

Two-dimensional metrics admitting precisely one projective vector field*

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Abstract

We give a complete list of 2-dimensional metrics that admit an essential projective vector field. This solves a problem explicitly posed by Sophus Lie in 1882.

1 Introduction

1.1 Main definitions and results

Let g be a smooth Riemannian or pseudo-Riemannian metric on a 2-dimensional disc D^2 .

Definition 1. A vector field v is called *projective*, if its flow takes (unparameterized) geodesics to geodesics.

As Lie showed [22], the set of vector fields projective with respect to a given g forms a Lie algebra (for our paper it is sufficient that it is a vector space). We will denote this Lie algebra by $\mathfrak{p}(g)$.

The following two problems were posed by Sophus Lie¹ in 1882:

Problem 1 (Lie). *Find all metrics g such that $\dim(\mathfrak{p}(g)) = 1$.*

Problem 2 (Lie). *Find all metrics g such that $\dim(\mathfrak{p}(g)) \geq 2$.*

The second problem of Lie was completely solved in [9]. The present paper gives a solution of the first problem of Lie. The reader should consult [9, 10] for the history of the question, for the connection with the results of Aminova [1, 2], and for the description of the circle of ideas, though we recall some of them in §2.1.

The biggest family of metrics admitting projective vector field consists of metrics admitting infinitesimal homotheties. Recall that a vector field v is a *infinitesimal homothety* for a metric g , if $L_v g = \lambda g$ for a certain constant $\lambda \in \mathbb{R}$, where L_v denotes the Lie derivative. In this definition, we allow $\lambda = 0$, so that Killing vector fields are also infinitesimal homotheties.

This “biggest” family of metrics is very well understood: it is well known and it was explicitly mentioned by Lie in the paper [22], that in the coordinates (x, y) such that $v = \frac{\partial}{\partial x}$ such metric g is given by the matrix $e^{\lambda x} \begin{pmatrix} E(y) & F(y) \\ F(y) & G(y) \end{pmatrix}$, where E, F, G are functions of y only.

Thus, the first Lie Problem as Sophus Lie himself understood it is to find all g admitting no infinitesimal homotheties, but such that $\dim(\mathfrak{p}(g)) = 1$. From the paper [22] it is clear that Lie considered this problem only locally, in small neighborhood of a generic point.

The next three theorems solve the Problem 1 (of Lie) above.

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Problem I: *Es wird verlangt, die Form des Bogenelementes einer jeden Fläche zu bestimmen, deren geodätische Kurven eine infinitesimale Transformation gestatten.*

Problem II: *Man soll die Form des Bogenelementes einer jeden Fläche bestimmen, deren geodätische Kurven mehrere infinitesimale Transformationen gestatten.*

Definition 2. Two metrics g and \bar{g} on D^2 are called *projectively equivalent*, if they have the same geodesics considered as unparameterized curves.

Theorem 1. *Assume the metric \check{g} on D^2 admits a projective vector field v . Assume in addition that for any open $U \subset D^2$ the restriction of \check{g} to U admits no infinitesimal homothety.*

Then, in a neighborhood of almost every point there exists a coordinate system (x, y) (in certain cases we consider the correspondent complex coordinates $(z = x + i \cdot y, \bar{z} = x - i \cdot y)$), such that in this neighborhood the vector field v and a certain metric g projectively equivalent to \check{g} are given by the formulas below.

1. **(Liouville Case)** $ds_g^2 = (X(x) - Y(y))(X_1(x)dx^2 + Y_1(y)dy^2)$, $v = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, where

$$(a) \quad X(x) = \frac{1}{x}, \quad Y(y) = \frac{1}{y}, \quad X_1(x) = c \cdot \frac{e^{-3x}}{x}, \quad Y_1(y) = \frac{e^{-3y}}{y}.$$

$$(b) \quad X(x) = \tan(x), \quad Y(y) = \tan(y), \quad X_1(x) = c \cdot \frac{e^{-3\lambda x}}{\cos(x)}, \quad Y_1(y) = \frac{e^{-3\lambda y}}{\cos(y)}.$$

$$(c) \quad X(x) = c \cdot e^{\nu x}, \quad Y(y) = e^{\nu y}, \quad X_1(x) = e^{2x}, \quad Y_1(y) = \varepsilon e^{2y}.$$

2. **(Complex-Liouville Case)** $ds_g^2 = (h(z) - \overline{h(z)})(h_1(z)dz^2 - \overline{h_1(z)}d\bar{z}^2)$, $v = \frac{\partial}{\partial x}$ ($= \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$), where

$$(a) \quad h(z) = \frac{1}{z}, \quad h_1(z) = C \cdot \frac{e^{-3z}}{z}.$$

$$(b) \quad h(z) = \tan(z), \quad h_1(z) = C \cdot \frac{e^{-3\lambda z}}{\cos(z)}.$$

$$(c) \quad h(z) = C \cdot e^{\nu z}, \quad h_1(z) = e^{2z}.$$

3. **(Jordan-block Case)** $ds_g^2 = (Y(y) + x)dxdy$, $v = v_1(x, y)\frac{\partial}{\partial x} + v_2(y)\frac{\partial}{\partial y}$, where

$$(a) \quad Y = e^{\frac{3}{2y}} \cdot \frac{\sqrt{|y|}}{y-3} + \int_{y_0}^y e^{\frac{3}{2\xi}} \cdot \frac{\sqrt{|\xi|}}{(\xi-3)^2} d\xi, \\ v_1 = \frac{y-3}{2} \left(x + \int_{y_0}^y e^{\frac{3}{2\xi}} \cdot \frac{\sqrt{|\xi|}}{(\xi-3)^2} d\xi \right), \quad v_2 = y^2.$$

$$(b) \quad Y = e^{-\frac{3}{2}\lambda \arctan(y)} \cdot \frac{\sqrt[4]{y^2+1}}{y-3\lambda} + \int_{y_0}^y e^{-\frac{3}{2}\lambda \arctan(\xi)} \cdot \frac{\sqrt[4]{\xi^2+1}}{(\xi-3\lambda)^2} d\xi, \\ v_1 = \frac{y-3\lambda}{2} \left(x + \int_{y_0}^y e^{-\frac{3}{2}\lambda \arctan(\xi)} \cdot \frac{\sqrt[4]{\xi^2+1}}{(\xi-3\lambda)^2} d\xi \right), \quad v_2 = y^2 + 1.$$

$$(c) \quad Y(y) = y^{\frac{1}{\eta}}, \quad v_1(x, y) = x, \quad v_2 = \eta y,$$

$$(d) \quad Y(y) = y^2, \quad v_1(x, y) = 2x, \quad v_2 = y,$$

where $c \in \mathbb{R} \setminus \{0\}$, $y_0 \in \mathbb{R}$, $\lambda \in \mathbb{R}$, $\nu, \eta \in (0, 4]$, $\nu \neq 1$, $\eta \notin \{\frac{1}{2}, 1\}$, $C \in \mathbb{C}$, $|C| = 1$, $\varepsilon \in \{-1, 1\}$ are constants, and \bar{h} and $\overline{h_1}$ denote the complex-conjugate to h and h_1 .

