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Introduction to Nijenhuis Geometry and its applications

Lecture 2

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Recap of Lecture 1

- Nijenhuis Geometry studies manifolds equipped with Nijenhuis operators.
  - In this study, no other structure is a priori assumed. In many applications, results proved about Nijenhuis operators will be combined with "partner structures".
  - In Lecture 1, we understood local structure of Nijenhuis operators near points where with $n = \dim(M)$ different eigenvalues (Haantjes Theorem)
  - As partner structure, we took a geodesically compatible metric g. In the Riemannian case, the operator L does not have Jordan blocks and Haantjes Theorem is applicable.
  - We describe such pairs (g, L) near generic points and also on compact two-dimensional manifolds.
    - Understanding singular points was the key step of the proof

Goal of this lecture: more complicated Segre characteristics

- Multiple eigenvalues
  - As an example of application in this lection I will always take projective equivalence. If g has a non-Riemannian characteristic, geodesically compatible L may have complex eigenvalues and Jordan blocks.
  - In the next lecture I go in another application: I will speak about integrable systems of hydrodynamic type.
- Theoretic part: Splitting Theorem
  - We will see that in the local study one can restrict ourself to the situation when L at a point p in which neighborhood we work has one real eigenvalue, or a pair of complex-conjugated eigenvalues.
- Complex eigenvalues
  - Global statements: nonexistence of Nijenhuis operator with complex eigenvalues on S^4
  - Nonexistence of projectively equivalent metrics on S^3 such that the corresponding L has complex eigenvalues.

Goal of the lecture continued

- Jordan blocks
  - Projectively equivalent metrics such that L is conjugate to a Jordan block
  - Projectively equivalent metrics of splitted signature on T^2
- If time allows: Singular points
  - Differentially nondegenerate singular points, their local description and stability.
  - Geodesically compatible metrics near differentially nondegenerate points
  - Nonexistence of differentially nondegenerate singular points on closed manifolds.

Theoretic part: Splitting Theorem

Let $\chi_{L(p)}(t) = \chi_1(t)\,\chi_2(t)$ be a factorisation of the characteristic polynomial of L at a point $p \in M$ into two factors with no common roots (over $\mathbb{R}$). We call such factorisations *admissible*. This factorisation can be naturally extended to a neighborhood of p.
Consider the distributions $\mathcal{D}_i = \operatorname{Ker} \chi_i(L)$ $(i = 1, 2)$ that provide a natural decomposition of the tangent bundle $TM = \mathcal{D}_1 \oplus \mathcal{D}_2$.

Theorem

Let L be a Nijenhuis operator. The distributions $\mathcal{D}_i = \operatorname{Ker} \chi_i(L)$ are both integrable. Moreover, in any adapted coordinate system $(x_1, \dots, x_r, y_{r+1}, \dots, y_n)$:

$$L(x, y) = \begin{pmatrix} L_1(x) & 0 \\ 0 & L_2(y) \end{pmatrix} = \text{blockdiagonal}(L_1(x), L_2(y)). \qquad (1)$$

In other words, L splits into a direct sum of two Nijenhuis operators: $L(x, y) = L_1(x) \oplus L_2(y)$. By construction, $\chi_{L_1} = \chi_1$ and $\chi_{L_2} = \chi_2$

Corollary

Every Nijenhuis operator L locally splits into a direct sum of Nijenhuis operators $L = L_1 \oplus L_2 \oplus \dots \oplus L_k$ each of which at the point $p \in M$ has either a single real eigenvalue or a single pair of complex eigenvalues.

Result of Haantjes used in Lecture 1 and repeated in the box below is a direct corollary of our result

Theorem (Haanties 1955). Let L be a Nijenhuis operator such that it is diagonalisable at every point and such that all eigenvalues are real.
Then, $L = \text{blockdiagonal}\,(\lambda(X_1)\,\mathrm{Id}_{m_1}, ..., \lambda_k(X_k)\,\mathrm{Id}_{m_k})$.

