proof *Step 1 (special case $Y = I_n$).* Since $\operatorname{rank}(X - I_n) \leqslant 1$, we find $a, b \in \mathbb{R}^n$ such that $X - I_n = a \otimes b$. If $b = 0$, we are done. Let us assume that $b \neq 0$. As $\det X = 1$, we find

$$1 = \det X = \det(I_n + a \otimes b) = 1 + \langle a; b\rangle\,,$$

which shows that $\langle a; b\rangle = 0$. Note that

$$\begin{aligned} A = X^t A\, X &= (I_n + a \otimes b)^t\, A\, (I_n + a \otimes b) = (I_n + b \otimes a)\, A\, (I_n + a \otimes b) \\ &= (I_n + b \otimes a)\,(A + Aa \otimes b) = A + Aa \otimes b + (b \otimes a)\, A + (b \otimes a)\,(Aa \otimes b) \\ &= A + Aa \otimes b + b \otimes A^t a + \langle Aa; a\rangle\,(b \otimes b)\,, \end{aligned}$$

which implies that

$$Aa \otimes b + b \otimes Aa + \langle Aa; a\rangle\,(b \otimes b) = 0. \tag{34}$$

Taking inner product with a, it follows from (34) that

$$Aa\,\langle b; a\rangle + b\,\langle Aa; a\rangle + \langle Aa; a\rangle\,\langle b; a\rangle\, b = b\,\langle Aa; a\rangle = 0,$$

which shows that $\langle Aa; a\rangle = 0$. Hence, $Aa \otimes b + b \otimes Aa = 0$, appealing to (34). Therefore, $Aa = \lambda b$, where $\lambda = -\frac{\langle Aa; b\rangle}{|b|^2}$. If $\lambda = 0$, we get $a = 0$ as A is invertible and we are done. If $\lambda \neq 0$, then $2\lambda\,(b \otimes b) = 0$, which shows that $b = 0$, a contradiction. This proves the lemma in the special case $Y = I_n$.

Step 2 (general case). Set $Z = X\,Y^{-1}$ and observe that

$$Z^t A\, Z = A, \quad \det Z = 1 \quad \text{and} \quad \operatorname{rank}(Z - I_n) \leqslant 1;$$

the last inequality coming from the fact that

$$\operatorname{rank}(Z - I_n) = \operatorname{rank}(Z\,Y - Y) = \operatorname{rank}(X - Y)\,.$$

Step 1 implies then the result. This proves the lemma. ■

The following lemma is easy to verify and the proof is skipped.

Lemma 25 *Let $n \geqslant 2$, $H \in \mathbb{R}^{n \times n}$, $T \in \mathbb{R}^{(n-1)\times(n-1)}$ be symmetric, $\mathbf{a} \in \mathbb{R}^{n-1}$ and $\alpha \in \mathbb{R}$ be such that, in some ordered orthonormal basis $\{a_1, \cdots, a_{n-1}, \nu\}$ of $\mathbb{R}^n$,*

$$H = \begin{pmatrix} T & \mathbf{a} \\ \mathbf{a}^t & \alpha \end{pmatrix}.$$

Then the following statements hold true.

(i) *The Cauchy expansion formula holds, namely*

$$\det H = \alpha \det T - \left\langle T^{\otimes} \mathbf{a}; \mathbf{a} \right\rangle \tag{35}$$

where $T^{\otimes}$ is the adjugate of T.

(ii) *If H is invertible, then $\left\langle H^{-1}\nu; \nu \right\rangle \neq 0$ if and only if T is invertible.*

(iii) *Furthermore, if* H *and* T *are invertible,*

$$H^{-1}=\begin{pmatrix} T^{-1}+\frac{\det T}{\det H}\left[T^{-1}\mathbf{a}\otimes T^{-1}\mathbf{a}\right] & -\frac{\det T}{\det H}\left(T^{-1}\mathbf{a}\right) \\ -\frac{\det T}{\det H}\left(T^{-1}\mathbf{a}\right)^{t} & \frac{\det T}{\det H} \end{pmatrix}. \tag{36}$$

We recall that for a symmetric and invertible $G\in\mathbb{R}^{n\times n}$, $\operatorname{sig}(G)$ denotes the signature of G, i.e. the number of positive eigenvalues. We conclude with another elementary lemma.

Lemma 26 *Let* $n\geqslant 2$ *and* $A,B\in GL_n(\mathbb{R})$ *be symmetric having the same leading principal* $(n-1)\times(n-1)$ *submatrix. Then,* $\operatorname{sig}(A)=\operatorname{sig}(B)$ *if and only if* $\det A\det B>0$.

