

SUBMAXIMAL METRIC PROJECTIVE AND METRIC AFFINE STRUCTURES

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Dedicated to Mike Eastwood on the occasion of his 60th birthday

ABSTRACT. We prove that the next possible dimension after the maximal $n^2 + 2n$ for the Lie algebra of local projective symmetries of a metric on a manifold of dimension $n > 1$ is $n^2 - 3n + 5$ if the signature is Riemannian or $n = 2$, $n^2 - 3n + 6$ if the signature is Lorentzian and $n > 2$, and $n^2 - 3n + 8$ otherwise. We also prove that the Lie algebra of local affine symmetries of a metric has the same submaximal dimensions (after the maximal $n^2 + n$) unless the signature is Riemannian and $n = 3, 4$, in which case the submaximal dimension is $n^2 - 3n + 6$.

INTRODUCTION

Consider a linear torsion-free connection $\Gamma = (\Gamma_{jk}^i)$ on a smooth connected manifold M^n of dimension $n \geq 2$. A vector field v is called a *projective symmetry*, or a *projective vector field*, if its local flow sends geodesics (considered as unparameterized curves) to geodesics. Since S.Lie [L] it is known that projective vector fields form a Lie algebra, which we denote by $\mathfrak{p}(\Gamma)$. A vector field v is called an *affine symmetry*, or an *affine vector field*, if its local flow preserves Γ ; affine vector fields also form a Lie algebra $\mathfrak{a}(\Gamma)$, which we call *affine algebra*. Obviously $\mathfrak{a}(\Gamma) \subseteq \mathfrak{p}(\Gamma)$ is a subalgebra.

It follows from E.Cartan [C] that $\dim(\mathfrak{p}(\Gamma)) \leq n^2 + 2n$ and a connection with the maximal dimension of the projective algebra is *projectively flat*, i.e. in a certain local coordinate system the geodesics are straight lines. I.Egorov [E₁] proved that the next possible dimension of $\mathfrak{p}(\Gamma)$, the so-called submaximal dimension (this is the maximal dimension among all non-flat structures), is $n^2 - 2n + 5$ for $n > 2$. For $n = 2$, it was known since S. Lie [L] and A.Tresse [T] that the submaximal dimension is 3.

However, for $n > 2$, the projective structures realizing this dimension are non-metric, in the sense there exists no (local) metric such that its Levi-Civita connection has $\dim(\mathfrak{p}) = n^2 - 2n + 5$. This observation follows for example from [EM, (3.5)], which can be viewed as a system

of linear equations on the components of the metric g , whose coefficients come from the components of the projective Weyl tensor W . By Egorov [E₁], in a certain coordinate system the connection with the submaximal dimension of the projective algebra has only two non-zero term $\Gamma_{23}^1 = \Gamma_{32}^1 = x_2$. Direct calculation shows that the only nonvanishing components of the projective Weyl tensor are $W_{232}^1 = 1 = -W_{322}^1$; substitution of this into [EM, (3.5)] yields a system of linear equations such that any solution g is a degenerate symmetric tensor.

Non-metrizability of the Egorov's submaximal projective structure was obtained independently (and by another method) by S. Casey and M. Dunajski. In fact, it follows instantly from our first main result (below δ_n^2 is the Kronecker symbol, i.e. 1 for $n = 2$ and 0 else):

Theorem 1. *Let Γ be the Levi-Civita connection of a metric g on M^n . Assume that Γ is not projectively flat at least at one point (i.e. g is not a metric of constant sectional curvature). Then the maximal possible dimension of the symmetry algebra $\mathfrak{p}(g) = \mathfrak{p}(\Gamma)$ is equal to*

- $n^2 - 3n + 5$, if g has Riemannian signature,¹
- $n^2 - 3n + 6 - \delta_n^2$, if g has Lorentzian signature,
- $n^2 - 3n + 8$, if g has the general signature.