Proof. We consider the following decomposition of the characteristic polynomial:

$$\chi_L = \underbrace{(t-\lambda_1)^{m_1}}_{\chi_1} \cdot ... \cdot \underbrace{(t-\lambda_k)^{m_k}}_{\chi_k}.$$

The decomposition is admissible: the zeros of different χ_i's are different. Then, there exists a coordinate system such that

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_k$$

where L_k has only one eigenvalue $\lambda_k(X_k)$.

Since L is diagonalizable, $L_k = \lambda(X_k) \cdot \mathrm{Id}$, □

Important consequence of the Splitting Theorem

- In the local study of Nijenhuis operators, near a point p, one can assume that at p the operator L has one real eigenvalues, or a pair of complex-conjugated eigenvalues.

Generalization of Thompson 2002 (also Turiel 1996 in real-analytic case) for blocks with nonconstant eigenvalues

Theorem (Thompson 2002) Let L be a Nijenhuis operator conjugated to a single Jordan block with a constant eigenvalue. Then, there exists a coordinate system such that all components of L are constants.

Theorem. Suppose that in a neighbourhood of a point $p \in M$, a Nijenhuis operator L is algebraically generic and similar to the standard Jordan block with a non-constant real eigenvalue $\lambda(x)$. Then there exists a local coordinate system $x_1, \dots, x_n$ in which the matrix $L(x)$ takes the following form:

$$L = \begin{pmatrix} \lambda(x_n) & 1 & & & a_1 \\ & \lambda(x_n) & \ddots & & a_2 \\ & & \ddots & 1 & \vdots \\ & & & \lambda(x_n) & a_{n-1} \\ & & & & \lambda(x_n) \end{pmatrix}, \quad \text{where}$$

$$\begin{aligned} a_1 &= \lambda'_{x_n} x_1, \\ a_2 &= 2\lambda'_{x_n} x_2, \\ &\dots \\ a_{n-2} &= (n-2)\lambda'_{x_n} x_{n-2}, \\ a_{n-1} &= 1 + (n-1)\lambda'_{x_n} x_{n-1}, \end{aligned}$$

and $\lambda' = \frac{\partial \lambda}{\partial \dots}$. If $\mathrm{d}\,\lambda(p) \neq 0$, then we may set $\lambda(x_1) = x_1$ and $\lambda' = 1$.

Nijenhuis operators with complex eigenvalues: Generalized Nirenberg-Newlander theorem

Theorem

Let L be a Nijenhuis operator on M with no real eigenvalues, i.e., its spectrum at every point $x \in M$ belongs to $\mathbb{C} \setminus \mathbb{R}$. Then

1. *M is a complex manifold w.r.t. a complex structure J canonically associated with L (in fact $J = f(L)$ where f is an analytic function on $\mathbb{C} \setminus \mathbb{R}$).*
2. *L is a complex holomorphic tensor field on M w.r.t. J, i.e. can be written in the form*

$$L^{\mathbb{C}} = \sum_{i,j=1}^{n} \ell^i_j(z) \, \mathrm{d}\, z^j \otimes \partial_{z^i}$$

   with all the functions $\ell^i_j(z)$ being holomorphic in complex coordinates $z_1, \dots, z_n$.
3. *The complex Nijenhuis torsion of L vanishes, i.e.*

$$\left(\mathcal{N}_L^{\mathbb{C}}\right)^i_{jk} = \ell^m_j \frac{\partial \ell^i_k}{\partial z^m} - \ell^m_k \frac{\partial \ell^i_j}{\partial z^m} - \ell^i_m \frac{\partial \ell^m_k}{\partial z^j} + \ell^i_m \frac{\partial \ell^m_j}{\partial z^k} = 0.$$

Let us compare the formula for $\mathcal{N}_L$ for the complex Nijenhuis torsion with the formula of the Nijenhuis torsion:

$$(\mathcal{N}_L^{\mathbb{C}})^i_{jk} = \ell^m_j \frac{\partial \ell^i_k}{\partial z^m} - \ell^m_k \frac{\partial \ell^i_j}{\partial z^m} - \ell^i_m \frac{\partial \ell^m_k}{\partial z^j} + \ell^i_m \frac{\partial \ell^m_j}{\partial z^k} = 0.$$

$$(\mathcal{N}_L)^i_{jk} = L^m_j \frac{\partial L^i_k}{\partial x^m} - L^m_k \frac{\partial L^i_j}{\partial x^m} - L^i_m \frac{\partial L^m_k}{\partial x^j} + L^i_m \frac{\partial L^m_j}{\partial x^k}.$$

Here ℓ is the matrix of L written in complex coordinates;
$L = \sum_{i,j=1}^n \ell^i_j(z)\, \mathrm{d}\, z^j \otimes \partial_{z^i}$.