Moreover, in the case 1b, if $\lambda = 0$, then $c \neq \pm 1$. In the case 2b, if $\lambda = 0$, then $C \neq \pm 1$. In the case 1c, if $\nu = 2$, then $c \neq -\varepsilon$. In the case 2c, if $\nu = 2$, then $C \neq \pm 1$.

Remark 1. We do not claim in Theorem 1 that all metrics projectively equivalent to g admit no infinitesimal homotheties. In view of Theorem 3, it is easy to understand whether a metric \bar{g} projectively equivalent to g from Theorem 1 admits an infinitesimal homothety: indeed, by Theorem 3, the metrics from Theorem 1 have unique (up to multiplication by a constant) projective field field. Thus, it is sufficiently to check whether v is an infinitesimal homothety.

Moreover, from the proof of Theorems 1, 2 it will be clear that for every metric g from Theorem 1 the set of metrics projectively equivalent to g and admitting an infinitesimal homothety is very small (has dimension at most 1 in the two- or three-dimensional space of metrics projectively equivalent to g .)

But certain metrics projectively equivalent to g may admit infinitesimal homotheties. For example, in the case 1c, the vector field v is already an infinitesimal homothety for g .

Clearly, projective equivalence is a symmetric, reflexive and transitive relation on the space of all metrics on $U \subseteq D^2$, i.e., it is an equivalence relation. The equivalence class of a metric g with respect to projective equivalence will be called the *projective class* of a metric (we give an equivalent analytic definition in §2.1), it contains all metrics projectively equivalent to g . Clearly, if v is a projective vector field for a metric from a projective class, it is a projective vector field for every metric from the projective class. Theorem 1 describes (in a neighborhood of almost every point) all projective classes admitting essential projective vector fields. The next theorem describes all metrics of such projective classes.

For two metrics (three metrics, respectively) g and \bar{g} on $U \subseteq D^2$ (g , \bar{g} , and \tilde{g} , respectively) and for $\alpha, \beta \in \mathbb{R}$ ($\alpha, \beta, \gamma \in \mathbb{R}$, respectively) such that the formula (1) ((2), respectively) has sense, let us denote by $\hat{g}[g, \bar{g}, \alpha, \beta]$ ($\hat{g}[g, \bar{g}, \tilde{g}, \alpha, \beta, \gamma]$, respectively) the metric (1) ((2), respectively):

$$\hat{g}[g, \bar{g}, \alpha, \beta] := \frac{\alpha \cdot g / (\det(g))^{2/3} + \beta \cdot \bar{g} / (\det(\bar{g}))^{2/3}}{(\det(\alpha \cdot g / (\det(g))^{2/3} + \beta \cdot \bar{g} / (\det(\bar{g}))^{2/3}))^2} \quad (1)$$

$$\hat{g}[g, \bar{g}, \tilde{g}, \alpha, \beta, \gamma] := \frac{\alpha \cdot g / (\det(g))^{2/3} + \beta \cdot \bar{g} / (\det(\bar{g}))^{2/3} + \gamma \cdot \tilde{g} / (\det(\tilde{g}))^{2/3}}{(\det(\alpha \cdot g / (\det(g))^{2/3} + \beta \cdot \bar{g} / (\det(\bar{g}))^{2/3} + \gamma \cdot \tilde{g} / (\det(\tilde{g}))^{2/3}))^2} \quad (2)$$

In these formulas, g , \bar{g} , and \tilde{g} should be understood as (2×2) -matrices of metrics in a local coordinate system. In §2.1 and §4.1, we will explain the geometry and the hidden linear structure staying behind this formula. In particular, it will be clear that the formula is independent of the coordinate system (though one can check it “by hands”). Moreover, if the metrics g and \bar{g} (g , \bar{g} , and \tilde{g} , respectively) are projectively equivalent, then $\hat{g}[g, \bar{g}, \alpha, \beta]$ ($\hat{g}[g, \bar{g}, \tilde{g}, \alpha, \beta, \gamma]$, respectively) is also projectively equivalent to g . Of course, the metrics $\hat{g}[g, \bar{g}, \alpha, \beta]$ ($\hat{g}[g, \bar{g}, \tilde{g}, \alpha, \beta, \gamma]$, respectively) are defined only for $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\det(\alpha \cdot g / (\det(g))^{2/3} + \beta \cdot \bar{g} / (\det(\bar{g}))^{2/3}) \neq 0$ ($\det(\alpha \cdot g / (\det(g))^{2/3} + \beta \cdot \bar{g} / (\det(\bar{g}))^{2/3} + \gamma \cdot \tilde{g} / (\det(\tilde{g}))^{2/3}) \neq 0$, respectively).

Denote by $G(g, \bar{g})$ ($G(g, \bar{g}, \tilde{g})$, respectively) the following set of metrics:

$$G(g, \bar{g}) := \{ \hat{g}[g, \bar{g}, \alpha, \beta] \mid \alpha, \beta \in \mathbb{R} \text{ such that } \hat{g}[g, \bar{g}, \alpha, \beta] \text{ is defined} \}. \quad (3)$$

$$G(g, \bar{g}, \tilde{g}) := \{ \hat{g}[g, \bar{g}, \tilde{g}, \alpha, \beta, \gamma] \mid \alpha, \beta, \gamma \in \mathbb{R} \text{ such that } \hat{g}[g, \bar{g}, \tilde{g}, \alpha, \beta, \gamma] \text{ is defined} \}. \quad (4)$$

As we explained above, if the metrics g and \bar{g} (g , \bar{g} , and \tilde{g} , respectively) are projectively equivalent, then $G(g, \bar{g})$ ($G(g, \bar{g}, \tilde{g})$, respectively) is a subset of their projective class.

Now, every metric g from Theorem 1 always admits a nontrivial projectively equivalent metric: as we explain in Appendix, for arbitrary data $X(x)$, $X_1(x)$, $Y(y)$, $Y_1(y)$, $h(z)$, $h_1(z)$,

- the metric g from the Liouville Case of Theorem 1 is projectively equivalent to the metric

$$ds_{\bar{g}} = \left(\frac{1}{X(x)} - \frac{1}{Y(y)} \right) \left(\frac{X_1(x)}{X(x)} dx^2 + \frac{Y_1(y)}{Y(y)} dy^2 \right), \quad (5)$$

- the metric g from the Complex-Liouville Case of Theorem 1 is projectively equivalent to the metric

$$ds_{\bar{g}} = \left(\frac{1}{\overline{h(z)}} - \frac{1}{h(z)} \right) \left(\frac{\overline{h_1(z)}}{\overline{h(z)}} d\bar{z}^2 - \frac{h_1(z)}{h(z)} dz^2 \right), \quad (6)$$

- the metric g from the Jordan-block-Case of Theorem 1 is projectively equivalent to the metric

$$ds_{\bar{g}} = -\frac{2(Y(y) + x)}{y^3} dx dy + \frac{(Y(y) + x)^2}{y^4} dy^2. \quad (7)$$

Such metric \bar{g} projectively equivalent to g will be called *canonical projectively equivalent metric*².