The bound for the general signature was obtained by Mikes [Mi₁]. Our approach however differs from his.

Notice that in the global setting, i.e. if we replace the projective algebra by a projective group, the sub-maximal projective connection is metric. The reason is that there are locally-projectively-flat manifolds whose projective group actually has dimension $n^2 + n$ [Y] (this is the global submaximal bound).

If M is closed and the metric g is Riemannian of non-constant sectional curvature, then the sub-maximal bound is $\binom{n}{2} + 1$ for all $n \neq 4$; for $n = 4$ this dimension is $\binom{n}{2} + 2 = 8$. Indeed by [M₁, M₂], on closed Riemannian manifolds of nonconstant sectional curvature, the projective group acts by isometries, so the claim follows from [Yan, E₂, KN]. The corresponding models are precisely $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ for $n \neq 4$, possibly quotient by a finite group, and $\mathbb{C}P^2$ for $n = 4$.

The problem of determining the dimension gaps (lacunes in the terminology of the Russian geometry school) between the maximal and sub-maximal structures is classical, see the discussion in [K₂].

¹Within this paper we consider the metrics up to multiplication by ± 1 (since multiplication by a nonzero constant does not change the projective and affine algebras). In particular both signatures $(+, -, \dots, -)$ and $(-, +, \dots, +)$ are Lorentzian for us, and we view positively and negatively definite metrics as Riemannian.

Let us now discuss an analogous question for the affine algebra. The maximal dimension of the space of affine symmetries of an affine connection is classically known to be $n^2 + n$. The submaximal dimension is n^2 and this was also found by I.Egorov [E₃]. The corresponding connections are projectively flat, and for projectively non-flat connections the affine algebra has maximal dimension $n^2 - 2n + 5$ [E₁].

Again, all these submaximal connections are non-metric. Our second main result concerns submaximal dimensions of the affine algebras $\mathfrak{a}(\Gamma)$ of Levi-Civita connections Γ .

Theorem 2. *Non-flat metrics g on a manifold M^n have maximal dimension of the affine algebra $\mathfrak{a}(g) = \mathfrak{a}(\Gamma)$ equal to*

- $n^2 - 3n + 5 + \delta_n^3 + \delta_n^4$, if g has Riemannian signature,
- $n^2 - 3n + 6 - \delta_n^2$, if g has Lorentzian signature,
- $n^2 - 3n + 8$, if g has the general signature.

In the process of the proof of Theorems 1 and 2 we essentially describe all metrics for the submaximal dimension of the projective and affine algebras of a metric connection.

Throughout the paper, all our considerations are local (so isomorphisms to the models are understood either locally or up to a covering).

Let us finally specify the gap Δ_1^p between the maximal dimension of the projective algebra and the submaximal one, and the gap Δ_2^p between the submaximal projective and submaximal metric projective dimensions (of un-restricted signature):

n	2	3	4	5	6	7	8	9	...
Δ_1^p	5	7	11	15	19	23	27	31	...
Δ_2^p	0	2	1	2	3	4	5	6	...

For affine algebras the corresponding gaps are the following (with the obvious modification to define Δ_i^a)

n	2	3	4	5	6	7	8	9	...
Δ_1^a	2	3	4	5	6	7	8	9	...
Δ_2^a	1	3	4	7	10	13	16	19	...