Proof The direct part is straightforward, so we only prove the converse part. Assume that $\det A\det B>0$ and let us show that $\operatorname{sig}(A)=\operatorname{sig}(B)$. Let $T\in\mathbb{R}^{(n-1)\times(n-1)}$ be the common leading principal submatrix of A and B. Let $\lambda_1(T)\leq\cdots\leq\lambda_{n-1}(T)$ be the eigenvalues of T and let $\lambda_1(A)\leq\cdots\leq\lambda_n(A)$, $\lambda_1(B)\leq\cdots\leq\lambda_n(B)$ be the eigenvalues of A, B respectively. Using Cauchy Interlacing Theorem, we find

$$\begin{cases} \lambda_1(A)\leq\lambda_1(T)\leq\lambda_2(A)\leq\cdots\leq\lambda_{n-1}(A)\leq\lambda_{n-1}(T)\leq\lambda_n(A) \\ \lambda_1(B)\leq\lambda_1(T)\leq\lambda_2(B)\leq\cdots\leq\lambda_{n-1}(B)\leq\lambda_{n-1}(T)\leq\lambda_n(B). \end{cases} \tag{37}$$

The proof of the lemma follows from the four following cases.

(i) When $\lambda_1(T)>0$, it follows from (37) that $\lambda_i(A),\lambda_i(B)>0$ for every $i=2,\cdots,n$. Since $\det A\det B>0$, this implies that $\lambda_1(A)\lambda_1(B)>0$. Hence, A,B have the same signature.

(ii) When $\lambda_{n-1}(T)<0$, the argument is similar to the aforementioned one. Using (37), $\lambda_i(A),\lambda_i(B)<0$ for every $i=1,\cdots,n-1$. Since $\det A\det B>0$, we have $\lambda_n(A)\lambda_n(B)>0$ which, again, shows that A,B have the same signature.

(iii) When $\lambda_k(T)<0<\lambda_{k+1}(T)$ for some $k\in\{1,\cdots,n-2\}$, we use (37) to observe that $\lambda_i(A),\lambda_i(B)<0$ for every $i=1,\cdots,k$, and $\lambda_j(A),\lambda_j(B)>0$ for every $j=k+2,\cdots,n$. As $\det A\det B>0$, we have $\lambda_{k+1}(A)\lambda_{k+1}(B)>0$ which forces A,B to have the same signature.

(iv) Finally, if $\lambda_k(T)=0$ for some $k\in\{1,\cdots,n\}$, then using (37) again, $\lambda_i(A),\lambda_i(B)<0$ for every $i=1,\cdots,k$, and $\lambda_j(A),\lambda_j(B)>0$ for every $j=k+1,\cdots,n$ which shows that A,B have the same signature ■

5.2 Constrained congruence problem

In the sequel, given $\nu\in\mathbb{R}^n$ with $|\nu|=1$, we write $P_\nu\in\mathbb{R}^{n\times n}$ to denote

$$P_\nu=I_n-\nu\otimes\nu,\quad \text{i.e. } P_\nu(x)=x-\langle x;\nu\rangle\,\nu,\ \text{for every } x\in\mathbb{R}^n.$$

We therefore have the following result.

Theorem 27 *Let* $\nu\in\mathbb{R}^n$ *with* $|\nu|=1$. *Let* $G,H\in\mathbb{R}^{n\times n}$ *be symmetric and invertible. Then there exists* $\Phi\in\mathbb{R}^{n\times n}$ *invertible satisfying*

$$\Phi^t H\,\Phi=G,\quad \det\Phi>0,\quad \Phi a=a \text{ for every } \langle a;\nu\rangle=0 \tag{38}$$

if and only if

(i) $P_\nu G\,P_\nu=P_\nu H\,P_\nu$, *i.e.* $\langle Ga;b\rangle=\langle Ha;b\rangle$ *for every* $\langle a;\nu\rangle=\langle b;\nu\rangle=0$

(ii) $\det G\det H>0$

(iii) *there exists* a *with* $\langle a;\nu\rangle = 0$ *such that*

$$\left(G+\sqrt{\frac{\det G}{\det H}}\,H\right)a=\left[G-\frac{\det G}{\det H}\,H+\sqrt{\frac{\det G}{\det H}}P_{\nu}\left(G-H\right)\right]\nu. \tag{39}$$

Moreover when Φ *exists, it is unique and has the following form*

$$\Phi = I_n+\left[a+\left(\sqrt{\frac{\det G}{\det H}}-1\right)\nu\right]\otimes\nu. \tag{40}$$