The manifold is even-dimensional; $dimM = 2n$; in the first formula the summation is from 1 to n, in the second from 1 to $2n$. Recall that the components of ℓ are holomophic.

We see that the formulas are virtually the same (the second is the n-dimensional analog of the 2n-dimensional first). Since algebraic and differential manipulations with holomorphic functions formally coincide with that of usual real functions, we obtain:

Corollary. Nijenhuis operator L with only complex eingenvalues is given by the same matrix as for the real case, but in complex coordinates: if $dimM = 2n$, then $\ell \in Mat(n \times n, \mathbb{C})$

First "global" applications

Theorem

Let L be a Nijenhuis operator on a closed connected manifold M with a non-real eigenvalue $\lambda \in \mathbb{C} \setminus \mathbb{R}$ at least at one point. Then this number λ is an eigenvalue of L with the same algebraic multiplicity at every point of M. Shortly: a Nijenhuis operator on a closed manifold may not have non-constant complex eigenvalues.

Idea of the proof: Holomorphic functions on closed manifolds are expected to be constant by the maximum principle.

Corollary

The eigenvalues of a Nijenhuis operator on the 4-dimensional sphere S^4 are all real.

Corollary

On S^3, all eigenvalues of L geodesically compatible with g (of any signature) are real.

Projective equivalence if L is a Jordan block

In our theorem presented above, we described Nijenhuis operators L which are conjugate to a Jordan block with nonconstant eigenvalue. Plugging them in the equation for geodesic equivalence (which is a linear system of equations and which can be immediately solved), we obtain

Theorem. Suppose Nijenhuis operator L is conjugate to a Jordan block in a neighborhood of a point and g is geodesically compatible to it. Then, in a neighborhood of the point, they are given by

$$g = \begin{pmatrix} & & & 1 & & a_{n-1} \\ & & & & & a_{n-2} \\ & & \cdot^{\cdot^{\cdot}} & & 0 & \vdots \\ & 1 & & & & a_1 \\ a_{n-1} & a_{n-2} & \dots & a_1 & & \sum_{i=1}^{n-2} a_i a_{n-i-1} \end{pmatrix}$$

$$L = \begin{pmatrix} \lambda(x_n) & 1 & & & a_1 \\ & \lambda(x_n) & \ddots & & a_2 \\ & & \ddots & 1 & \vdots \\ & & & \lambda(x_n) & a_{n-1} \\ & & & & \lambda(x_n) \end{pmatrix}, \text{ where}$$

$$a_1 = \lambda'_{x_n} x_1,$$

$$a_2 = 2\lambda'_{x_n} x_2,$$

$$\dots$$

$$a_{n-2} = (n-2)\lambda'_{x_n} x_{n-2},$$

General splitting principle

Suppose Nijenhuis operator L has a partner geometric structure (e.g., a geodesically compatible metric) or consider a natural associated object (examples will come in Lecture 3). If L has the product structure in a coordinate system,

$$L = \mathrm{blockdiagonal}(L_1(X_1), L_2(X_2)),$$

then one "may expect" that this splitting induces a splitting of the partner structure.

How the GSP works in projective equivalence

Theorem (BM 2012). Suppose (g_1, L_1) on M_1 and (g_2, L_2) on M_2 are geodesically compatible and $\text{Spectrum}(L_1) \cap \text{Spectrum}(L_2) = \emptyset$. Then,

$$\left(g := \text{blockdiagonal}\left(g_1 \chi_{L_2}(L_1), g_1 \chi_{L_1}(L_2)\right)\ ,\quad \text{blockdiagonal}\left(L_1, L_2\right)\right)$$

is geodesically compatible. Moreover, any pair compatible to $\text{blockdiagonal}\left(L_1, L_2\right)$ has this form.