1. DEGREE OF MOBILITY AND USEFUL ESTIMATES.

Two metrics g and \bar{g} on a manifold M are *projectively equivalent*, if any g -geodesic is a reparameterization of a \bar{g} -geodesic. This can be expressed [S₂] through the (1, 1)-tensor $a = \bar{g}^{-1}g \cdot \left| \frac{\det(\bar{g})}{\det(g)} \right|^{1/(n+1)}$, where \bar{g}^{-1} is the inverse of \bar{g} ($\bar{g}^{ik}\bar{g}_{kj} = \delta_j^i$), and $\bar{g}^{-1}g$ is the contraction ($\bar{g}^{ik}g_{kj}$): the metrics g and \bar{g} are geodesically equivalent if and only if

$$(n+1)a_{j,k}^i = a_{,s}^{is}g_{jk} + a_{j,s}^s\delta_k^i. \quad (1)$$

In the coordinate-free notation the above formula reads

$$(n+1)\nabla a = \operatorname{div}(g^{-1}a) \otimes g + \operatorname{div}(a) \otimes \operatorname{Id}.$$

Dimension $D(g)$ of the solution space $\operatorname{Sol}(1)$ of this linear PDE system on unknown a is called the *degree of mobility* of g .

Let us denote by $I(g)$ the algebra of infinitesimal isometries (Killing vector fields), by $H(g)$ the algebra of homotheties, and by $C(g)$ the algebra of the conformal vector fields. These are given by the equation $L_v g = \lambda \cdot g$ on the vector field v , where λ is respectively zero, a constant or an arbitrary function. Clearly $I(g) \subseteq H(g) \subseteq C(g)$.

In this paper we shall actively use the following two estimates due to [Mi₁, Theorem 1 and Theorem 2]:

$$\dim \mathfrak{p}(g) \leq \dim I(g) + D(g) \tag{2}$$

$$\dim \mathfrak{p}(g) \leq \dim H(g) + D(g) - 1. \tag{3}$$

Let us prove these estimates (our proof is different from that of Mikes and is much simpler). It is well known (see [M₂] and references therein) that $v \in \mathfrak{p}(g)$ iff

$$a = g^{-1}L_v g - \frac{1}{n+1} \operatorname{Trace}(g^{-1}L_v g) \cdot \operatorname{Id} \tag{4}$$

is a solution of (1). Denote by $\phi : \mathfrak{p}(g) \rightarrow \operatorname{Sol}(1)$ the linear map sending v to the right hand side of (4). Since $\operatorname{Ker}(\phi) = I(g)$, the rank theorem yields inequality (2).

In order to obtain the second estimate, we observe that (1) has the obvious one-dimensional subspace of constant solutions $\mathbb{R} \cdot \operatorname{Id}$. Let $\pi : \operatorname{Sol}(1) \rightarrow \operatorname{Sol}(1)/(\mathbb{R} \cdot \operatorname{Id})$ be the projection. Then the linear map $\pi \circ \phi : \mathfrak{p}(g) \rightarrow \operatorname{Sol}(1)/(\mathbb{R} \cdot \operatorname{Id})$ has kernel $H(g)$ and (3) follows.

Below we will use the following results on the degree of mobility. It is always is bounded so:

$$D(g) \leq \binom{n+2}{2},$$

the equality corresponds to the space of constant curvature [S₂]. The next biggest (submaximal) dimension in any signature is [Mi₂]:

$$D(g)_{\text{sub.max}} = \binom{n-1}{2} + 1. \tag{5}$$

Finally, let us recall from [KM, Lemma 6] that if the Weyl conformal curvature tensor of the metric g vanishes, but g is not of constant sectional curvature, then $D(g) \leq 2$.

2. RIEMANNIAN CASE, DIMENSION $n > 3$

In this section we assume that the metric is Riemannian, $n > 3$. Denote by S_c^n the space form of constant curvature c , i.e. the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ of radius $c > 0$, the Euclidean space \mathbb{R}^n for $c = 0$, or the hyperbolic space \mathbb{H}_c^n for $c < 0$ equipped with the standard metrics.