Moreover, the metric g from the case 3d of Theorem 1 admits one more metric projectively equivalent to it and essentially different from the canonically projectively equivalent metric given by (7), namely \tilde{g} given by

$$ds_{\tilde{g}}^2 = \frac{9 dx^2}{(y^2 + x)^2 (3x - y^2)^6} - 4 \frac{y (9x + y^2) dx dy}{(3x - y^2)^6 (y^2 + x)^3} + \frac{12x dy^2}{(y^2 + x)^2 (3x - y^2)^6} \quad (8)$$

Theorem 2. *The projective class of every metric g from cases 1a–3c of Theorem 1 coincides with $G(g, \bar{g})$, where \bar{g} is the canonical projectively equivalent metric. The projective class of the metric g from the case 3d of Theorem 1 coincides with $G(g, \bar{g}, \tilde{g})$, where \bar{g} is the canonical projectively equivalent metric, and \tilde{g} is the metric given by (8).*

We see that Theorem 1 describes all projective classes admitting essential projective vector field, and Theorem 2 describes all metrics of these projective classes. Theorem 3 below implies that all these metrics actually have $\dim(\mathfrak{p}) = 1$, because by [9] a metric admitting two independent projective vector fields admits a Killing vector field.

Theorem 3. *None of the metrics from Theorem 1 admits a nontrivial Killing vector field.*

Theorems 1, 2, 3 give a complete solution of the Problem 1 (of Lie) above.

1.2 New ideas compared with [9]

The theory of projective transformations and projectively equivalent metrics has a long and fascinating history. First non-trivial examples of projectively equivalent metrics and projective transformations were discovered by Lagrange [21] and Beltrami [4]. Recently, there has been a considerable growth in interest in projective differential geometry, due to new methods that allow one to solve interesting new and classical problems, see for example [7, 16, 28, 29, 34, 35].

The present paper also solves an interesting classical problem explicitly stated by Sophus Lie. In a certain sense, this paper is a continuation of [9], where another problem stated by Sophus Lie (Problem 2 from §1.1) was solved; let us recall the main idea of [9] and comment in brief on new (with respect to [9]) ideas of the present paper.

Let S be a *projective structure* on a surface, i.e., equivalence class of torsion-free affine connections with the same unparameterized geodesics (in §2.1 we will explain that in a coordinate system projective structures are parametrized by four functions). Certain projective structures contain a Levi-Civita connection of a metric, in this case we say that the metric is *compatible* with the projective structure, and the projective structure is *metrizable*.

Projective structures with projective vector fields are easy to classify: in particular, projective structures admitting two projective vector fields were essentially described by Lie himself [22] and Tresse [44]. In order to solve Problem 2 of Lie, one needs to understand which projective structures admitting two projective vector fields are metrizable.

By an old and now well-understood observation of R. Liouville [23], metrics compatible with a projective structure can be found as nondegenerate solutions of an overdetermined system of linear partial differential equation, whose coefficients are given by the projective structure. There exists an algorithmic way (sometimes called prolongation-projection method) how to understand whether an overdetermined system of linear partial differential equation has a nontrivial solution. In [9], the (shortcut of this) algorithm was applied and all metrics whose projective structures admit two projective vector fields were described. Moreover, recently the algorithmic way how to understand whether a given projective structure is metrizable was essentially simplified in [10].

Unfortunately, it is hard to apply this method to find all metric admitting one projective vector field. Indeed, projective structures admitting one projective vector field depend explicitly on arbitrary functions of one variable. Prolongation-projection method (or the results of [10]) applied to such projective structures results in a completely intractable nonlinear system of ODEs, yielding no insight.

²The notion is not coordinate independent and has sense only if the metric has the form as in Theorem 1

In order to solve Problem 1 of Lie we used another method. We used that the system of PDE defining compatible metrics is projectively invariant, hence the solution space is invariant under Lie derivative by a projective vector field. If a metric admits an essential projective vector field, the family of compatible metrics is at least two-dimensional. The question of finding projective structures with a 2-dimensional family of compatible Riemannian metrics was posed by Beltrami [4] and solved by Dini [12]. The solution depends explicitly on two arbitrary functions, each of one variable. However, we are interested in all signatures: even if the original metric g is Riemannian, other metrics in the family need not be. In signature $(+,-)$, there are more solutions: in addition to straightforward an analogue of the Riemannian solution, there is a “complex” form and a “degenerate” case. Although this may have been known to Darboux and other authors, Appendix of the paper provides a straightforward and self-contained description of pairs of projectively equivalent metrics.

Returning to the main thread of the paper, the strategy now is to analyse the linear action of the Lie derivative along a projective vector field on the space of solutions to the equation for compatible metrics. In the nontrivial case, this action turns out to be invertible with a 2-dimensional invariant subspace, see §2.2. The form of the metrics corresponding to this subspace is thus given by one of the three cases in Appendix. There are also three possibilities for the linear action: it may have a single 1-dimensional eigenspace, complex conjugate eigenvalues or two real eigenvalues. Consequently there are nine separate cases to consider, each of which gives rise to a system of six PDEs for the at most two unknown functions (of one variable) in the metrics and two unknown components of the projective vector field. The Leibniz rule for the Lie derivative implies that the system is linear in the six first derivatives of these functions. It turns out that the system is solvable for these first derivatives (with independent variables not appearing explicitly). This leads to their explicit integration in each case. The result is the explicit classification of metrizable projective structures admitting at least one projective vector field and compatible with at least two nonproportional metrics, i.e., essentially, Theorem 1.

However, this is not yet an explicit classification of metrics with a nontrivial vector field, as an explicit form has only been found for one metric in the same projective class. The classification is completed by Theorem 2, in which all metrics projectively equivalent to those in Theorem 1 are described, and Theorem 3, which shows that the dimension of the space of projective vector fields is exactly 1. The proofs of Theorems 2, 3 is standard (though quite nontrivial technically). In order to prove Theorem 2, we apply the (adapted version of the) prolongation-projection method to show that no other solution exists. In order to prove Theorem 3, we use a certain trick known to Darboux and Eisenhart, see Section 5 for more details.

2 Schema of the proof of Theorem 1

Roughly speaking, we reformulate our problem as 9 systems of PDE and solve them. In this section we will explain how we do it. More precisely,

- in §2.1, we review the theoretic results we will use.
- In §2.2, we prove two additional (relatively simple) results.
- In §2.3, we explain the main trick that allowed us to reduce our problem to 9 systems of PDE which are relatively easy and can be solved explicitly, probably after an appropriate coordinate change. We will also explain in what sense the systems are easy.

In Section 3, we solve these 9 systems of PDE.

2.1 General theory

can be found in [9, 10, 24, 25, 26, 27, 42] and in more classical sources which in particular are acknowledged in [9]. The present paper should be viewed as a continuation of [9], it could be useful for a reader to have [9] at hand while reading the present paper.

We will work on a small disc D^2 in local coordinates (x, y) .

Definition 3. *Projective connection* is a second order ordinary differential equation of the form

$$y'' = K_0(x, y) + K_1(x, y) y' + K_2(x, y) (y')^2 + K_3(x, y) (y')^3, \quad (9)$$

where the functions $K_i : D^2 \rightarrow \mathbb{R}$.

