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Introduction to Nijenhuis Geometry and its applications

Lecture 3

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Recap of Lectures 1,2

- Nijenhuis Geometry studies manifolds equipped with Nijenhuis operators.
  - In this study, no other structure is a priori assumed. In many applications, results proved about Nijenhuis operators will be combined with "partner structures".
  - In Lectures 1 and 2, we understood local structure of $\mathfrak{gl}$-regular Nijenhuis operators
  - As partner structure, we considered a geodesically compatible metric g.
  - We describe such pairs (g, L) near generic points and also on compact two-dimensional manifolds.

Goal of this lecture: symmetries and conservation laws of Nijenhuis operators and applications

- Nijenhuis operators have natural objects associated to them. We did see some in the first two lectures: eigenvalues, eigenvectors, coefficients of characteristic polynomial. All of them were of *agebraic* nature.
- In this and next lecture we consider natural *differential-geometric* objects associated to Nijenhuis operator.
    - Conservation laws
    - Symmetries (next lecture)
- Application: integration in quadratures of certain systems of hydrodynamic type

The application selected for this lecture : (integrable) systems of hydrodynamic type

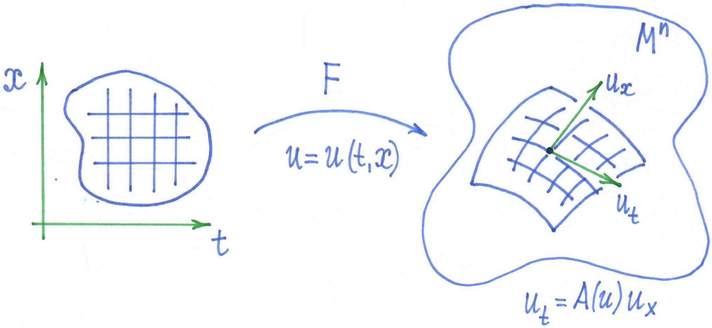
- As mathematical objects, they are PDE systems of the form

$$\frac{\partial}{\partial t}u^i(t,x) = A(u)^i_s \frac{\partial}{\partial x}u^s(t,x) \quad \text{(shortly: } u_t = Au_x\text{)}.$$

  Here $u(t,x) = (u^1(t,x), ..., u^n(t,x))$ is unknown vector-function of two variables (t,x) and A is a matrix depending on $(u^1, ..., u^n)$ with no explicit dependence on t and x. (Here x is one-dimensional variable)

- $(u^1, ..., u^n)$ should be viewed as local coordinate system on a manifold. The matrix A is then an operator (=$(1,1)$-tensor); after a transformation $(u^1, ..., u^n) \mapsto (\tilde{u}^1, ..., \tilde{u}^n)$ the equation above has the same form with $\tilde{A}(\tilde{u}) = JA(u)J^{-1}$.

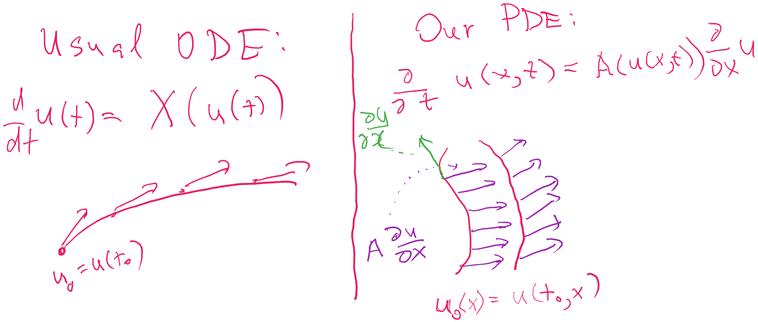
Geometric interpretation



Solution of the hydrodynamic type system $u_t = A(u)u_x$ as a surface
in the manifold M^n endowed with the field of endomorphisms A.

Let us compare the ODE $\frac{d}{dt}u(t) = V(u)$ describing dynamics of particles with the PDE $u_t = Au_x$:

- Infinite-dimensional “particles” are curves $u(x)$; the “configuration space” is the set of all (real-analytic curves)
- In $u_t = Au_x$: the t-derivatives for fixed $t = t_0$ are expressed in terms of $u(t_0, x)$.
- For initial real-analytic data $\hat{u}(x) = u(t_0, x)$ there exists a unique local solution $u(t, x)$ by the Kovalevskaya Theorem.