Riemannian manifolds with abundant isometries were studied (among others) by H.C.Wang [W], K.Yano [Yan], I.Egorov [E₂], S.Kobayashi and T.Nagano [KN]. According to the local versions of their results, the Riemannian metrics g of non-constant sectional curvature on M^n with $\dim I(g) \geq \binom{n-1}{2} + 3$, $n > 2$, are from the following list:

- (a) $\dim I(g) = \binom{n}{2} + 2 = 8$, $n = 4$. The corresponding g is a Kähler metric on a complex surface with constant nonzero holomorphic sectional curvature (e.g. Fubini-Study metric on $\mathbb{C}P^2$).
- (b) $\dim I(g) = \binom{n}{2} + 1$. The corresponding g is the standard metric on the product $M^n = \mathbb{R}^1 \times S_c^{n-1}$ ($c \neq 0$).
- (c₁) $\dim I(g) = \binom{n}{2}$. The corresponding $g = dt^2 + a(t)^2 ds_0^2$ is the warped product metric, where ds_0^2 is the standard metric on S_c^{n-1} and $a(t)$ is a generic function (such that g is not of constant sectional curvature and not as in (b)).
- (c₂) $\dim I(g) = \binom{n}{2}$, $n = 6$. M is a Kähler manifold of complex dimension 3 of constant nonzero holomorphic sectional curvature.
- (d₁) $\dim I(g) = \binom{n-1}{2} + 3$. The corresponding $M^n = S_c^2 \times S_{\bar{c}}^{n-2}$, and the constants c, \bar{c} are not simultaneously zero.
- (d₂) $\dim I(g) = \binom{n-1}{2} + 3$, $n = 8$. The corresponding M is a Kähler manifold of complex dimension 4 with constant nonzero holomorphic sectional curvature.

In all these cases, except possibly (c₁), $H(g) = I(g)$. Thus the submaximal dimension of the homothety algebra is $\dim H(g) = \binom{n}{2} + 1$ for $n \neq 4$. Consequently in the cases (b) and (c₁), where the metric g is conformally flat and so $D(g) \leq 2$, we obtain from (3):

$$\dim \mathfrak{p}(g) \leq \binom{n}{2} + 1 + 1 < n^2 - 3n + 5.$$

Consider the spaces (a,c₂,d₂) of nonzero constant holomorphic sectional curvature. Then $D(g) = 1$ (since it is always so for the irreducible symmetric spaces [S₁]) and we conclude similarly that $\dim \mathfrak{p}(g)$ is strictly less than the bound from Theorem 1.

For the case (d₁) we have, combining (3) and (5):

$$\dim \mathfrak{p}(g) \leq \binom{n-1}{2} + 3 + \binom{n-1}{2} = n^2 - 3n + 5.$$

Finally, if the metric has $\dim I(g) < \binom{n-1}{2} + 3$, then $\dim H(g) < \binom{n-1}{2} + 3$. Indeed, if $I(g)$ acts transitively, then $H(g) = I(g)$ unless the metric is flat everywhere². In the intransitive case, M is (locally) foliated by codimension 1 leaves, to which the Killing fields are tangent. On these leaves of dimension $(n-1)$ the metric is not of constant sectional curvature (otherwise we have the case (c)), and thus $\dim I(g) \leq \binom{n-2}{2} + 1$ and hence $\dim H(g) \leq \binom{n-2}{2} + 2$. Thus in this case $\dim \mathfrak{p}(g)$ is strictly less than the bound of Theorem 1.

It follows from the above estimates and the analysis of the obtained models that the equality for $n > 3$ is attained only in the following sub-case of (d₁):

$$M = S_c^2 \times \mathbb{R}^{n-2}, \quad c \neq 0$$

(in the case of the second factor S_c^{n-2} having curvature $\bar{c} \neq 0$ all projective transformations are isometries). The projective transformations of this M consist of 3-dimensional space of isometries of the first factor (that is $so(3)$ or $sl(2)$ for $c > 0$ or $c < 0$ respectively) plus arbitrary affine transformations $x \mapsto A \cdot x + b$ of the second.

3. LORENTZIAN SIGNATURE, DIMENSION $n > 3$

In this section we consider Lorentzian manifolds of dimension $n > 3$. Complete classification of such metrics with the largest dimensions of $I(g)$ is not known to us, so we approach the problem differently.