In particular, when $\operatorname{rank}\left[P_\nu G\,P_\nu\right]=\operatorname{rank}\left[P_\nu H\,P_\nu\right]=n-1$, *then* $\left\langle H^{-1}\nu;\nu\right\rangle\left\langle G^{-1}\nu;\nu\right\rangle\neq 0$ *and*

$$\Phi = I_n+\left[\sqrt{\frac{\det G}{\det H}}\left(\frac{H^{-1}}{\langle H^{-1}\nu;\nu\rangle}\right)-\frac{G^{-1}}{\langle G^{-1}\nu;\nu\rangle}\right]\nu\otimes\nu. \tag{41}$$

If $H=I_n$, *then (41) gets further simplified to*

$$\Phi = I_n+\left[G+\left(\sqrt{\det G}-1-\langle G\nu;\nu\rangle\right)I_n\right]\nu\otimes\nu. \tag{42}$$

Remark 28 *(i)* If in (38) one drops the condition $\det\Phi>0$, i.e. if we consider the problem of finding Φ satisfying

$$\Phi^t H\,\Phi = G \quad\text{and}\quad \Phi a = a \text{ for every } \langle a;\nu\rangle = 0,$$

the result is then an easy consequence (see Step 7 below) of Witt theorem (see, for example, Artin [3, Theorem 3.9]). Note also that (39) is not required in this case.

(ii) Note that the condition $\Phi a = a$ for every $\langle a;\nu\rangle=0$, is equivalent to the existence of $\alpha\in\mathbb{R}^n$ such that

$$\Phi = I_n+\alpha\otimes\nu.$$

This in turn is equivalent to $\nu\wedge\Phi=\nu\wedge I_n$, i.e.

$$\nu_j\left(\Phi_{ik}-\delta_{ik}\right)=\nu_k\left(\Phi_{ij}-\delta_{ij}\right)\quad\text{for every } i,j,k=1,\cdots,n.$$

(iii) We observe that we always have

$$\operatorname{rank}\left[P_\nu G\,P_\nu\right]=\operatorname{rank}\left[P_\nu H\,P_\nu\right]\in\{n-2,n-1\}\,.$$

Moreover, $\operatorname{rank}\left[P_\nu G\,P_\nu\right]=n-1$ if and only if $\left\langle G^{-1}\nu;\nu\right\rangle\neq 0$ (see Lemma 25).

(iv) In Step 6 below, we prove that

$$\left[P_\nu G\,P_\nu = P_\nu H\,P_\nu \text{ and } \det G\det H>0\right]\quad\Rightarrow\quad \operatorname{sig}(G)=\operatorname{sig}(H)$$

where $\operatorname{sig}(G)$ and $\operatorname{sig}(H)$ denote the signature of G and H respectively.

(v) The a with $\langle a;\nu\rangle=0$ satisfying (39) is unique; this follows from the uniqueness of Φ. It could also be seen directly, but we do not enter into details.

Proof (Theorem 27) Let $\{a_1,\cdots,a_{n-1},\nu\}$ be an orthonormal basis of $\mathbb{R}^n$. With respect to this basis, P_ν,H,G have the following matrix representations

$$P_\nu=\begin{pmatrix} I_{n-1} & \mathbf{0}\\ \mathbf{0}^t & 0\end{pmatrix},\quad H=\begin{pmatrix} T_H & \mathbf{a}_H\\ \mathbf{a}_H^t & \alpha_H\end{pmatrix},\quad G=\begin{pmatrix} T_G & \mathbf{a}_G\\ \mathbf{a}_G^t & \alpha_G\end{pmatrix},$$

where $T_H, T_G \in \mathbb{R}^{(n-1)\times(n-1)}$ are symmetric, $\mathbf{a}_H, \mathbf{a}_G \in \mathbb{R}^{n-1}$ and $\alpha_H, \alpha_G \in \mathbb{R}$. Set

$$S = \{\nu\}^{\perp} = \operatorname{span}\{a_1, \cdots, a_{n-1}\} \quad \text{and} \quad \lambda = \sqrt{\frac{\det G}{\det H}} > 0.$$

We divide the proof into six steps.