Exercise. In Lecture 1 we have proven the Levi-Civita Theorem (description of (g, L) such that $L = (\,diag)(x^1, ..., x^n)$This L the direct product of 1-dimensional blocks. Let us see, in dimension 2 for simplicity, that Levi-Civita Theorem follows from Theorem BM 2012 above.

Take 1-dimensional $\left(g_1 = \alpha_1(x^1)(dx^1)^2, L_1 = x^1 \frac{\partial}{\partial x^1} \times dx^1\right)$, $\left(g_2 = \alpha_2(x^2)(dx^2)^2, L_2 = x^2 \frac{\partial}{\partial x^2} \times dx^2\right)$. Then, $\chi_{L_2}(t) = (x^2 - t)$; so $\chi_{L_2}(L_1) = (x^2 - x^1)\frac{\partial}{\partial x^1} \times dx^1$; similarly, $\chi_{L_1}(L_2) = (x^1 - x^2)\frac{\partial}{\partial x^2} \times dx^2$ so the metric looks

“Man soll die Feste feiern, wie sie fallen!”

The frametitle is a famouse german saying with no good equivalent in English; a possible one is “One should celebrate the festivals as they occur”; let us celebrate what we achieved, in the theory of projectively equivalent metrics, so far.

- If $g \sim \bar{g}$, then (g, L) is geodesically compatible (for a Nijenhuis operator L)

(2) In a neighborhood of almost every point, L splits in the product of blocks such that the block has only one real or two complex-conjugate eigenvalues.

(3) For such “one-block” L’s, geodesically compatble metics can be (easily, since the equation for g is linear) described: for 1-dim block, any g is compatible; for Jordan block it was done my lecture, for complex eigenvalue it is also done (BM +Pucacco 2009 or even Darboux 1884; BM 2015), for eigenvalues with bigger geometric multiplicity Boubel 2015

(4) The splitting of L’s induces the splitting of g’s

(2) +(3)+(4) gives us a complete local description (normal forms) of geodesically compatible pairs (g, L) in a

Celebration continued

- The problem of local description of projectively equivalent metrics was explicitly asked by E. Betrami in 1865:

  Original: La seconda ... generalizzazione ... del nostro problema, vale a dire: riportare i punti di una superficie sopra un'altra superficie in modo che alle linee geodetiche della prima corrispondano linee geodetiche della seconda.

  English translation: Describe all pairs of geodesically equivalent metrics.

- Its two dimensional Riemannian version was solved by U. Dini 1869 (We saw his solution as Exercise few slides above)
- Pseudo-Riemanian case was actively studied (e.g., in the context of general relativity, e.g. G. Hall, ...). Petrov solved the 2-dimensional and 3-dimensional Lorentzian cases (Lenin prize 1972).
- A wrong solution reported by A. Aminova 1993.
- Pseudo-Riemannian version were solved in only in BM 2015 and the proof went through Nijenhuis geometry (and based as a motivation for Nijenhuis geometry programme)

A global statement: projectively equivalent metrics of splitted signature on the 2-torus

Theorem (M$\sim$ 2012)

Let $g \sim \bar{g}$ be projectively equivalent metrics on the 2-torus T^2. Assume they are not proportional with a constant coefficient, and that the metric g does not have a Killing vector field. Then, there exists a Riemannian metric projectively equivalent to g.

Remark. Geodesically compatible (g, L) such that g is Riemannian on closed 2-surfaces were discussed in Lecture 1. In particular, the case of 2-torus was completely understood: On the universal cover $\mathbb{R}^2$ the pair (g, L) is given by $((X(x) - Y(y))\left(dx^2 + dy^2\right), L = \mathrm{diag}(X(x), Y(y))$ such that X and Y are periodic functions (I will show the pictures later). One quotient $\left(\mathbb{R}^2, g, L\right)$ by a lattice respecting the periods of the functions X and Y.