Examples of systems of hydrodynamic type

- Systems of hydrodynamic type naturally arise in continuum mechanics, in the theory of nonlinear dispersive waves and in the theory of shock waves. In mathematics it was studied before, e.g. by Riemann and by Hopf.
- I will give two mathematical examples.

Example: Cauchy-Riemann conditions $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ can be written in the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_y = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x .$$

This is a system of hydrodynamic type with $n = 2$, $t = y$ and $A = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$.

Of course in this example the matrix A does not depend on u, and is relatively simple. In many mathematical problems A is complicated.

A more complicated example: polynomial integrals for geodesic flows on surfaces

Example: Consider a 2-dimensional Riemannian metric in the conformal form $g = \lambda(x_1, x_2)(dx_1^2 + dx_2^2)$ and let us look for an integral F of the geodesic flow which is homogeneous in momenta of degree 3. Then, by a known (e.g. Kolokoltsov 1982) coordinate transformation we may think that

$$F = {p_1}^3 a(x_1, x_2) + {p_1}^2 p_2 b(x_1, x_2) + p_1 {p_2}^2 (a(x_1, x_2) - 1) + {p_2}^3 b(x_1, x_2)$$

The condition that $\{F, H\} = 0$ is then the system $u_t = Au_x$ of hydrodynamic type with $n = 3$, $t = x_1$, $x_2 = x$,

$$u = \begin{pmatrix} \lambda(x_1, x_2) \\ a(x_1, x_2) \\ b(x_1, x_2) \end{pmatrix} \text{ and } A = \begin{bmatrix} -3\frac{b}{a-1} & 0 & 2\frac{\lambda}{a-1} \\ -\frac{1}{2}\frac{b(8a+1)}{\lambda(a-1)} & 0 & 3\frac{a}{a-1} \\ -\frac{3b^2 - a^2 + 2a - 1}{\lambda(a-1)} & -1 & 2\frac{b}{a-1} \end{bmatrix}.$$

What does integrability mean in this context?

- Let me first recall the definition in the finite-dimensional case.
  - The established definition is as follows: an ODE-system $\dot{u} = V(u)$ is integrable, if it has sufficiently many
  - functions $f_1, f_2, ..., f_{k-1}$ that are constant on the solutions,
  - vectors fields $W_1, ..., W_{n-k}$ which commute with V.
- In the finite-dimensional case, integrability implies that one can reduce solving of the system to integration in quadratures, i.e., to integration of closed 1-forms and solving the systems of functional equations (S. Lie 1884; sometimes called Arnold-Kozlov Theorem).
- The infinite-dimensional case:
  - There exists an established definition, which is visually similar (= uses similar words) to the finite-dimensional case.
  - There is also a collection of methods to deal with integrable infinite-dimensional integrable systems
  - Unfortunately in most cases this collection of methods does not help: it merely reduces one problem to an equally complicated another one

- **In my lecture I will call a system of hydrodynamic type integrable, if one can integrate it in quadratures (for most initial data, or at least for sufficiently many (e.g. dense subset) initial data).**
- This definition of integrable systems was used by classical mathematicians (e.g., Jacobi, Lie, Pailnleve, Cartan, Eisenhart) and physicists
- Other notions of integrability are used in literature. For example, the PDE-system for the existence of polynomial integral is integrable in certain case, but nobody solved it or found sufficiently many solutions so far
- The list of known integrable systems of hydrodynamic type, in the sense of my definition, is very short; moreover, in all known cases the operator A is diagonal in a coordinate system. I will give the list on the next slide

List of known integrable systems of hydrodynamic type

- Simplest nontrivial example: assume L is Nijenhuis operator with n different eigenvalues. Then, the equation $u_t = Lu_x$ can be integrated in quadratures.
  - **Proof (Folklore).** By Haantjes Theorem, we may assume $L = \mathrm{diag}(u^1, \dots, u^n)$, since one can do it by a diffeomorphic coordinate change. The equation $u_t = Lu_x$ decouples then in n independent Hopf equations $\begin{cases} u^1_t = u^1 u^1_x \\ \vdots \\ u^n_t = u^n u^n_x \end{cases}$
- Bi-Hamiltonian systems are integrable (Dubrovin-Novikov, Tsarev). Many examples of Bi-Hamiltonian systems of hydrodynamic type such that A has n different eigenvalues were constructed by Magri et al.
- Weakly-nonlinear integrable systems (of hydrodynamic type) are integrable in my sense (Ferapontov 1991–1992).