It is still true that for g of non-constant sectional curvature $\dim I(g)$ does not exceed $\binom{n}{2} + 1$ in the Lorentzian signature (in fact, the inequality fails only when $n = 4$ for the Riemannian and split signature [E₃]). If the metric g is conformally flat, but not of constant sectional curvature, then by (2) we get

$$\dim \mathfrak{p}(g) \leq \binom{n}{2} + 1 + 2 < n^2 - 3n + 6.$$

On the other hand, if (M, g) is not conformally flat, then by [DT, KT] $\dim H(g) \leq \dim C(g) \leq \binom{n-1}{2} + 4$ and so, using (3) and (5), we get

$$\dim \mathfrak{p}(g) \leq \binom{n-1}{2} + 4 + \binom{n-1}{2} = n^2 - 3n + 6.$$

This estimate is achieved on the Lorentzian pp-wave metric, trivially extended to dimension n , $M^n = M_{pp}^3(x, y, z) \times \mathbb{R}^{n-3}(u_4, \dots, u_n)$, see

²Indeed, if $\varphi^*g = \lambda \cdot g$ for a map $\varphi : M \rightarrow M$, then $\varphi^*\|R_g\|^2 = \lambda^{-2}\|R_g\|^2$. Thus either $\lambda = 1$ or $R_g(x) = 0$ for any fixed point x of φ . Given transitivity of $I(g)$, for any x there is a λ -homothety φ with $\varphi(x) = x$, whence the claim.

[KT]:

$$g = 2dx dy + z^2 dy^2 + dz^2 + \sum_{i=4}^n du_i^2.$$

This metric has $\frac{1}{2}(n^2 - 3n + 8)$ Killing vector fields

$$\partial_x, \partial_y, e^y(\partial_z - z\partial_x), e^{-y}(\partial_z + z\partial_x), \partial_{u_i}, u_i\partial_x - y\partial_{u_i}, u_i\partial_{u_j} - u_j\partial_{u_i},$$

1 pure homothety

$$2x\partial_x + z\partial_z + \sum_{i=4}^n u_i\partial_{u_i}$$

and $\frac{1}{2}(n^2 - 3n + 2)$ pure affine fields

$$y\partial_x, u_i\partial_x, u_i\partial_{u_j} + u_j\partial_{u_i}$$

(the latter is easy to check as the connection is rather simple: $\Gamma_{23}^1 = \Gamma_{32}^1 = 2z$, $\Gamma_{22}^3 = -z$ and the other Christoffel symbols are zero). There are no non-affine projective fields. Thus the totality of the linearly independent projective symmetries is $\dim \mathfrak{p}(g) = n^2 - 3n + 6$. This proves the claim for $n > 3$.

Let us note that the submaximal model described above is (visibly) not unique. We can take any metric from the list of Kručkovič [Kr] (see Section 5) that has 4 Killing fields and 1 essential homothety/affine field, and extend it trivially to n dimensions, achieving the same result: The new metric g will have $(n-2)$ -dimensional space of parallel vector fields, yielding $(n-1)(n-2)$ affine symmetries, to which we add the 4 fields coming from the 3D metric (3 Killing fields + 1 homothety, as one Killing field that is parallel was already counted). All maximal models are obtained in this way (this observation is based on the fact [FM, Theorem 5] that parallel (0,2) tensors are linear combinations of symmetric products of parallel vectors).