Remark. The case when the metric admits a Killing vector field is easier; a local description is in M$\sim$ 2012, there is a more detailed description in P. Mounoud 2020

Two remarks

- Theorem on the previous slide was proved before we started the Nijenhuis geometry programme; the proof is quite complex and some steps surprised some experts in Lorentzian (2-dim) geometry
  - I will explain the results using newer (two papers of BKM 2023) results in Nijenhuis geometry. The steps of the proof will be similar to that of in the Riemannian case (Lecture 1) and is also based on the description of singular points of Nijenhuis operator.
- Why the 2-torus?
  - If the metric has splitted signature, then the (closed) surface carrying this metric has zero Euler characteristic (because a null (light-like) direction gives us a nowhere vanishing vector field, possibly on a finite cover).
  - The case of the Klein bottle is reduced to the case of the torus.

Global example: two-dimensional torus T^2

Example

- Two constant eigenvalues λ_1 and λ_2 (this example does not admit compatible nonflat g). Let ξ and η be two vector fields on T^2, that are linearly independent at each point. Then we define L as follows:

$$L(\xi) = \lambda_1 \xi \quad \text{and} \quad L(\eta) = \lambda_2 \eta. \tag{2}$$

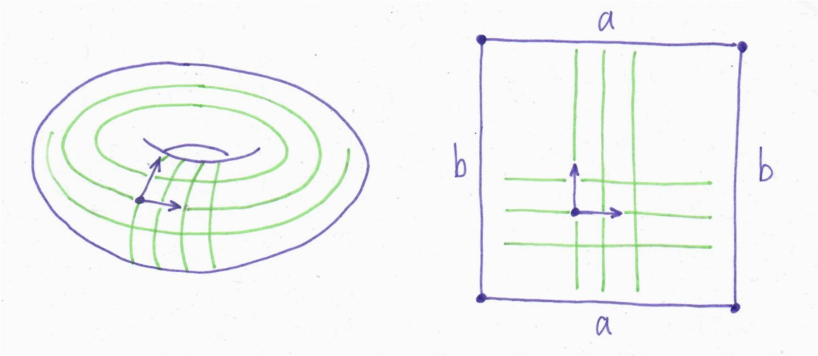
- One of eigenvalues is constant (and equals zero), the other is not. In usual angle coordinates (ϕ_1, ϕ_2), we set

$$L = \begin{pmatrix} 0 & g(\phi_1, \phi_2) \\ 0 & f(\phi_2) \end{pmatrix}. \tag{3}$$

- Two non-constant eigenvalues λ_1 and λ_2. Obvious example:

$$L = \begin{pmatrix} f(\phi_1) & 0 \\ 0 & g(\phi_2) \end{pmatrix}, \qquad f(\phi_1) < c < g(\phi_2). \tag{4}$$

Illustration 1

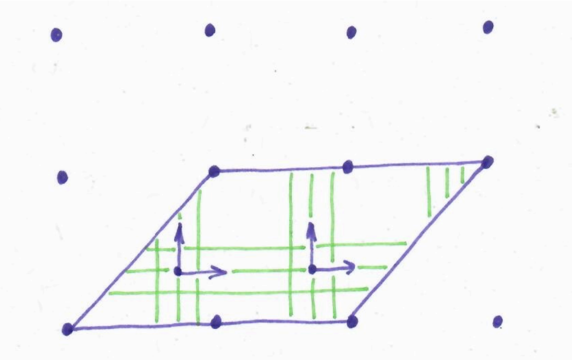


$$L = \begin{pmatrix} \sin\phi_1 & 0 \\ 0 & 2+\sin\phi_2 \end{pmatrix}$$

$$\mathsf{T}^2 = \mathbb{R}/\Gamma, \quad \Gamma = 2\pi\mathbb{Z} \times 2\pi\mathbb{Z}$$

This example can be generalised by replacing function sin by any other periodic function and by taking finite covering over this "standard" torus.