Nijenhuis geometry allowed us to generalise all results from the list above to nondiagonal (gl-regular) operators

- The case $u_t = Lu_x$, where L is a gl-regular Nijenhuis operator, possibly nondiagonalisable, was done in BKM 2023.
- Nondiagonal bi-Hamiltonian examples were constructed in BKM 2021. I will speak about them in the next lecture.
- **The remaining part of this lecture will be about the nondiagonal analogy of the weakly-nonlinear gl-regular case.**

What problem I will discuss (solve and give proofs) in the next part of the talk

- I start with gl-regular Nijenhuis operator L with nonconstant eigenvalues (we locally described them in the first part).
- I consider the PDE-system $u_t = Au_x$ with

$$A = \det(L)L^{-1}.$$

- I explain how to solve this PDE in quadratures, that is, using integration of closed 1-forms and solving of systems of functional equations
- If time allows, I will speak about finite-dimensional reductions of such systems.

Remark. The system with this A is just one system (for a given Nijenhuis operator A). It is appears naturally in many problems in physics, especially in the theory of finite-dimensional reductions of infinite-dimensional integrable PDE-systems.

E.g. finite-gap solutions of the KdV equations can be interpreted as certain solutions of our system (Dubrovin 1987, Moser 1986). The same

A nondegeneracy assumption: gl-regular operators

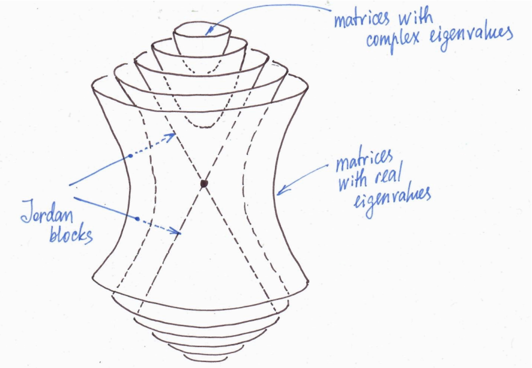
Definition

A linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$ is called gl-regular, if one of the following equivalent conditions hold

- the dimension of its orbit $\mathcal{O}(L) = \{ALA^{-1} \mid A \in \mathrm{GL}(n, \mathbb{R})\}$ is maximal (and equals $n^2 - n$),
- Jordan normal form of L contains exactly one Jordan block for each eigenvalue.

gl-regularity in dimension 2

Traceless 2×2-matrices $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{sl}(2, \mathbb{R})$ as points in $\mathbb{R}^3(a, b, c)$:



In all dimensions: The gl-regular operators with fixed eigenvalues of fixed algebraic multiplicity form the orbit of largest dimension of the action of $SL(n)$ on $GL(n)$ by conjugation.

Why the gl-regularity condition

- As I explained on the previous slide, the condition is generic.
- It still allows Jordan blocks, bifurcation of eigenvalues and singular points
- The hodograph methods which will be discussed in Lecture 4 will require gl-regularity
- All examples from the literature we found, in particular two mathematical examples I discussed in the introduction (Cauchi-Riemann equations and equation for existence of polynomial integral) assume/satisfy gl-regularity

Preliminary work: conservation laws of the Nijenhuis operators

Def. Let $A = A^i_j$ be any, not necessary Nijenhuis, operator. By its conservation law we understand a function $f : M \to \mathbb{R}$ such that $d(A^* df) = 0$.