Step 1 (uniqueness of Φ). We start with the uniqueness of Φ satisfying (38). Let us suppose that there exist $\Psi, \Phi \in GL_n(\mathbb{R})$ satisfying (38). Since $\Psi x = x = \Phi x$ for every $x \in S$, there exist $a, b \in \mathbb{R}^n$ such that

$$\Psi = I_n + a \otimes \nu \quad \text{and} \quad \Phi = I_n + b \otimes \nu.$$

Set $X = \Psi\,\Phi^{-1}$. Then, for some $c \in \mathbb{R}^n$, $X = I_n + c \otimes \nu$. Note that, $\det X > 0$ and

$$X^t H\, X = \Phi^{-t}\Psi^t H\, \Psi\, \Phi^{-1} = \Phi^{-t} G\, \Phi^{-1} = H.$$

Since $X^t H\, X = H$ and $\det X > 0$, we deduce, according to Lemma 23 that $X = I_n$, which settles the uniqueness of Φ satisfying (38).

Step 2 (necessary and sufficient conditions). The equivalence of the existence of Φ satisfying (38) with (i), (ii) and (iii) of the theorem follows from the three following immediate facts.

Fact 1. The next equivalence is straightforward

$$\begin{bmatrix} \Phi a = a, \text{ for every } a \in S \\ \det \Phi = \lambda \end{bmatrix} \quad \Leftrightarrow \quad \left[\Phi = \begin{pmatrix} I_{n-1} & \mathbf{b} \\ \mathbf{0}^t & \lambda \end{pmatrix} \text{ where } \mathbf{b} \in \mathbb{R}^{n-1}\right] \tag{43}$$

Fact 2. We also have the following equivalence by direct computation using the uniqueness of Φ and (43)

$$\begin{bmatrix} \Phi^t H\, \Phi = G \\ \det \Phi = \sqrt{\frac{\det G}{\det H}} > 0 \\ \Phi a = a, \text{ for every } a \in S \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} \Phi = \begin{pmatrix} I_{n-1} & \mathbf{b} \\ \mathbf{0}^t & \lambda \end{pmatrix} \text{ where } \mathbf{b} \in \mathbb{R}^{n-1} \\ T_H = T_G := T \\ T\,\mathbf{b} = \mathbf{a}_G - \lambda\,\mathbf{a}_H \\ \langle \mathbf{b}; \mathbf{a}_G + \lambda\,\mathbf{a}_H \rangle = \alpha_G - \lambda^2 \alpha_H \end{bmatrix}. \tag{44}$$

Fact 3. We finally have

$$[\text{Condition (i) in Theorem 27}] \quad \Leftrightarrow \quad [T_H = T_G := T]$$

while

$$[\text{Conditions (i), (iii) in Theorem 27}] \quad \Leftrightarrow \quad \begin{bmatrix} T_H = T_G := T \\ T\,\mathbf{b} = \mathbf{a}_G - \lambda\,\mathbf{a}_H \\ \langle \mathbf{b}; \mathbf{a}_G + \lambda\,\mathbf{a}_H \rangle = \alpha_G - \lambda^2 \alpha_H \end{bmatrix}.$$

This is obtained by straightforward computation, noting that $a \in S$ if and only if there exists $\mathbf{b} \in \mathbb{R}^{n-1}$ such that $a = \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix}$.

Step 3 (proof of (40)). We already know that Φ has the following form

$$\Phi = \begin{pmatrix} I_{n-1} & \mathbf{b} \\ \mathbf{0}^t & \lambda \end{pmatrix} \quad \text{where } \mathbf{b} \in \mathbb{R}^{n-1}.$$

We also know that $a = \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix}$. We therefore deduce that

$$\begin{aligned} \Phi &= I_n + [a + (\lambda - 1)\,\nu] \otimes \nu = I_n + a \otimes \nu + (\lambda - 1)\,\nu \otimes \nu \\ &= \begin{pmatrix} I_{n-1} & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix} + \begin{pmatrix} 0_{n-1} & \mathbf{b} \\ \mathbf{0}^t & 0 \end{pmatrix} + (\lambda - 1) \begin{pmatrix} 0_{n-1} & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix} = \begin{pmatrix} I_{n-1} & \mathbf{b} \\ \mathbf{0}^t & \lambda \end{pmatrix} \end{aligned}$$

as wished.

Step 4 (proof of (41)). We now assume that $\operatorname{rank}[P_\nu H P_\nu] = \operatorname{rank}[P_\nu G P_\nu] = n-1$. Recall below that $\lambda = \sqrt{\det G / \det H}$.