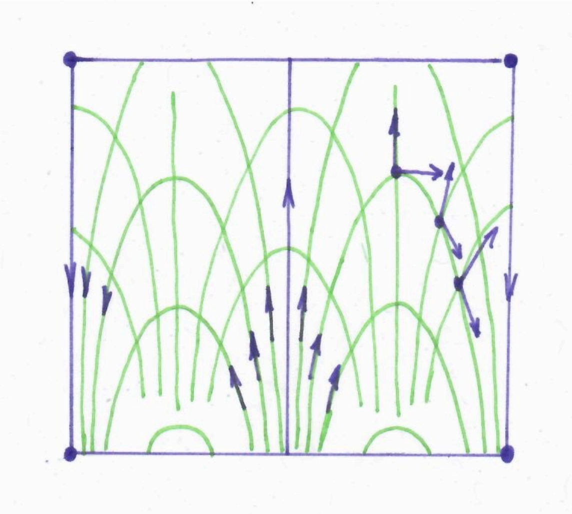
Illustration 2



$$L = \begin{pmatrix} \sin\phi_1 & 0 \\ 0 & 2 + \sin\phi_2 \end{pmatrix}$$

$$\begin{aligned} \mathsf{T}^2 = \mathbb{R}/\Gamma, \quad \Gamma &= \mathrm{Span}_{\mathbb{Z}}(2e_1, e_1 + e_2) \\ &= \{(2k + m)e_1 + me_2,\ k.m \in \mathbb{Z}\} \end{aligned}$$

Illustration 3



An example of two linearly independent vector fields on the torus

$\mathfrak{gl}$-regular operators

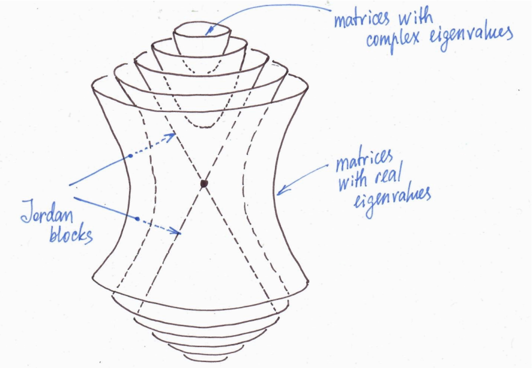
Definition

A linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$ is called $\mathfrak{gl}$-regular, if one of the following equivalent conditions hold

- the dimension of its orbit $\mathcal{O}(L) = \{ALA^{-1} \mid A \in \mathrm{GL}(n, \mathbb{R})\}$ is maximal (and equals $n^2 - n$),
- Jordan normal form of L contains exactly one Jordan block for each eigenvalue.

gl-regularity in dimension 2

Traceless 2×2-matrices $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{sl}(2, \mathbb{R})$ as points in $\mathbb{R}^3(a, b, c)$:



Conclusion. An operator L on a two-dimensional manifold is gl-regular if and only if L admits no singular points of scalar type, i.e., $L(x) \neq \lambda \cdot \mathrm{Id}$ at any point.

$\mathfrak{gl}$-regular linear operators in dimension 2

In dimension 2, we have $\mathfrak{gl}$-regular linear operators of the following algebraic types:

- two distinct real eigenvalues: $L = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, $\lambda \neq \mu$;
- two complex conjugate eigenvalues $L = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix}$, $\mu \neq 0$;
- Jordan block: $L = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$;
- scalar matrix: $L = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.

We see that the first three types are $\mathfrak{gl}$-regular, and the only exception are scalar matrices $\lambda \cdot \mathrm{Id}$.

Why the gl-regularity condition

- As I explain on the next slide, the condition is somehow generic.
- In the context of projectively metrics on the 2-torus, one can show (will not do because of the time restrictions, the proof is not complicated but uses other circle of ideas) that the correspoding L is always gl-regular
- gl-regular operators will be used in the next lecture

Reminder: Local classification in dimension 2, near points where eigenvalues do not bifurcate

First three types of non-singular points

- Two real eigenvalues:

$$L = \begin{pmatrix} f(x) & 0 \\ 0 & g(y) \end{pmatrix},$$

  where $f(x)$ and $g(y)$ are smooth functions such that $f(x) \neq g(y)$ for all (x, y).

- Two complex conjugate eigenvalues:

$$L = \begin{pmatrix} f(x,y) & -g(x,y) \\ g(x,y) & f(x,y) \end{pmatrix},$$

  where $h = f + \mathrm{i}\, g$ is holomorphic in $z = x + \mathrm{i}\, y$, $g(x, y) \neq 0$ for all (x, y).