$A^* : T^*M \to T^*M$ is the dual operator, $A^*\xi(v) = \xi(Av)$. In coordinates, $A^*\xi$ is given by

$$(\xi_1, ..., \xi_n)A^t.$$

The name "conservation law" will be explained on the next slide; they are infinite-dimensional analogues of integrals (=functions constant on trajectories) for finite-dimensional integrable systems. The first examples or conservation laws are related to the term "conservation laws" used in physics, e.g. in thermodynamics.

Why conservation law is called conservation law?

Let M^n be a manifold and $A = A^i_j$ an operator on it.

Consider $\gamma : S^1 \to M^n$ to be a closed loop in M^n. In coordinates, it is given parametrically as $\gamma(x) = \big(u^1(x), \dots, u^n(x)\big)$, $x \in \mathbb{R} \bmod 2\pi \simeq S^1$. Then the quasilinear equation

$$u_t = A\, u_x,$$

describes a certain evolution $\gamma_t : S^1 \to M^n$ of this loop in M determined by the operator A.

For a function $g : M \to \mathbb{R}$, we can consider a natural functional on the loop space defined by

$$\gamma \;\mapsto\; \oint_\gamma g\big(\gamma(x)\big)\,\mathrm{d}\,x \qquad (***)$$

Proposition

This functional $(***)$ (or the function g as its density) is a conservation law for the PDE system $u_t = A\, u_x$ if and only if for every solution $u(x, t)$ the value of this functional $\oint_\gamma g\big(u(t, x)\big)\,\mathrm{d}\,x$ does not change during evolution in t.

Conservation laws (continued...)

Proposion. The functional $\gamma(x) \mapsto \oint_\gamma g\big(\gamma(x)\big)\,\mathrm{d}\,x$ (or the function g as its density) is a conservation law for the PDE system $u_t = A\,u_x$ if and only if for every solution $u(x,t)$ the value of this functional $\oint_\gamma g\big(u(t,x)\big)\,\mathrm{d}\,x$ does not change during evolution in t.

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_0^{2\pi} g(u(t,x))\,\mathrm{d}\,x = \int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}\,t} g(u(t,x))\,\mathrm{d}\,x = \int_0^{2\pi} \frac{\partial g}{\partial u^i} u_t^i\,\mathrm{d}\,x$$

$$= \int_0^{2\pi} \frac{\partial g}{\partial u^i} A_j^i u_x^j\,\mathrm{d}\,x = \oint_\gamma \frac{\partial g}{\partial u^i} A_j^i\,\mathrm{d}\,u^j = \oint_\gamma A^*(\mathrm{d}\,g)$$

This integral vanishes on any loop if and only if $A^*(\mathrm{d}\,g)$ is exact.

In this lecture we study conservation laws of Nijenhuis operators only

Theorem (Folklore; known by Magri and Kosman-Schwarzbach (1990th), proof e.g. BKM 2021). If f is conservation law for a Nijenhuis operator L, then it is conservation law for L^k with any k.

Starting with one conservation law , we then construct infinitely many of them (from which we will need the first n):
$f_1 := f$, f_2 satisfying $df_2 = L^* df_1$, f_3 satisfying
$df_3 = L^* df_2 = (L^2)^* df_1, \dots$

We will call the sequence of functions satisfying $df_i = L^* df_{i-1}$ hierarchy of conservation laws.

Our first new result is a local description of conservation laws for gl-regular Nijenhuis operators.

- I will first give a series of examples and then
- claim/explain that there are no other examples.

"Splitting Lemma" for conservation laws (GSP in action)

Splitting Lemma for Nijenhuis operators. Suppose eigenvalues of a Nijenhuis operator L at $p \in M^n$ are decomposed into k disjunct subsets:

$$\mathrm{spectrum}(L)(p) = \mathrm{spectrum}_1 \dot\cup \cdots \dot\cup\, \mathrm{spectrum}_k.$$

Then, locally there exists a local coordinate system such that L is blockdiagonal, $L = \mathrm{diag}(L_1, \dots, L_k)$, each L_i is a Nijenhuis operator depending on the own coordinates only, and at the point p we have

$$\mathrm{spectrum}(L_1)(p) = \mathrm{spectrum}_1, \dots, \mathrm{spectrum}(L_k)(p) = \mathrm{spectrum}_k.$$

Theorem. Consider the "blockdiagonal" gl-regular Nijenhuis operator $L = \mathrm{diag}(L_1, \dots, L_k)$. Then, every conservation law for L is the sum $h_1 + ... + h_k$, where for every $i = 1, ..., k$ the function h_i depends on the coordinates of the ith block and is a conservation law for L_i.