Finally we remark that $\mathfrak{p}(g) = \mathfrak{a}(g)$ for all Lorentzian metrics g of submaximal dimension of the projective algebra. Indeed, if there was an essentially projective symmetry, then by [FM, Theorem 3]

$$\dim \mathfrak{p}(g) \leq \dim I(g) + D(g) - 1. \quad (6)$$

According to [DT] for a conformally non-flat metric g $\dim C(g) \leq \binom{n-1}{2} + 4$. It also follows from the construction in [DT] that at any point $x \in M$ the grading 0 component $\mathfrak{s}_0(x)$ of the (associated graded to the naturally filtered) symmetry group \mathfrak{s} satisfies: $\mathfrak{so}(1, n-1) \not\subset \mathfrak{s}_0 \subset \mathfrak{co}(1, n-1)$. Thus $I(g) \neq C(g)$ ³ and so $\dim I(g) \leq \binom{n-1}{2} + 3$.

³Consider $v \in C(g)$ that satisfies $v_x = 0$, $d_x v \in \mathfrak{s}_0(x) \setminus \mathfrak{so}(1, n-1)$. Then $L_v g(x) = \lambda g_x$ for $\lambda \neq 1$, and so v is not an infinitesimal isometry.

Combining this estimate, (6) and (5) we obtain $\dim \mathfrak{p}(g) \leq n^2 - 3n + 5$, which contradicts submaximality of the projective algebra.

4. THE PROOF FOR THE GENERAL SIGNATURE

Consider now the metric of signature (p, q) , where both $p, q \geq 2$, so $n = p + q \geq 4$. If the metric g is conformally flat and not of constant sectional curvature, then by (2) we get

$$\dim \mathfrak{p}(g) \leq \binom{n}{2} + 2 + 2 < n^2 - 3n + 8.$$

Next, by the results of [KT], for non-conformally flat metric structure we have

$$\dim C(g) \leq \binom{n-1}{2} + 6.$$

and so, using $H(g) \subseteq C(g)$, by (2) and (5) we get

$$\dim \mathfrak{p}(g) \leq \binom{n-1}{2} + 6 + \binom{n-1}{2} = n^2 - 3n + 8.$$

The equality is attained on the metric of the split signature pp-waves trivially extended from 4 to n dimensions:

$$g_{pp} = dx dw + dy dz + y^2 dw^2 + \sum_{i=5}^n \epsilon_i du_i^2.$$

This metric has conformal Weyl curvature tensor $CW = (dy \wedge dw)^2$, and is Einstein (Ricci-flat). Moreover the projective symmetries coincide with its affine symmetries. We have $C(g_{pp}) = H(g_{pp})$, and the generators of this algebra were calculated in [KT]:

$$\begin{aligned} & \partial_x, \partial_z, \partial_w, \partial_y - 2yw \partial_x + w^2 \partial_z, y \partial_x - w \partial_z, \\ & (z + yw^2) \partial_x - w \partial_y - \frac{1}{3}w^3 \partial_z, x \partial_z - y \partial_w + \frac{2}{3}y^3 \partial_x, \\ & x \partial_x + y \partial_y - z \partial_z - w \partial_w, 2x \partial_x + y \partial_y + z \partial_z, \\ & \partial_{u_i}, \epsilon_i u_i \partial_{u_j} - \epsilon_j u_j \partial_{u_i}, 2\epsilon_i u_i \partial_z - y \partial_{u_i}, 2\epsilon_i u_i \partial_x - w \partial_{u_i}. \end{aligned}$$

In addition, g_{pp} has the following genuine affine symmetries (not homotheties)

$$\begin{aligned} & y \partial_z, w \partial_x, y \partial_x + w \partial_z, 2\epsilon_i u_i \partial_z + y \partial_{u_i}, \\ & 2\epsilon_i u_i \partial_x + w \partial_{u_i}, \epsilon_i u_i \partial_{u_j} + \epsilon_j u_j \partial_{u_i}. \end{aligned}$$

Thus the total number of linearly independent projective symmetries is $\dim \mathfrak{p}(g) = n^2 - 3n + 8$ as required in Theorem 1.

5. DIMENSION $n = 3$

Consider the special case $n = 3$, where g necessarily has Riemannian or Lorentzian signature.