- Jordan block:

$$L = \begin{pmatrix} f(y) & 1 \\ 0 & f(y) \end{pmatrix},$$

  where $f(y)$ is a smooth function.

Local classification in dimension 2 near singular points

Theorem below is long, two pages; essential message is that we have a full description. In the paper, we assume real analyticity, but we do not need it in the context of projectively equivalent metrics.

Theorem (BKM 2023)

Let L be a Nijenhuis operator such that $L(\mathrm{p}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then in suitable local coordinates (x, y), this operator takes one of the following forms:

1. *Series L:* $L_{\mathrm{nil}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L_{\mathrm{nd}} = \begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix},$
2. *Series M:* $M_{2k-1} = \begin{pmatrix} 0 & 1 \\ 0 & y^{2k-1} \end{pmatrix}, \quad M_{2k}^{\epsilon} = \begin{pmatrix} 0 & 1 \\ 0 & \epsilon y^{2k} \end{pmatrix},$
3. *Series N:* $N_{2k-1} = \begin{pmatrix} y^{2k-1} & 1 \\ 0 & y^{2k-1} \end{pmatrix}, \quad N_{2k}^{\epsilon} = \begin{pmatrix} \epsilon y^{2k} & 1 \\ 0 & \epsilon y^{2k} \end{pmatrix}.$

Here $k \in \mathbb{N}$, $\epsilon = \pm 1$.

(to be continued ...)

Theorem about local classification in dimension 2 (continued)

Magic formula: L can be uniquely recovered from the coefficients of its characteristic polynomial $v = \mathrm{tr}\, L$ and $u = -\det L$ provided they are functionally independed as follows:

$$L = \begin{pmatrix} v_x & v_y \\ u_x & u_y \end{pmatrix}^{-1} \begin{pmatrix} v & 1 \\ u & 0 \end{pmatrix} \begin{pmatrix} v_x & v_y \\ u_x & u_y \end{pmatrix}, \qquad v = \mathrm{tr}\, L,\ u = -\det L.$$

4. Series $O^{d,\epsilon}_{k,\mathrm{c}}$, $k \geq 1$, $d \geq 2k+1$, $\epsilon = \pm 1$, $\mathrm{c} = (c_0, \dots, c_{k-1}) \in \mathbb{R}^k$ and we set $\epsilon = 1$, if $d = 2m+1$ is odd:

$$v = \alpha x y^{2k-1} + y^k \left(c_{k-1} y^{k-1} + \dots + c_1 y + c_0\right), \quad u = \epsilon\, y^d, \quad \alpha = k c_0^2 \left(1 - \tfrac{k}{d}\right) \neq 0.$$

5. Series $P^{k,\epsilon}_{s,\mathrm{c}}$, $k \geq 1$, $s \geq 2k$, $\epsilon = \pm 1$, $\mathrm{c} = (c_0, \dots, c_{k-1}) \in \mathbb{R}^k$:

$$v = \alpha x y^s + y^{s-k+1} \left(c_{k-1} y^{k-1} + \dots + c_1 y + c_0\right) + 2\epsilon\, y^k, \quad u = -y^{2k}, \quad \alpha = 2\epsilon\, k c_0 \neq 0.$$

6. Series $S^{2k,\epsilon}_{\mathrm{c}}$ and S^{2k+1}_{c}, $k \geq 1$, $\mathrm{c} = (c_0, \dots, c_{k-1}) \in \mathbb{R}^k$:

$$v = \alpha x y^{2k-1} + y^k \left(c_{k-1} y^{k-1} + \dots + c_1 y + c_0\right), \quad u = \epsilon\, y^{2k}, \quad \alpha = \tfrac{k}{2} \left(c_0^2 + 4\epsilon\right) \neq 0,$$

Singular set

Let (M^2, L) be a gl-regular Nijenhuis manifold of dimension 2.
Let Sing be the set of singular points of L where the algebraic type of L changes. In our case, this means that the eigenvalues of L collide, i.e.

$$\mathsf{Sing} = \{\mathsf{p} \in \mathrm{M}^2 \mid v^2 + 4u = 0\}, \quad \text{where } v = \operatorname{tr} L,\ u = -\det L,$$

unless $v^2 + 4u \equiv 0$ on M^2 meaning that L is similar to a Jordan block at each point.
From Classification Theorem we immediately obtain a *local* description of Sing in canonical coordinates x, y:

- for L_{nil}, N_{2k-1} and N^{ϵ}_{2k}, the singular set is empty;
- for L_{nd} the singular set is $\mathsf{Sing} = \{x^2 + 4y = 0\}$;
- for all the other series M, O, P and S: $\quad \mathsf{Sing} = \{y = 0\}$.