Success report: By Theorem above, in order to describe all conservation laws for gl-regular Nijenhuis operators in a neighbourhood of almost every point, it is sufficient to describe

"Diagonal" example (well known)

Conservation law is a function $f : M \to \mathbb{R}$ such that $d(L^* df) = 0$. For gl-regular $L = diag(L_1, \dots, L_k)$ every conservation law is the sum of conservation laws for L_i.

In dimension 1, a Nijenhuis operator with nonconstant eigenvalue is up to a coordinate change $udu \otimes \frac{\partial}{\partial u} = \left(u\right)$.
Any function $h(u)$ is a conservation law since any 1-form is closed. The hierarchy of the conservation laws is then $h(u)$, $\int_{\hat u}^{u} \xi h'(\xi) d\xi$, $\int_{\hat u}^{u} \xi^2 h'(\xi) d\xi$,..., $\int_{\hat u}^{u} \xi^k h'(\xi) d\xi$,... .

Corollary. Suppose $L = \mathrm{diag}(u^1, ..., u^n)$. Then, near a point where $u^1, ..., u^n$ are mutually different, every conservation law has the following form

$$f = h_1(u^1) + ... + h_n(u^n)$$

Then, corresponding hierarchy is

$$f_2 = \int_{\hat u^1}^{u^1} \xi h'_1(\xi) d\xi + ... + \int_{\hat u^n}^{u^n} \xi h'_n(\xi) d\xi, \ \dots$$

$$f_k = \int_{\hat u^1}^{u^1} \xi^{k-1} h'_1(\xi) d\xi + \dots + \int_{\hat u^n}^{u^n} \xi^{k-1} h'_n(\xi) d\xi, \ \dots$$

Conservation laws for the Jordan-block case

Now consider the Nijenhuis operators in the form

$$L_c = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad L_{nc} = \begin{pmatrix} u^n & u^{n-1} & u^{n-2} & \dots & u^1 \\ 0 & u^n & u^{n-1} & \dots & u^2 \\ & & \ddots & \ddots & \\ 0 & \dots & 0 & u^n & u^{n-1} \\ 0 & 0 & 0 & \dots & u^n \end{pmatrix}.$$

Next, for any n functions $h_1, \dots, h_n$ of one variables we consider the functions $f_1, \dots, f_n$ given by

$$h_1(L_{nc})L_c^{n-1} + \dots + h_{n-1}(L_{nc})L_c + h_n(L_{nc}) = f_1 L_c^{n-1} + f_2 L_c^{n-2} + \dots + f_n \mathrm{Id}$$

Theorem. The functions $f_1, \dots, f_n$ are conservation laws both for L_c and for L_{nc}. Moreover, every conservation law of L_c or of L_{nc} can be constructed in this way.

We see that the "freedom" is the same as in the "diagonal case": a conservation law is determined by n functions $h_1, \dots, h_n$ of 1 variable each.

Success report. We described all conservation laws of gl-regular Nijenhuis operators (at almost every point). Starting from one of them (we call it h), one constructs the whole hierarchy by integrating of closed forms $L^* dh$, $(L^2)^* dh$, $(L^3)^* dh$, ...

“Splitting lemma” for conservation laws and preliminary success report:

Splitting Lemma for Nijenhuis operators. Suppose eigenvalues of a Nijenhuis operator L at $p \in M^n$ are decomposed into k disjunkt subsets:

$$\mathrm{spectrum}(L)(p) = \mathrm{spectrum}_1 \dot\cup \cdots \dot\cup\, \mathrm{spectrum}_k.$$

Then, locally there exists a local coordinate system such that L is blockdiagonal, $L = \mathrm{diag}(L_1, \dots, L_k)$, each L_i is a Nijenhuis operator depending on the own coordinates only, and at the point p we have

$$\mathrm{spectrum}(L_1)(p) = \mathrm{spectrum}_1, \dots, \mathrm{spectrum}(L_k)(p) = \mathrm{spectrum}_k.$$

Theorem. Let $L = \mathrm{diag}(L_1, \dots, L_k)$ where L_i are gl-regular Nijenhuis operators with pairwise disjunct spectra (= the operator L is gl-regular).