For any symmetric affine connection $\Gamma = (\Gamma_{jk}^i(x, y))$, the projective connection associated to Γ is

$$y'' = -\Gamma_{11}^2 + (\Gamma_{11}^1 - 2\Gamma_{12}^2) y' - (\Gamma_{22}^2 - 2\Gamma_{12}^1) (y')^2 + \Gamma_{22}^1 (y')^3. \quad (10)$$

We say that a metric g belongs to the *projective class of the projective connection* (9), if the projective connection (9) is associated to the Levi-Civita connection of g .

As it has been known since the time of Beltrami [4], the projective connection associated to Γ carries all information about unparameterized geodesic of Γ . More precisely, for every solution $y(x)$ of (10), the curve $(x, y(x))$ is, up to reparametrization, a geodesic of Γ . In particular, two metrics are projectively equivalent, if and only if they belong to the projective class of the same projective connection. Therefore, according to the definition in §1.1, the projective class of g is the projective class of the projective connection associated to the Levi-Civita connection of g .

Let us reformulate (following [23, 9]) the condition

“the metric $E(x, y) dx^2 + 2F(x, y) dx dy + G(x, y) dy^2$ belongs to the projective class of (9)” as a system of PDE on E, F, G .

Consider the symmetric nondegenerate matrix

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} := \det(g)^{-2/3} \cdot g = \frac{1}{(EG - F^2)^{2/3}} \begin{pmatrix} E & F \\ F & G \end{pmatrix}. \quad (11)$$

Lemma 1 ([23, 9]). *The projective connection associated to the Levi-Civita connection of the metric g is (9) if and only if the entries of the matrix $a = \det(g)^{-2/3} \cdot g$ satisfy the linear PDE system*

$$\left. \begin{aligned} a_{11x} - \frac{2}{3} K_1 a_{11} + 2 K_0 a_{12} &= 0 \\ a_{11y} + 2 a_{12x} - \frac{4}{3} K_2 a_{11} + \frac{2}{3} K_1 a_{12} + 2 K_0 a_{22} &= 0 \\ 2 a_{12y} + a_{22x} - 2 K_3 a_{11} - \frac{2}{3} K_2 a_{12} + \frac{4}{3} K_1 a_{22} &= 0 \\ a_{22y} - 2 K_3 a_{12} + \frac{2}{3} K_2 a_{22} &= 0 \end{aligned} \right\} \quad (12)$$

In the formula (11) above, a should be understood as a section of

$$S_2D \otimes (\Lambda_2D)^{-\frac{4}{3}}, \quad (13)$$

where Λ_2 is the one-dimensional bundle of volume forms. Indeed, after a coordinate change the transformation rule of an element of (13) and of $\det(g)^{-2/3} \cdot g$ coincide.

In particular, the Lie derivative of $a = \det(g)^{-2/3} \cdot g$ is well defined (as a mapping from the sections of $S_2D \otimes (\Lambda_2D)^{-\frac{4}{3}}$ to itself), is independent on the coordinate system, and is given by the formula

$$L_v a = L_v \left(\det(g)^{-2/3} \cdot g \right) = \det(g)^{-2/3} \cdot L_v g - \frac{2}{3} \det(g)^{-2/3} \text{trace}_g(L_v g) \cdot g, \quad (14)$$

where $\text{trace}_g(L_v g) := \sum_{i,j} (L_v g)_{ij} g^{ij}$.

Remark 2. The formula (11) is invertible: $a = g / \det(g)^{2/3}$ if and only iff $g = a / \det(a)^2$. The mapping $a \mapsto a / \det(a)^2$ can be viewed as a mapping from $S_2D \otimes (\Lambda_2D)^{-\frac{4}{3}}$ to S_2D , which is defined for nondegenerate points of $S_2D \otimes (\Lambda_2D)^{-\frac{4}{3}}$ only, and sends them into nondegenerate points of S_2D . In particular, if a nondegenerate a is a solution of (12), then the metric $g = a / \det(a)^2$ belongs to the projective class of (9).

The system (12) has the following nice properties, which will be used later:

- It is linear and of finite type. In particular, its space of solutions (that will be denoted by \mathcal{A}) is a finite-dimensional ($\dim(\mathcal{A}) \leq 6$, [23]) vector space.
- Moreover, if $\dim(\mathcal{A}) \geq 4$, then every metric from the projective class admits a Killing vector field [18].
- The system (12) depends on the projective connection only and is therefore projectively invariant. In particular, for every projective vector field v and for every solution $a \in \mathcal{A}$ we have $L_v a \in \mathcal{A}$. Thus, L_v is a (linear) mapping from \mathcal{A} to itself.

We will also use the following two statements: first is due to Knebelman [17], another proof could also be found in [9, 20, 24, 42, 33, 5], one more proof easily follows from the theory of invariant operators, see for example [3]. The second is combination of the formula (11) and the connection between projectively equivalent metrics and integrable systems due to [25, 24], see also Darboux [11, §608], see also of [9, §2.4].

- If a metric g admits a Killing vector field, then every metric projectively equivalent to g also admits a Killing vector field.
- a is a solution of the system (12) corresponding to (the projective connection associated to the Levi-Civita connection of) a metric g , if and only if the function

$$I : TD^2 \rightarrow \mathbb{R}, \quad I(\xi) = \det(g)^{2/3} \cdot \sum_{i,j} a_{ij} \xi^i \xi^j \quad (15)$$

is an integral for the geodesic flow of g .

Remark 3.

1. Tensor products with powers of $(\Lambda_2 D)^{\frac{1}{3}}$ appear naturally in the theory of projectively equivalent metrics and projective connections, see [13].
2. A multidimensional generalization of the formula $a := \det(g)^{-2/3} g$ and of Lemma 1 can be found in [14], see also [2, 6, 37, 41].
3. The formula (14) appears naturally in investigation of projective transformations of surfaces, see [30, 31, 32], and can be generalized for all dimensions, see [36, 43].

2.2 Every nontrivial solution a of the system (12) is nondegenerate at almost every point, and $L_v : \mathcal{A} \rightarrow \mathcal{A}$ is nondegenerate.

Within this paragraph we assume that the restriction of g to every open neighborhood $U \subseteq D^2$ admits no Killing vector field. We denote by \mathcal{A} the space of solutions of the system (12) corresponding to the projective connection associated to the Levi-Civita connection of g .

Lemma 2. *Assume $a \in \mathcal{A}$ is not identically zero. Then, the set of the points where a is degenerate is nowhere dense (in the topological sense, i.e., the complement to this set is open and everywhere dense.).*

Proof. The set of the points where a is degenerate is evidently a closed set. Assume there exists a neighborhood U such that a is degenerate at every point of U . In a local coordinate system (x^1, x^2) in the neighborhood U a is given by a symmetric (2×2) -matrix $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$. If the kernel of a is two-dimensional at every point of a certain neighborhood $U \subseteq D^2$, then the restriction of a to U is identically zero. Since the PDE system (12) is linear and is of finite type, $a \equiv 0$ on the whole D^2 . Then, the set of the points where the kernel of U is precisely one-dimensional is everywhere dense in U . Without loss of generality, passing to a smaller neighborhood if it is necessary, we may assume that the kernel of a is precisely one-dimensional at every point of U . Take a local coordinate system (x, y)

on an open subset $U' \subseteq U$ such that the kernel of a is the linear hull of $\frac{\partial}{\partial y}$. Then, in this coordinate system the matrix a has the form $a = \begin{pmatrix} \alpha(x, y) & 0 \\ 0 & 0 \end{pmatrix}$, where α vanishes at no point of U' .