In this dimension the Weyl conformal curvature vanishes identically, and so (even for non-conformally flat metrics) $D(g) \leq 2$, see [KM]. Since the submaximal $\dim I(g) \leq 4$ we get by (2): $\dim \mathfrak{p}(g) \leq 6$.

On the other hand, in the Riemannian case⁴ $H(g) \leq 4$, so (2) implies $\dim \mathfrak{p}(g) \leq 5$. Also if g is not conformally flat, then $\dim C(g) \leq 4$ by [KT]. Henceforth we get the bound $\dim \mathfrak{p}(g) \leq 5$ in this case too.

The local metrics in 3 dimensions with $\dim I(g) = 4$ were classified by G.Kručkovič [Kr]. There are 8 different cases. The first three are the Lorentzian metrics:

- (1) $g_1 = k dx^2 + 2(2 - c)e^{cx} dx dy + e^{2x} dz^2$, ($c \neq 2$)
- (2) $g_2 = k dx^2 + e^{2x}(2 dx dy - dz^2)$,
- (3) $g_3 = k dx^2 + e^{x\sqrt{4-\omega^2}}(2 dx dy - \frac{4}{\omega^2} \cos^2(\frac{\omega x}{2}) dz^2)$.

For all these metrics $\dim I(g) = 4$, $\dim H(g) = 5$, $D(g) = 2$, so both estimates (2) and (3) state $\dim \mathfrak{p}(g) \leq 6$.

And in fact, this bound is achieved for all 3 cases. The infinitesimal automorphisms can be shown explicitly. For instance, for the first metric g_1 ($c \neq 2$) the algebra of projective symmetries has generators

$$\partial_y, \partial_z, z \partial_y + e^{x(c-2)} \partial_z, \partial_x - cy \partial_y - z \partial_z, \left(2y + \frac{k e^{-cx}}{(c-2)c} \right) \partial_y + z \partial_z, e^{cx} \partial_y.$$

The first four are Killing fields, the fifth is a homothety and the last is a projective field for the metric g (in fact it is an affine field for g).

For $c = 0$ the homothety has to be changed to $(2y + \frac{k}{2}x) \partial_y + z \partial_z$, and the genuine projective field has to be changed to $2y \partial_y + z \partial_z$.

These formulae were obtained using the *DifferentialGeometry* package of MAPLE, and can be easily verified by hand. The formulae for g_2 and g_3 are obtained similarly.

The other metrics (4)-(8) in [Kr] can be of both possible signatures.

The metrics of cases (4), (5) and (6) in the list of Kručkovič are not conformally flat, and for them direct calculation yields $H(g) = I(g)$, $D(g) = 1$. Thus by (3) we get $4 \leq \dim \mathfrak{p}(g) \leq 4$, i.e. all projective transformations in these cases are isometries, and the metrics are not of submaximal projective symmetry.

The metrics in cases (7) and (8) of loc.cit. are conformally flat, but for them $H(g) = I(g)$, $D(g) = 2$, whence $\dim \mathfrak{p}(g) \leq 5$. In fact, in these cases the manifold is locally $M^3 = \mathbb{R}^1 \times S_c^2$, where $c \neq 0$. There is one

⁴Recall from Section 2 that $H(g) = I(g)$ if $I(g)$ is transitive. It is easy to check that $\dim I(g) \leq 3$ if $I(g)$ is intransitive.

genuine affine symmetry (scaling along \mathbb{R}^1), so we have $\dim \mathfrak{p}(g) = 5$. Again for the Lorentzian signature these M^3 are not submaximal, but for Riemannian signature they are. This finishes investigation of the 3-dimensional case.

6. DIMENSION $n = 2$

Dimension 2 is another exception to the above arguments. Again here the problem is classical: the 2D projective structures were studied by S. Lie [L], R. Liouville [Li] and A. Tresse [T]. In the latter reference it was proven that the submaximal dimension of the symmetry algebra is 3 and the submaximal projective structures were classified.