Then, every conservation law of L is the sum of conservation laws of L_i.

Success report. Theorem and examples above describe all

An integrable system of hydrodynamic type, its symmetries and integration in quadratures

Let L be a gl-regular Nijenhuis operator. Define $(1,1)$ tensors A_i via

$$\det(\lambda \mathrm{Id} - L)(\lambda \mathrm{Id} - L)^{-1} = \lambda^{n-1} \underbrace{\mathrm{Id}}_{A_n} + \lambda^{n-2} A_{n-1} + \dots + (-1)^{n-1} \underbrace{\det(L) L^{-1}}_{A_1}.$$

Theorem (BKM 2022; Lorenzoni-Magri 2005). The systems of hydrodynamic type corresponding to the operators $A_1, \dots, A_n$ is in involution, that is, the system of n^2 PDEs on n functions $u^1(t_1, \dots, t_n)$ given by

$$\frac{\partial u}{\partial t_i} = A_i \frac{\partial u}{\partial x} \qquad (*)$$

is compatible, that is, for any initial real analytic curve $\hat{u}(x)$ there exists a real-analytic solution $u(t_1, ...t_{n-1}, t_n)$ such that $u(0, ..., 0, x) = \hat{u}(x)$.

Remark. The variable t_n above should be viewed as the variable x. That is, the ith equation is actually $\frac{\partial u}{\partial t_i} = A_i \frac{\partial u}{\partial t_n}$. Note that since $A_n = \mathrm{Id}$, the nth equation from Theorem above reads $\frac{\partial u}{\partial t_n} = \frac{\partial u}{\partial x}$.

Historical remark and credits. For diagonal L, Theorem was known in 1990th due to a series of works of Ferapontov and Pavlov. For arbitrary operator L, Theorem can be obtained from the results of Lorenzoni-Magri

Integration in quadratures of the system $(*)$

$$\frac{\partial u}{\partial t_i} = A_i \frac{\partial u}{\partial x} \qquad (*)$$

Main Theorem. Let L be a gl-regular Nijenhuis operator and $f_1, \dots, f_n$ the first n elements of a hierarchy of conservation laws (i.e., $df_i = L^* df_{i-1}$). Then, near the points where functions $f_1, \dots, f_n$ are functionally independent, any vector-valued function $u(t_1, \dots, t_n)$ satisfying $(**)$ is a solution of $(*)$.

$$f_i(u(t_1, \dots, t_n)) = t_i + \mathrm{const}_i \qquad (**)$$

Moreover, for almost every solution $u(t_1, \dots, t_n)$ of $(*)$ there exists a hijerarchy of the conservation laws $f_1, \dots, f_n$ such that they are functionally independent and such that $u(t_1, \dots, t_n)$ satisfies $(**)$.

Success report: We reduced solving of the PDE-system $(*)$ to solving the system $(**)$, that is, to algebraic operations and integrations. In the classical literature, it refers to as "solving, or integrating, in quadratures". Almost all solutions $(u(t_1, \dots, t_n))$ of $(*)$ can be obtained by this procedure.

Proof of Main Theorem

Main Theorem. Let L be a gl-regular Nijenhuis operator and $f_1, \dots, f_n$ the first n elements of a hierarchy of conservation laws such that the functions $f_1, \dots, f_n$ are functionally independent. Then, any vector-valued function $u(t_1, \dots, t_n)$ satisfying $f_i(u(t, \dots, t_n)) = t_i + \text{const}_i$ is a solution of $\frac{\partial u}{\partial t_i} = A_i \frac{\partial u}{\partial x}$ with $x = t_n$.