Then, the integral (15) of the geodesic flow of g is equal to $\det(g)^{2/3} \cdot \alpha(\xi^1)^2$. Then, the function $I_{lin} := \sqrt{\det(g)^{2/3} \cdot |\alpha|} \xi^1$ is also an integral. Since the integral I_{lin} is linear in velocities, the metric $g|_{U'}$ admits a Killing vector field. The contradiction proves Lemma 2. \square

Lemma 3. *For every projective field v , the mapping $L_v : \mathcal{A} \rightarrow \mathcal{A}$ is nondegenerate.*

Remark 4. See §2.1 for interpretation L_v as a linear mapping from \mathcal{A} to \mathcal{A} .

Proof of Lemma 3. Assume there exists a nontrivial $\bar{a} \in \mathcal{A}$ such that $L_v \bar{a} = 0$. In a neighborhood of the point such that $v \neq 0$ take a coordinate system (x, y) such that $v = \frac{\partial}{\partial x}$. Then, the components of $L_v \bar{a}$ are the x -derivatives of the components of \bar{a} , and the condition $L_v \bar{a} = 0$ implies that the components of \bar{a} are independent of x . Then, the components of the metric $\bar{g} := \bar{a}/(\det(\bar{a}))^2$, which is defined almost everywhere by Lemma 2, are independent of x . Then, v is a Killing vector field for \bar{g} . Since, as we explained in §2.1, see Remark 2 there, the metric g is projectively equivalent to \bar{g} , then, by result of Knebelman [17] we recalled in §2.1, the metric g also admits a Killing vector field in a neighborhood of almost every point. The contradiction proves Lemma 3. \square

2.3 How to reduce Theorem 1 to 9 = 3 × 3 Frobenius systems of PDE

Recall that a PDE-system of the first order is *Frobenius*, if the derivatives of all unknown functions are explicitly given as functions of the unknown functions. Frobenius systems are easy to handle: there exists an algorithmic way to reduce them to ODE. In our case, the Frobenius systems are simple enough so we could explicitly solve them. Note that the most straightforward way to reformulate the problem as a system of PDE, i.e., if we write down the conditions that a vector field $\frac{\partial}{\partial x}$ is projective with respect to g as a system of 4 PDE on 3 unknown components of the metric, leads to a much more complicated system of PDE which is impossible (= we did not find a way one can do it) to solve by standard methods. In fact, the system is only slightly overdetermined (4 equations on 3 unknowns), and the standard prolongation-projection method will require too many (more than 20) operations of prolongation and prolongation-projections.

The reduction of Theorem 1 to 9 Frobenius systems of PDE is based on the description of projectively equivalent metrics in Appendix, and on the following two trivial statements from linear algebra:

- For every nondegenerate linear mapping $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ there exists a basis in \mathbb{R}^2 such that for the appropriate $\text{const} \in \mathbb{R}$ the matrix of $\text{const} \cdot L$ is given by

$$(a) \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}, \quad (b) \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}, \quad \text{or} \quad (c) \begin{pmatrix} \lambda & \\ & 1 \end{pmatrix}. \quad (16)$$

Moreover in the matrix (c) we can assume $\lambda \in (-\infty, -1] \cup [1, \infty)$.

- Every nondegenerate linear mapping $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ has a two-dimensional invariant subspace such that the matrix of the restriction of $\text{const} \cdot L$ to this subspace is one of the matrices (16) in a certain basis.

Let us explain how we reduced Theorem 1 to analysing 9 Frobenius systems of PDE.

Suppose the metric g has an essential projective vector field v and admits no Killing vector field. Consider the projective connection associated to the Levi-Civita connection of the metric, and the space \mathcal{A} of the solutions of (12). Since the system (12) is projectively invariant, for every $a \in \mathcal{A}$ the Lie derivative $L_v a$ is also a solution. Thus, L_v can be viewed as a linear mapping $L_v : \mathcal{A} \rightarrow \mathcal{A}$.

The case $\dim(\mathcal{A}) = 1$ is not interesting for us. Indeed, in this case, all metrics projectively equivalent to g have the form $\text{const} \cdot g$, which in particular imply that all projective vector fields are infinitesimal homotheties, and in Theorem 1 we excluded such metrics.

The case $\dim(\mathcal{A}) \geq 4$ is also not interesting for us. Indeed, in this case, as we recalled in §2.1, the metric g admits a Killing vector field.

If $\dim(\mathcal{A}) = 2$ **or** 3 , then, as we explained above, \mathcal{A} has a two-dimensional invariant subspace $\hat{\mathcal{A}}$ such that the restriction of L_v to $\hat{\mathcal{A}}$ is given by one of the matrices (16). If $\{a, \bar{a}\} \in \hat{\mathcal{A}}$ is the basis such that L_v is given by, say, the matrix (b) from (16), we have (the matrices (a) and (c) will be treated in §3.1 and §3.3, respectively)

$$\left. \begin{aligned} L_v a &= \lambda a - \bar{a} \\ L_v \bar{a} &= a + \lambda \bar{a}. \end{aligned} \right\} \quad (17)$$

By Lemma 2 from §2.2, without loss of generality we can assume that the the matrices of a, \bar{a} are nondegenerate, since they are so at almost every point. Then, a and \bar{a} come from two certain metrics by formula (11), see Remark 2. By Lemma 1, the metrics are projectively equivalent to g ; without loss of generality we can think that the metric corresponding to a is the initial metric g . We will call \bar{g} the metric corresponding to \bar{a} , so that

$$a = \det(g)^{-2/3} \cdot g, \quad \bar{a} = \det(\bar{g})^{-2/3} \cdot \bar{g}.$$

Then, in view of (14), the condition (17) reads

$$\left. \begin{aligned} L_v g &= \frac{2}{3} \text{trace}_g(L_v g)g + \lambda g - \left(\frac{\det(g)}{\det(\bar{g})} \right)^{2/3} \bar{g} \\ L_v \bar{g} &= \frac{2}{3} \text{trace}_{\bar{g}}(L_v \bar{g})\bar{g} + \left(\frac{\det(\bar{g})}{\det(g)} \right)^{2/3} g + \lambda \bar{g}. \end{aligned} \right\} \quad (18)$$

From other side, by Theorem A from Appendix, there exists a coordinate system (x, y) such that the metrics g and \bar{g} are given by one of the model forms. Substituting the model metrics g, \bar{g} from Appendix, we obtain the system of $6 = 3 + 3$ PDE³ of the first order on the data of the metrics and on the unknown projective vector field v .

Let us now count the number of first derivatives of the unknown functions in this system of 6 PDE. In every model case, the data of metrics g and \bar{g} , i.e., X and Y in Liouville case, h in the Complex-Liouville Case, and Y in the Jordan-block Case, have at most two first derivatives. Together with four derivatives of the components of the vector field v , it gives us at most 6 first derivatives of the unknown functions.