Step 4.1 (definition of a). We are now in the case where

$$H = \begin{pmatrix} T & \mathbf{a}_H \\ \mathbf{a}_H^t & \alpha_H \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} T & \mathbf{a}_G \\ \mathbf{a}_G^t & \alpha_G \end{pmatrix}$$

with T invertible. So let $\mathbf{b}$ be the unique solution of

$$T\,\mathbf{b} = \mathbf{a}_G - \lambda\,\mathbf{a}_H\,. \tag{45}$$

Let us show that the equation

$$\langle \mathbf{b}; \mathbf{a}_G + \lambda\,\mathbf{a}_H \rangle = \alpha_G - \lambda^2 \alpha_H \tag{46}$$

follows automatically. Using (45) and Lemma 25, we get (since $T^{-1}\det T = T^{\otimes}$)

$$\begin{aligned}\langle \mathbf{b}; \mathbf{a}_G + \lambda\,\mathbf{a}_H \rangle + \lambda^2\alpha_H &= \frac{1}{\det T}\left\langle \mathbf{a_G}; T^{\otimes}\mathbf{a_G}\right\rangle - \frac{\lambda^2}{\det T}\left\langle \mathbf{a_H}; T^{\otimes}\mathbf{a_H}\right\rangle + \lambda^2\alpha_H \\ &= \frac{1}{\det T}\left(\alpha_G \det T - \det G\right) + \frac{\lambda^2}{\det T}\left(\det H - \alpha_H \det T\right) + \lambda^2\alpha_H = \alpha_G\end{aligned}$$

as wished. Therefore, $a = \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix}$ satisfies (45) and (46), and hence (39), by Fact 3 in Step 2.

Step 4.2. Note that

$$a = \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix} = \lambda\,\frac{P_\nu H^{-1}\nu}{\langle H^{-1}\nu;\nu\rangle} - \frac{P_\nu G^{-1}\nu}{\langle G^{-1}\nu;\nu\rangle}$$

since, invoking Lemma 25,

$$\begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix} = \begin{pmatrix} T^{-1}\mathbf{a}_G - \lambda T^{-1}\mathbf{a}_H \\ 0 \end{pmatrix} = -\left(\frac{\det G}{\det T}\right) P_\nu G^{-1}\nu + \lambda \left(\frac{\det H}{\det T}\right) P_\nu H^{-1}\nu$$

while, still by Lemma 25,

$$\left\langle H^{-1}\nu;\nu\right\rangle = \frac{\det T}{\det H} \quad \text{and} \quad \left\langle G^{-1}\nu;\nu\right\rangle = \frac{\det T}{\det G}\,.$$

Hence a defined above satisfies (39). That Φ then takes the form of (41) is straightforward.

Step 5 (proof of (42)). We now establish that if $H = I_n$, then (41) gets simplified to (42). We therefore have to prove that

$$G\nu - \nu - \langle G\nu;\nu\rangle\,\nu = -\frac{G^{-1}\nu}{\langle G^{-1}\nu;\nu\rangle} \quad \text{i.e.} \quad \left\langle G^{-1}\nu;\nu\right\rangle\left[G\nu - \langle G\nu;\nu\rangle\,\nu\right] + \left[G^{-1}\nu - \left\langle G^{-1}\nu;\nu\right\rangle\nu\right] = 0$$

or equivalently

$$\left\langle G^{-1}\nu;\nu\right\rangle P_\nu G\,\nu + P_\nu G^{-1}\nu = 0. \tag{47}$$

Using (i) (i.e. $P_\nu G\,P_\nu = P_\nu$) we deduce that

$$G\left(P_\nu G^{-1}\nu\right) - \left\langle G\left(P_\nu G^{-1}\nu\right);\nu\right\rangle\nu = P_\nu G\left(P_\nu G^{-1}\nu\right) = P_\nu G^{-1}\nu$$

and thus

$$G\left(P_\nu G^{-1}\nu\right) - P_\nu G^{-1}\nu = \left\langle G\left(P_\nu G^{-1}\nu\right);\nu\right\rangle\nu.$$

Simplifying the above equality (noting that $G\left(P_\nu G^{-1}\nu\right) = \nu - \left\langle G^{-1}\nu;\nu\right\rangle G\,\nu$), we obtain

$$\nu - \left\langle G^{-1}\nu;\nu\right\rangle G\,\nu - P_\nu G^{-1}\nu = \left[1 - \left\langle G^{-1}\nu;\nu\right\rangle \langle G\,\nu;\nu\rangle\right]\nu,$$

which is exactly (47).