Proof. Since $f_1, ..., f_n$ are functionally independent, they form a coordinate system. In this coordinate system, the covectors $df_1, ..., df_n$ are given by

$$\begin{aligned} df_1 &= (1 \quad 0 \quad \dots \quad 0) \\ df_2 &= (0 \quad 1 \quad \dots \quad 0) \\ &\vdots \\ df_n &= (0 \quad 0 \quad \dots \quad 1)\ . \end{aligned}$$

The condition $L^* df_i = df_{i+1}$ implies that in this coordinate system

$$L = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & 1 \\ \sigma_n & \sigma_{n-1} & \dots & \sigma_2 & \sigma_1 \end{pmatrix}$$

The matrices of these form (called companion matrices by Frobenius (Germ: Begleitmatrix)) are frequently used e.g. in the standard University course of basic linear algebra, in particular because the

Preliminary algebraic Lemma

Lemma. For L given by

$$L = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ \sigma_n & \sigma_{n-1} & \dots & \sigma_2 & \sigma_1 \end{pmatrix}$$

define $A_1, ..., A_n$ by

$$\det(\lambda \mathrm{Id} - L)(\lambda \mathrm{Id} - L)^{-1} = \lambda^{n-1} \underbrace{\mathrm{Id}}_{A_n} + \lambda^{n-2} A_{n-1} + \dots + (-1)^{n-1} \underbrace{\det(L) L^{-1}}_{A_1}.$$

Then, the last column of A_i is $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1_i \\ \vdots \\ 0 \end{pmatrix}$.

Proof is an exercise for the 1st year students in linear algebra which you give to them when you read the linear algebra course next time.

Next, consider our system $(*)$:

$$\frac{\partial u}{\partial t_i} = A_i \frac{\partial u}{\partial x} \qquad (*)$$

In our coordinate system, since the last column of A_i is e_i, we have that $A_i e_n = e_i$. This implies that the vector valued functions

$$\begin{pmatrix} f_1(t_1, ..., t_n) \\ \vdots \\ f_n(t_1, ..., t_n) \end{pmatrix} \text{ given by } \begin{pmatrix} f_1(t_1, ..., t_n) \\ \vdots \\ f_n(t_1, ..., t_n) \end{pmatrix} = \begin{pmatrix} t_1 + \text{const}_1 \\ \vdots \\ t_n + \text{const}_n \end{pmatrix}$$

solves the system $(*)$. Indeed, $\frac{\partial}{\partial t_i} \begin{pmatrix} t_1 + \text{const}_1 \\ \vdots \\ t_n + \text{const}_n \end{pmatrix} = e_i.$

Next, note that the condition $(**)$ reads, in coordinates $f_1, ..., f_n$,

$$f_i(u(t_1, \ldots, t_n)) = t_i + \text{const}_i \qquad (**)$$

and solving these condition with respect to $f_1, ..., f_n$ we obtain

$$\begin{pmatrix} f_1(t_1, ..., t_n) \\ \vdots \\ f_n(t_1, ..., t_n) \end{pmatrix} = \begin{pmatrix} t_1 + \text{const}_1 \\ \vdots \\ t_n + \text{const}_n \end{pmatrix}.$$

We see that in the coordinate system $f_1, ..., f_n$ every solution of $(**)$ is a solution of $(*)$. Since the property to be a solutions of $(**)$ or a solution of $(*)$ does not depend on the coordinate

Recap of the lecture

- Nijenhuis operators have natural differential-geometric objects associated to them.
  - Conservation laws.
    - We described explicitly (the hierarchies) of conservation laws for for gl-regular Nijenhuis operators.
    - The "freedom" is n functions of one variable.
  - We considered a natural system of hydrodynamic type "$(*)$" associated to a Nijenhuis operator L. It generators A_i are given in terms of L, for example, $A_1 = L^{-1}\ \det(L)$. The initial data for this system is a choice of a curve $\hat{u}(x) = u(x, \underbrace{0, ..., 0}_{t})$.
  - For any hierarchy $f_1, ..., f_n$ of conservation laws we constructed a solution $u(x, t_1, ..., t_{n-1})$ of this system in terms of system of algebraic equations

$$f_i(u(t_1, \ldots, t_n)) = t_i + \mathrm{const}_i \qquad (**)$$

    - For almost every initial curve $\hat{u}$, one can construct the corresponding solution using this method.