Thus, in the system (18) the number of highest (= first) derivatives is not greater than the number of the equations. It appears that in all cases it is possible to solve⁴ the system with respect to the first derivatives, i.e., to bring the systems in the Frobenius form, and then to solve it using standard methods.

We see that we have three choices for the matrix from (16), and three choices for the model metrics g, \bar{g} . Thus, we have $3 \times 3 = 9$ Frobenius systems to solve. We will consequently do it in Section 3.

Remark 5. In a certain sense, certain system from these nine are closely related, and can be obtained one from another by a kind of complexification. Indeed, as Remark A from Appendix shows, Complex-Liouville Case could be understand as the complexification of the Liouville Case. Moreover, over the field of complex numbers, the matrix (a) from (16) has the same type as the matrix (b): they both have two different eigenvalues. One can indeed formalize these arguments and reduce the number of systems to solve to four. But the nine systems are so simple that it is shorter to solve them than to explain how to make a solution of one using a solution of another.

3 Calculations related to proof of Theorem 1

Within the whole section we assume that

- D^2 is a smooth disc with a (Riemannian or pseudoriemannian) metric g and coordinates (x, y) ,

³each of two equations of (18) is an equation on a symmetric (2×2) -matrix, i.e., is equivalent to three scalar equations

⁴since the system (18) and its analog for matrices (a), (c) of (16) is linear in the derivatives, it is an exercise in linear algebra

- the smooth vector field v is projective with respect to the metric g ,
- the restriction of the metric g to any open subset $U \subset D^2$ admits no Killing vector field.

Within the whole section we work in the coordinates (x, y) ; g will always denote the metric we work with, and $v = (v_1, v_2)$ its projective vector field. As in §1.1, we reserve the notation ε for ± 1 .

We consider the projective connection (9) associated to the Levi-Civita connection of the metric g , and denote by \mathcal{A} the space of the solutions of the equation (12). We assume $\dim(\mathcal{A}) = 2$ or 3 , see §2.1 for explanation why we can do it.

Let $\hat{\mathcal{A}} \subseteq \mathcal{A}$ be a two-dimensional subspace invariant with respect to the Lie derivative: $L_v(a) \in \hat{\mathcal{A}}$ for every $a \in \hat{\mathcal{A}}$ (we explained its existence in §2.3).

Then, in view of Lemma 3 and after the multiplication of v by an appropriate nonzero constant, in a certain basis $\{a, \bar{a}\}$ of $\hat{\mathcal{A}}$ the matrix of L_v is as in (16).

We have three possibilities for the matrices in (16), we will carefully consider them in §§3.1, 3.2, 3.3.

3.1 The matrix of L_v is as (a) in (16)

Assume that in the basis $\{a, \bar{a}\}$ the matrix of $L_v : \hat{\mathcal{A}} \rightarrow \hat{\mathcal{A}}$ is given by $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$.

Without loss of generality, in view of Lemma 2 and Remark 2, we can assume that $a = \det(g)^{-2/3} \cdot g$, $\bar{a} = \det(\bar{g})^{-2/3} \cdot \bar{g}$ for certain metrics g, \bar{g} from the projective class of (9). Then, as we explained in §2.3, the condition

$$L_v \begin{pmatrix} a \\ \bar{a} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \begin{pmatrix} a \\ \bar{a} \end{pmatrix} \quad \text{or, equivalently,} \quad \begin{cases} L_v a = a + \bar{a} \\ L_v \bar{a} = \bar{a} \end{cases} \quad (19)$$

is equivalent to the following condition:

$$\begin{cases} L_v g - \frac{2}{3} \text{trace}_g(L_v g) \cdot g = g + \left(\frac{\det(g)}{\det(\bar{g})} \right)^{2/3} \bar{g} \\ L_v \bar{g} - \frac{2}{3} \text{trace}_{\bar{g}}(L_v \bar{g}) \cdot \bar{g} = \bar{g} \end{cases} \quad (20)$$

As we explained in Appendix, in a neighborhood of almost every point the metrics g and \bar{g} have one of three normal forms. We will carefully consider all three cases.

3.1.1 Liouville Case

Assume they have the Liouville form, i.e.,

$$ds_g^2 = (X - Y)(dx^2 + \varepsilon dy^2), \quad ds_{\bar{g}}^2 = \left(\frac{1}{Y} - \frac{1}{X} \right) \left(\frac{dx^2}{X} + \varepsilon \frac{dy^2}{Y} \right). \quad (21)$$

After some calculations we obtain that the Lie derivatives of g and \bar{g} are given by the matrices

$$\begin{pmatrix} X'v_1 + 2X \frac{\partial v_1}{\partial x} - 2Y \frac{\partial v_1}{\partial x} - Y'v_2 & \left(\frac{\partial v_1}{\partial y} + \varepsilon \frac{\partial v_2}{\partial x} \right) (X - Y) \\ \left(\frac{\partial v_1}{\partial y} + \varepsilon \frac{\partial v_2}{\partial x} \right) (X - Y) & \varepsilon \left(X'v_1 - Y'v_2 + 2X \frac{\partial v_2}{\partial y} - 2Y \frac{\partial v_2}{\partial y} \right) \end{pmatrix},$$

$$\begin{pmatrix} \frac{YX'v_1(2Y-X) + 2XY \frac{\partial v_1}{\partial x} (X-Y) - Y'X^2v_2}{Y^2X^3} & \frac{(X-Y) \left(\frac{\partial v_1}{\partial y} Y + \varepsilon \frac{\partial v_2}{\partial x} X \right)}{Y^2X^2} \\ \frac{(X-Y) \left(\frac{\partial v_1}{\partial y} Y + \varepsilon \frac{\partial v_2}{\partial x} X \right)}{Y^2X^2} & \varepsilon \frac{X'Y^2v_1 + XY'v_2(Y-2X) + 2XY(X-Y) \frac{\partial v_2}{\partial y}}{Y^3X^2} \end{pmatrix}$$

and the system (20) is equivalent to the following system of 6 PDE's on the unknown functions $v_1(x, y)$, $v_2(x, y)$, $X(x)$, and $Y(y)$.

$$\left. \begin{aligned} \frac{Y'v_2}{3} - \frac{X'v_1}{3} + \frac{2}{3} \frac{\partial v_1}{\partial x} (X - Y) - \frac{4}{3} \frac{\partial v_2}{\partial y} (X - Y) &= (Y + 1)(X - Y) \\ (X - Y) \left(\frac{\partial v_1}{\partial y} + \varepsilon \frac{\partial v_2}{\partial x} \right) &= 0 \\ \frac{X'v_1}{3} - \frac{Y'v_2}{3} - \frac{2}{3} \frac{\partial v_2}{\partial y} (X - Y) + \frac{4}{3} \frac{\partial v_1}{\partial x} (X - Y) &= (1 + X)(Y - X) \\ YX'v_1 - 2Y \frac{\partial v_1}{\partial x} (X - Y) - v_2 Y' (3X - 2Y) + 4Y \frac{\partial v_2}{\partial y} (X - Y) &= 3Y(Y - X) \\ (X - Y) \left(\frac{\partial v_1}{\partial y} Y + \varepsilon \frac{\partial v_2}{\partial x} X \right) &= 0 \\ X'v_1(3Y - 2X) - Y'Xv_2 - 2X \frac{\partial v_2}{\partial y} (X - Y) + 4 \frac{\partial v_1}{\partial x} X(X - Y) &= 3X(Y - X) \end{aligned} \right\} \quad (22)$$