Step 6 (proof of Remark 28 (iv)). Let $H, G \in GL_n(\mathbb{R})$ be symmetric satisfying

$$\langle Hx;y\rangle = \langle Gx;y\rangle\,,\ \text{for every } x,y\in S \quad \text{and} \quad \det H \det G > 0.$$

We have to show that $\operatorname{sig}(H) = \operatorname{sig}(G)$. According to our notations H, G are written as

$$H = \begin{pmatrix} T & \mathbf{a}_H \\ \mathbf{a}_H^t & \alpha_H \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} T & \mathbf{a}_G \\ \mathbf{a}_G^t & \alpha_G \end{pmatrix},$$

where $T \in \mathbb{R}^{(n-1)\times(n-1)}$ is symmetric, $\mathbf{a}_H, \mathbf{a}_G \in \mathbb{R}^{n-1}$ and $\alpha_H, \alpha_G \in \mathbb{R}$. The claim follows, at once, from Lemma 26.

Step 7 (proof of Remark 28 (i)). We have to find Φ satisfying

$$\Phi^t H\,\Phi = G \quad \text{and} \quad \Phi a = a \text{ for every } \langle a;\nu\rangle = 0 \tag{48}$$

Since H and G have same signature (see Step 6 above), we find an invertible $\Psi \in \mathbb{R}^{n\times n}$ satisfying $\Psi^t H\,\Psi = G$. We next define a linear map $f : S = \{\nu\}^\perp \to \mathbb{R}^n$ by $f = \Psi^{-1}\big|_S$. Invoking Hypothesis (i) of the theorem, we find, for every $x, y \in S$,

$$\langle Gf(x);f(y)\rangle = \left\langle \Psi^t H\,\Psi f(x);f(y)\right\rangle = \langle H\,\Psi f(x);\Psi f(y)\rangle = \langle Hx;y\rangle = \langle Gx;y\rangle\,.$$

Using Witt theorem, we find an invertible $F \in \mathbb{R}^{n\times n}$ such that, for every $x, y \in S$,

$$F|_S = f \quad \text{and} \quad \langle GF(x);F(y)\rangle = \langle Gx;y\rangle\,.$$

Finally define $\Phi \in \mathbb{R}^{n\times n}$ as $\Phi = \Psi\,F$ and observe that it solves (48), as wished. ■

6 Appendix 2: Some properties of Lipschitz sets

The following results have been communicated to us by Nicola Fusco [12]. In the sequel $\mathcal{H}^k$ denotes the $k-$dimensional measure in $\mathbb{R}^n$, while $\mathcal{L}^n$ stands for the Lebesgue measure in $\mathbb{R}^n$. We also let $B_R \subset \mathbb{R}^n$ be the ball centered at 0 and of radius $R > 0$. We start with the following result.

Lemma 29 *Let $n \geq 1$ and let $R > 0$. Given a set $E \subset B_R$ with $\mathcal{L}^n(E) = 0$, for every $x, y \in B_R$ there exists a piecewise smooth (and thus Lipschitz) curve $\gamma : [0,1] \to B_R$ such that $\gamma(0) = x$ and $\gamma(1) = y$ with the property that*

$$\mathcal{L}^1(\{t \in [0,1] : \gamma(t) \in E\}) = 0.$$

Proof We argue by induction on the dimension.

If $n = 1$ and $x, y \in (-R, R)$, with $x < y$, we have $\mathcal{L}^1([x,y] \cap E) = 0$. Therefore, since the curve $\gamma(t) = (1-t)\,x + t\,y$ is a diffeomorphism between $[0,1]$ and $[x,y]$, we have also that $\mathcal{L}^1(\{t \in [0,1] : \gamma(t) \in E\}) = 0$.

Assume $n \geq 2$ and that the result is true in dimension $n-1$. If $x \neq y$, we may assume without loss of generality that $-R < x_n < y_n < R$. Observe that for $\mathcal{H}^{n-1}-$a.e. $\nu \in \partial B_1$ the intersection between E and the straight line $L_{x,\nu}$ passing through x and parallel to ν has $\mathcal{H}^1-$measure zero.

Indeed, using polar coordinates, performing a change of variable and interchanging the order of integration, we have

$$0 = \mathcal{L}^n(E) = \int_0^\infty dr \int_{\partial B_r(x)} \chi_E(z) d\mathcal{H}_z^{n-1} = \int_0^\infty r^{n-1}\, dr \int_{\partial B_1} \chi_E(x + r\nu)\, d\mathcal{H}_\nu^{n-1}$$
$$= \int_{\partial B_1} d\mathcal{H}_\nu^{n-1} \int_0^\infty r^{n-1} \chi_E(x + r\nu)\ dr.$$

Thus, for $\mathcal{H}^{n-1}$-a.e. $\nu \in \partial B_1$

$$0 = \int_0^\infty r^{n-1} \chi_E(x + r\nu)\, dr = \int_0^\infty \chi_E(x + r\nu)\, dr = \mathcal{H}^1(E \cap L_{x,\nu}).$$