Thus, locally Sing is a smooth curve. Since $\mathsf{Sing} \subset \mathrm{M}^2$ is closed, we may think of it as a submanifold consisting, perhaps, of several connected components:

$$\mathsf{Sing} = \mathsf{S}_1 \cup \cdots \cup \mathsf{S}_\ell.$$

If M is compact, then each of them is an embedded circle.

Global results in dimension 2

Theorem

Let (M^2, L) *be a closed connected* gl*-regular Nijenhuis manifold. Then we have one of the following possibilities:*

1. M^2 *is orientable and* $L = \alpha \, \mathrm{Id} + \beta J$*, where* J *is a complex structure on* M^2 *and* $\alpha, \beta \in \mathbb{R}$ *are constants,* $\beta \neq 0$*.*
2. M^2 *is homeomorphic to either torus or Klein bottle, and* L *has two distinct real eigenvalues at each point of* M^2*.*
3. M^2 *is homeomorphic to a torus, and* L *is similar to a Jordan block at each point of* M^2*.*
4. M^2 *is homeomorphic to either torus or Klein bottle, and one of eigenvalues of* L *is constant.*

Corollary

1. *Let* M^2 *be orientable and* $\mathsf{M}^2 \not\approx \mathsf{T}^2$*. Then on* M^2 *there are no* gl*-regular Nijenhuis operators except for* $L = \alpha\,\mathsf{Id} + \beta J$*.*
2. *Let* M^2 *be non-orientable and* $\mathsf{M}^2 \not\approx \mathsf{K}^2$*. Then on* M^2 *there are no* gl*-regular Nijenhuis operators at all.*

Idea of the proof of Theorem about projectively equivalent metrics on the torus

Theorem Let (M^2, L) be a closed connected $\mathfrak{gl}$-regular Nijenhuis manifold. Then we have one of the following possibilities:

1. M^2 is orientable and $L = \alpha \,\mathrm{Id} + \beta J$, where J is a complex structure on M^2 and $\alpha, \beta \in \mathbb{R}$ are constants, $\beta \neq 0$.
2. M^2 is homeomorphic to either torus or Klein bottle, and L has two distinct real eigenvalues at each point of M^2.
3. M^2 is homeomorphic to a torus, and L is similar to a Jordan block at each point of M^2.
4. M^2 is homeomorphic to either torus or Klein bottle, and one of eigenvalues of L is constant.

Let (g, L) be geodesically compatible on T^2.

- In the case (1) of Theorem, the eigenvalues of L are constant which implies $\mathrm{tr}\,(L) = \mathrm{const}$. In this case, the equation of geodesic compatibility $L_{ij,k} = \frac{1}{2} g_{ik} \frac{\partial \mathrm{tr}\,(L)}{\partial x^j} + \frac{1}{2} g_{ij} \frac{\partial \mathrm{tr}\,(L)}{\partial x^k}$ implies $\nabla L = 0$ which in turn implies in dimension 2 that the metric is flat.
- In the case (2) one can find a Riemannian metric in the class of projectively equivalent metrics.
- In the case (3), one writes the formula for geodesically compatible (g, L) from the beginning of the lecture and sees that it can not live on closed manifolds.
- In the case (4), there exists a Killing vector field.

Results-oriented recap of the first two lectures

- We described $\mathfrak{gl}$-regular Nijenhuis operators locally
  - In dimension n: at regular points
- We combined this description with the existence of the partner structure, geodesically compatible g
  - In Riemannian case, we described compatible pairs (g, L) near all points
  - In any signature, we described (g, L) at all points; solving the classical Beltrami problem 1865
- We combined the local description in a global picture and as result described all projectively equivalent metrics on 2-dim manifolds.