The two projective structures arising in this way (see also [Ma]), when written as the 2nd order ODE on the plane, are

$$x y'' = \epsilon (y')^3 - \frac{1}{2} y' \quad (\epsilon = \pm 1). \quad (7)$$

It is easy to check they are metrizable; the corresponding metrics are

$$g = x dx^2 - 2 \epsilon x dy^2. \quad (8)$$

The projective symmetry algebra is $sl(2)$ realized on the plane $\mathbb{R}^2(x, y)$ via the vector fields $\partial_y, x \partial_x + y \partial_y, 2xy \partial_x + y^2 \partial_y$.

Another form of (7) was considered in [BMM]: $y'' = \epsilon e^{-2x} (y')^3 + \frac{1}{2} y'$. It is obtained from (7) by the transformation $x \mapsto -\epsilon e^x$.

Yet in [K₁] this projective connection was written differently:

$$y'' = \epsilon (xy' - y)^3.$$

The symmetry algebra $sl(2)$ in this realization has the standard linear representation on $\mathbb{R}^2(x, y)$: $x \partial_x - y \partial_y, x \partial_y, y \partial_x$. A metric corresponding to this projective connection is

$$g = \left(\frac{dx}{y^2} - \frac{x dy}{y^3} \right)^2 - \epsilon \frac{dy^2}{y^8}.$$

The complete list of the corresponding metrics is contained in [BMM].

This finishes the proof of Theorem 1.

7. AFFINE SYMMETRIES OF A METRIC

In this Section we prove Theorem 2, estimating $\mathfrak{a}(g) \subseteq \mathfrak{p}(g)$.

Recall that (for any $n \geq 2$) if the Levi-Civita connection Γ of the metric g is projectively flat, then g has constant curvature. This is a variant of Beltrami's theorem, see [EM]. Thus if g is not of constant sectional curvature, we get the bound

$$\dim \mathfrak{a}(g) \leq \dim \mathfrak{p}(g) \leq n^2 - 3n + \sigma, \quad (9)$$

where $\sigma = 5, 6$ or 8 is the same number, depending on the signature of g and dimension n of M , as in Theorem 1.

We have to show that this bound is achieved within the class of metrics of nonconstant sectional curvature. But indeed, the submaximal projective symmetry algebras of the models, studied in the preceding sections consisted of affine fields only, provided $n > 2$. Thus we conclude that the bound is actually sharp in the considered class.

If g has nonzero constant sectional curvature (in this case $\dim \mathfrak{a}(g)$ is also less than the maximal value $n^2 + n$), then $\mathfrak{a}(g) = I(g)$ and we get

$$\dim \mathfrak{a}(g) = \binom{n+1}{2}. \quad (10)$$

For $n \geq 5$ and for $n = 2$ this does not exceed the first bound (and is strictly less than it for $n > 5$) in the Riemannian case, and it never exceeds it (for $n > 4$, is strictly less than it) for the Lorentzian signature. For the Riemannian signature and $n = 3, 4$ the bound in (9) exceeds the bound in (10) by 1. Thus in this case the constant sectional curvature spaces give the submaximal dimension of $\mathfrak{p}(g)$.

In the general signature (p, q) , $p, q \geq 2$, $n = p + q \geq 4$, the bound of (10) is strictly less than that of (9), thus nothing new is added here.

Dimension 2 is again special. If the affine connection is not flat, then $\mathfrak{a}(g) = H(g)$. Indeed, since the curvature is nonzero, the holonomy algebra is full. The maximal dimension of $H(g)$ in the case of non-constant scalar curvature is 2 (this is achieved, for example, in the case of metric (8) via its first two symmetries). Consequently the submaximal dimension of $\mathfrak{a}(g)$ is 3 and is achieved for the round sphere, the Lobachevsky plane or de Sitter metric. Herewith the claim is proved.

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