Recall that by Fubini theorem there exists $x_n < t < y_n$ such that $\mathcal{H}^{n-1}(E \cap \pi) = 0$, where π is the horizontal plane $\pi = \{z_n = t\}$.
Let $\nu_1, \nu_2 \in \partial B_1$ be two directions such that

$$\mathcal{H}^1(E \cap L_{x,\nu_1}) = \mathcal{H}^1(E \cap L_{y,\nu_2}) = 0.$$

Using the property above it is clear that we may always choose ν_1 and ν_2 so that $L_{x,\nu_1} \cap \pi \cap B_R \neq \emptyset$ and $L_{y,\nu_2} \cap \pi \cap B_R \neq \emptyset$. Set $L_{x,\nu_1} \cap \pi \cap B_R = \{\bar{x}\}$ and $L_{y,\nu_2} \cap \pi \cap B_R = \{\bar{y}\}$. By the induction assumption there exists a piecewise smooth (and thus Lipschitz) curve $\tilde{\gamma} : [0,1] \to \pi$ such that

$$\tilde{\gamma}(0) = \bar{x}, \quad \tilde{\gamma}(1) = \bar{y} \quad \text{and} \quad \mathcal{H}^1(\{t \in [0,1] : \tilde{\gamma}(t) \in E \cap \pi\}) = 0.$$

Then, the curve γ is obtained by joining the segment connecting x and $\bar{x}$, the curve $\tilde{\gamma}$ and the segment connecting $\bar{y}$ and y, up to a suitable reparametrization. ■
We now conclude with the following proposition (for the definition of Lipschitz boundary, see Definition 6).

Proposition 30 *Let Ω be a bounded open set with Lipschitz boundary. If $\partial\Omega$ is connected, then for every $x, y \in \partial\Omega$ there exists a Lipschitz curve $\gamma : [0,1] \to \partial\Omega$ such that $\gamma(0) = x$, $\gamma(1) = y$ and*

$$\langle \nu(\gamma(t)), \gamma'(t) \rangle = 0 \qquad \mathcal{H}^1 \text{ a.e. } t \in [0,1],$$

where $\nu(\cdot)$ stands for the exterior normal to $\partial\Omega$.

Proof Since $\partial\Omega$ is connected, to prove the assertion it is enough to show that for every $x \in \partial\Omega$ there exists a neighbourhood U_x of x such that for every $y \in U_x \cap \partial\Omega$ there is a Lipschitz path connecting x and y with the required properties. To this end we fix $x \in \partial\Omega$; we may assume, without loss of generality, that $x = 0$. By definition of Lipschitz boundaries, we can find $r, \epsilon > 0$ and a Lipschitz function $\varphi : B_r \subset \mathbb{R}^{n-1} \to (-\epsilon, \epsilon)$ such that, upon rotation and relabeling of coordinates if necessary,

$$\Omega \cap C_{r,\epsilon} = \{x \in C_{r,\epsilon} : x_n < \varphi(x')\} \quad \text{and} \quad \partial\Omega \cap C_{r,\epsilon} = \{x \in C_{r,\epsilon} : x_n = \varphi(x')\}$$

where $C_{r,\epsilon} = B_r \times (\epsilon, \epsilon)$. Let

$$E = \{x' \in B_r : \varphi \text{ is not differentiable at } x'\}.$$

Clearly $\mathcal{L}^{n-1}(B_r \setminus E) = 0$. Recall that if $z = (z', z_n) \in \partial\Omega \cap C_{r,\epsilon}$ with $z' \in B_r \setminus E$, then the exterior normal to $\partial\Omega$ at z is given by

$$\nu(z) = \frac{1}{\sqrt{1 + |\nabla\varphi(z')|^2}} (-\nabla\varphi(z'), 1). \tag{49}$$

Set $U_x = C_{r,\epsilon}$ and let $y \in \partial\Omega \cap C_{r,\epsilon}$. From Lemma 29 there exists a piecewise smooth curve $\tilde{\gamma} : [0,1] \to B_r$ such that

$$\tilde{\gamma}(0) = 0, \quad \tilde{\gamma}(1) = y' \quad \text{and} \quad \tilde{\gamma}(t) \notin E \ \mathcal{H}^1 \text{ a.e. } t \in [0,1].$$

Finally define $\gamma : [0,1] \to \partial\Omega$ by

$$\gamma(t) = (\tilde{\gamma}(t), \varphi(\tilde{\gamma}(t))) \quad \text{for every } t \in [0,1].$$