We see that (in view of nondegeneracy of the metric $(X - Y)(dx^2 + \varepsilon dy^2)$) the second and fifth equations of (22) imply that v_1 depends on the variable x only, v_2 depends on the variable y only. Then, all unknown functions in the system (22) are functions of one variable only, so the system (22) is actually a system of ODE (of first order). We see that it is linear in the derivatives. Solving it for the derivatives of the unknown functions $X(x), Y(y), v_1(x), v_2(y)$, we obtain that (22) is equivalent to the following 4 ODE:

$$v_1' = -\frac{X}{2} - \frac{3}{2}, \quad v_2' = -\frac{3}{2} - \frac{Y}{2}, \quad Y'v_2 = -Y^2, \quad X'v_1 = -X^2. \quad (23)$$

These equations can already be solved; since the solution is quite complicated and is given in terms of Lambert functions, instead of solving the system we change the coordinates (probably passing to a smaller neighborhood) such that in the new coordinates the metrics g and \bar{g} and the vector field v are given by elementary functions.

Since by assumption the metric g admits no Killing vector field, the functions X, Y are not constant in every neighborhood, which in particular implies that for almost every point the functions v_1, v_2 are not zero in a neighborhood of the point. In such neighborhood consider the coordinate change

$$(x, y) = (x(x_{old}), y(y_{old})) \text{ given by } dx = \frac{1}{v_1} dx_{old} \quad dy = \frac{1}{v_2} dy_{old}. \quad (24)$$

After this coordinate change the ‘‘old’’ equation $X'v_1 = -X^2$ reads $\dot{X} = -X^2$, where $\dot{X} := \frac{dX}{dx}$, $\dot{Y} := \frac{dY}{dy}$, $\dot{v}_1 := \frac{dv_1}{dx}$, $\dot{v}_2 := \frac{dv_2}{dy}$ are derivatives with respect to the new coordinates. The equation can be solved, its nonconstant solution is $X(x) = \frac{1}{x+c}$. Since the formula (24) defines the coordinates up to addition of arbitrary constants, without loss of generality we assume $c = 0$, so that $X = \frac{1}{x}$. Similarly, in the new coordinates the equation $v_1' = -\frac{X}{2} - \frac{3}{2}$ reads $2\dot{v}_1 = -(X + 3)v_1$. After substitution $X = \frac{1}{x}$ we obtain $2\dot{v}_1 = -\left(\frac{1}{x} + 3\right)v_1$, which can be easily solved, the solution is $(v_1(x))^2 = \frac{C_1}{x} \cdot e^{-3x}$. Similarly, in the new coordinates the functions v_2, Y are given by $(v_2(y))^2 = \frac{C_2}{y} \cdot e^{-3y}$, $Y(y) = \frac{1}{y}$.

Thus, the metrics g and \bar{g} and the projective vector field v are given by

$$\begin{aligned} ds_g^2 &= (X - Y)(dx_{old}^2 \pm dy_{old}^2) = \left(\frac{1}{x} - \frac{1}{y}\right) \left(\frac{C_1}{x} e^{-3x} dx^2 + \frac{C_2}{y} e^{-3y} dy^2\right), \\ ds_{\bar{g}}^2 &= \left(\frac{1}{Y} - \frac{1}{X}\right) \left(\frac{dx_{old}^2}{X} \pm \frac{dy_{old}^2}{Y}\right) = (y - x) (C_1 e^{-3x} dx^2 + C_2 e^{-3y} dy^2), \\ v &= v_1 \frac{\partial}{\partial x_{old}} + v_2 \frac{\partial}{\partial y_{old}} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}. \end{aligned}$$

We see that the metric g and the vector field v are as in case 1a of Theorem 1.

3.1.2 Complex-Liouville Case

Assume the metric g and \bar{g} have the Complex-Liouville form from Theorem A of Appendix, i.e.

$$\begin{aligned} ds_g^2 &= 2\Im(h) dx dy, \\ ds_{\bar{g}}^2 &= -\left(\frac{\Im(h)}{\Im(h)^2 + \Re(h)^2}\right)^2 dx^2 + 2\frac{\Re(h)\Im(h)}{(\Im(h)^2 + \Re(h)^2)^2} dx dy + \left(\frac{\Im(h)}{\Im(h)^2 + \Re(h)^2}\right)^2 dy^2. \end{aligned} \quad (25)$$

Remark 6. It could be helpful for understanding to know the complex version of the formulas (25): it is

$$\left. \begin{aligned} ds_g^2 &= -\frac{1}{4}(\overline{h(z)} - h(z)) (d\bar{z}^2 - dz^2) \\ ds_{\bar{g}}^2 &= -\frac{1}{4}\left(\frac{1}{\overline{h(z)}} - \frac{1}{h(z)}\right) \left(\frac{d\bar{z}^2}{\overline{h(z)}} - \frac{dz^2}{h(z)}\right), \end{aligned} \right\} \quad (26)$$

where \bar{z} denotes the complex-conjugate to z , $\overline{h(z)}$ denotes the complex-conjugate to $h(z)$, and \bar{g} does not mean complex-conjugate to g , see Remark 1 from Appendix.

We see that the formula above is in a certain sense complexification of (21), the role of $X(x)$ plays $h(z)$ and the role of $Y(y)$ plays $h(\bar{z})$. We will see later, in all paragraphs related to Complex-Liouville Case, that all equations related to Complex-Liouville Case could be viewed as complexification of the correspondent equations from the Liouville Case. Actually, one can show it advance, and avoid the calculation, but it appears that it is shorter to do the calculations than to explain why they could be avoided.

Arguing as in the previous paragraphs, we obtain that the the conditions (20) are equivalent to a system of linear 6 PDE of the first order. Solving this system with respect to the first derivatives, and using the Cauchy-Riemann conditions for the holomorphic function h , we obtain that the system is equivalent to the system

$$\left. \begin{aligned} v_y^2 = v_x^1 &= -\frac{\Re(h)}{2} - \frac{3}{2} \\ -v_y^1 = v_x^2 &= -\frac{\Im(h)}{2} \\ \Re(h)_x = \Im(h)_y &= \frac{v^1(\Im^2(h) - \Re^2(h)) - 2v^2\Re(h)\Im(h)}{(v^2)^2 + (v^1)^2} \\ -\Re(h)_y = \Im(h)_x &= -\frac{v^2(\Im^2(h) - \Re^2(h)) + 2v^1\Re(h)\Im(h)}{(v^2)^2 + (v^1)^2} \end{aligned} \right\} \quad (27)$$

From the first two equations of (27) we see that the function $V := v^1 + i \cdot v^2$ is a holomorphic function of the variable $z := x + i \cdot y$. It is easy to check that the last two equations of (27) are equivalent to

$$Vh_z = -h^2, \quad (28)$$

and the first two equations of (27) are equivalent to

$$V_z = -\frac{h}{2} - \frac{3}{2} \quad (29)$$

(where V_z, h_z are the derivative of V and h with respect to z). We see that the equations (27 – 29) are direct analog of (23).