Clearly γ is Lipschitz, $\gamma(0) = 0$ and $\gamma(1) = y$. Moreover, for $\mathcal{H}^1$ a.e. $t \in (0,1)$, the map $\tilde{\gamma}$ is differentiable at t and the function φ is differentiable at $\tilde{\gamma}(t)$, therefore γ is differentiable at t and, using (49),

$$\langle \gamma'(t); \nu(\gamma(t)) \rangle = \langle (\tilde{\gamma}'(t), \nabla\varphi(\tilde{\gamma}(t)) \cdot \tilde{\gamma}'(t)); \nu(\gamma(t)) \rangle = 0.$$

The proposition is therefore proved. ■

7 Appendix 3: ellipticity

We here discuss the ellipticity (see [10] for details) of the operator

$$\mathcal{L}_H(u)(x) = (Du(x))^t H \, Du(x)$$

where $x \in \Omega \subset \mathbb{R}^n$ is an open set, $H \in \mathbb{R}^{n\times n}$ and

$$u \in \mathcal{S} = \left\{ u \in C^1(\Omega; \mathbb{R}^n) : \det Du(x) \neq 0, \ \forall x \in \Omega \right\}.$$

Define, for fixed $(x, \xi, u) \in \Omega \times \mathbb{R}^n \times \mathcal{S}$, the operator $\mathcal{A}_{x,\xi,u} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ through

$$\mathcal{A}_{x,\xi,u}(\lambda) = (\lambda \otimes \xi)^t H \, Du(x) + (Du(x))^t H (\lambda \otimes \xi).$$

Definition 31 (Ellipticity) *The differential operator $\mathcal{L}_H$ is said to be* elliptic *(over Ω and $\mathcal{S}$) if for every fixed $(x, \xi, u) \in \Omega \times \mathbb{R}^n \times \mathcal{S}$ with $\xi \neq 0$, then $\lambda = 0 \in \mathbb{R}^n$ is the only solution of*

$$\mathcal{A}_{x,\xi,u}(\lambda) = 0.$$

Proposition 32 *The operator $\mathcal{L}_H$ is elliptic (over Ω and $\mathcal{S}$) if and only if H_s is invertible.*

Proof Observe that $\mathcal{A}_{x,\xi,u}(\lambda) = 0$ is equivalent to

$$\left[\left((Du(x))^t H \right) \lambda \right] \otimes \xi + \xi \otimes \left[\left((Du(x))^t H^t \right) \lambda \right] = 0$$

and thus to

$$\begin{cases} \left[\left((Du(x))^t H_s \right) \lambda \right] \otimes \xi + \xi \otimes \left[\left((Du(x))^t H_s \right) \lambda \right] = 0 \\ \left[\left((Du(x))^t H_a \right) \lambda \right] \otimes \xi - \xi \otimes \left[\left((Du(x))^t H_a \right) \lambda \right] = 0. \end{cases}$$

(i) Assume first that H_s is invertible. Setting $\mu = \left((Du(x))^t H_s \right) \lambda$, we find that the first set of equations is equivalent to $\mu \otimes \xi + \xi \otimes \mu = 0$. When $\xi \neq 0$, the only solution is then $\mu = 0$. Since $\left((Du(x))^t H_s \right)$ is invertible, we have the claim.

(ii) If H_s is not invertible, we can find $\lambda \neq 0$ with $\lambda \in \ker \left((Du(x))^t H_s \right)$ and therefore

$$\left[\left((Du(x))^t H_s \right) \lambda \right] \otimes \xi + \xi \otimes \left[\left((Du(x))^t H_s \right) \lambda \right] = 0 \quad \text{for every } \xi \neq 0.$$

Then two cases can happen. Either $\lambda \in \ker\left((Du(x))^t H_a\right)$ and thus

$$\left[\left((Du(x))^t H_a\right)\lambda\right] \otimes \xi - \xi \otimes \left[\left((Du(x))^t H_a\right)\lambda\right] = 0 \quad \text{for every } \xi \neq 0$$

concluding the claim. Or $\lambda \notin \ker\left((Du(x))^t H_a\right)$ and hence $\xi = \left((Du(x))^t H_a\right)\lambda \neq 0$ satisfies trivially

$$\left[\left((Du(x))^t H_a\right)\lambda\right] \otimes \xi - \xi \otimes \left[\left((Du(x))^t H_a\right)\lambda\right] = 0.$$

Therefore $\mathcal{A}_{x,\xi,u}(\lambda) = 0$ has a non-trivial solution; concluding the proof of the proposition. ■

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