After the holomorphic coordinate change

$$dz_{\text{new}} = \frac{1}{V} dz_{\text{old}}, \quad (30)$$

the equation (28) is $h_{z_{\text{new}}} = -h^2$ implying

$$h(z_{\text{new}}) = \frac{1}{z_{\text{new}} + \text{const}}. \quad (31)$$

Since the the formulas (30) defines z_{new} up to addition of a complex constant, we can (and will) assume without loss of generality that $\text{const} = 0$. In this new coordinate the vector field v is $\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}$.

Now consider the equation (29). In the new coordinates it reads

$$2V_{z_{\text{new}}} = -\left(\frac{1}{z_{\text{new}}} + 3\right)V.$$

Solving it, we obtain

$$V^2 = \frac{Ce^{-3z_{\text{new}}}}{z_{\text{new}}}. \quad (32)$$

Finally, substituting the coordinate change and the solutions (31 – 32) in the metrics we obtain that, after the appropriate scaling, the metrics g and \bar{g} have the form

$$\begin{aligned} ds_g^2 &= 2\Im(h)dx_{\text{old}}dy_{\text{old}} = \frac{1}{4} \cdot (h(\bar{z}) - h(z)) (d\bar{z}_{\text{old}}^2 - dz_{\text{old}}^2) \\ &= \frac{1}{4} \cdot \left(\frac{1}{\bar{z}} - \frac{1}{z}\right) \left(\bar{C}e^{-3\bar{z}}\frac{d\bar{z}^2}{\bar{z}} - Ce^{-3z}\frac{dz^2}{z}\right), \\ ds_{\bar{g}}^2 &= \frac{1}{4} (z - \bar{z}) (\bar{C}e^{-3\bar{z}}d\bar{z}^2 - Ce^{-3z}dz^2), \end{aligned}$$

and the projective vector field v is $\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}$.

We see that the metric and the projective vector field v are as in case 2a of Theorem 1.

3.1.3 Jordan-block Case

Let the metrics g and \bar{g} be given by the formulas from the Remark 2 of Appendix:

$$\begin{aligned} ds_g^2 &= 2(Y(y) + x) dx dy \\ ds_{\bar{g}} &= -\frac{2(Y(y)+x)}{y^3} dx dy + \frac{(Y(y)+x)^2}{y^4} dy^2. \end{aligned} \quad (33)$$

Arguing as above, we obtain that the condition (20) is equivalent to a certain system of 6 PDE on the unknown functions v_1, v_2, Y . Solving first 5 PDE with respect to the derivatives of the unknown functions and substituting the solution in the remaining equation, we obtain that the system is equivalent to

$$\left. \begin{aligned} \frac{\partial v_1}{\partial x} &= \frac{1}{2}y - \frac{3}{2} \\ \frac{\partial v_1}{\partial y} &= \frac{1}{2}Y + \frac{1}{2}x, \\ v_2 &= y^2 \\ Y' &= -\frac{1}{2} \frac{-5yY + 3Y - 5yx + 3x + 2v_2'Y + 2v_2'x + 2v_1}{v_2} \end{aligned} \right\} \quad (34)$$

We see that the first three equations of (34) are equivalent to $v_2 = y^2$, $v_1 = \left(\frac{1}{2}y - \frac{3}{2}\right)x + \frac{1}{2}Y_1$, where $Y_1' = Y$. Substituting these in the last equation of (34), we obtain the following linear ODE on Y_1 :

$$y^2 Y_1'' = \frac{1}{2} (y Y_1' - 3 Y_1' - Y_1).$$

The equation can be solved, the general solution is

$$Y_1 = (y - 3) \left(C_1 + C_2 \int_{y_0}^y e^{\frac{3}{2\xi}} \frac{\sqrt{|\xi|}}{(\xi - 3)^2} d\xi \right).$$

Then, the function $Y = Y_1'$ is

$$Y = \left(C_2 e^{\frac{3}{2y}} \frac{\sqrt{|y|}}{(y - 3)} \right) + \left(C_1 + C_2 \int_{y_0}^y e^{\frac{3}{2\xi}} \frac{\sqrt{|\xi|}}{(\xi - 3)^2} d\xi \right).$$

The assumption that the metric admits no Killing vector field implies $C_2 \neq 0$. In view of the coordinate change $x_{\text{new}} = x_{\text{old}} + C_1$, we can assume $C_1 = 0$.

Then, the components v_1, v_2 of the projective vector field are

$$v_2 = y^2, \quad v_1 = \left(\frac{1}{2}y - \frac{3}{2}\right)x + C_2 \frac{y - 3}{2} \int_{y_0}^y e^{\frac{3}{2\xi}} \frac{\sqrt{|\xi|}}{(\xi - 3)^2} d\xi$$

We see that the metric (after the appropriate coordinate change and the scaling) and the projective vector field v are as in case 3a of Theorem 1.

3.2 The matrix of L_v is as (b) in (16)

Assume that in the basis $\{a, \bar{a}\}$ the matrix of $L_v : \hat{\mathcal{A}} \rightarrow \hat{\mathcal{A}}$ is given by $\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$.

Without loss of generality, in view of Lemma 2, we can assume that $a = \det(g)^{-2/3} \cdot g$, $\bar{a} = \det(\bar{g})^{-2/3} \cdot \bar{g}$ for certain metrics g, \bar{g} from the projective class of (9). Then, as we explained in §2.3, the condition

$$L_v \begin{pmatrix} a \\ \bar{a} \end{pmatrix} = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} \begin{pmatrix} a \\ \bar{a} \end{pmatrix}$$

is equivalent to the condition (18).

As we explained in Appendix, in a neighborhood of almost every point the metrics g and \bar{g} have one of three normal forms. We will carefully consider all three cases.

3.2.1 Liouville Case

Assume the metrics g and \bar{g} have the Liouville form (21). Then, the condition (18) is equivalent to a system of 6 PDE's on the unknown functions $v_1(x, y)$, $v_2(x, y)$, $X(x)$, and $Y(y)$.

Solving the equations with respect to the derivatives, we obtain that the equations are equivalent to the following system of 6 PDE's in Frobenius form:

$$\left. \begin{aligned} \frac{\partial v_1}{\partial x} &= X/2 - 3/2 \lambda \\ \frac{\partial v_2}{\partial y} &= Y/2 - 3/2 \lambda \\ \frac{\partial v_1}{\partial y} &= 0 \\ \frac{\partial v_2}{\partial x} &= 0 \\ X' v_1 &= 1 + X^2 \\ Y' v_2 &= 1 + Y^2 \end{aligned} \right\} \quad (35)$$

We see that the functions v_1 and v_2 are functions of one variable only⁵, so that all the equations (35) are actually ODE's. Moreover, the assumption that there exists no Killing vector field implies that X and Y are not constant. Then, in view of the first two equations of (35), the components v_1 and v_2 are not zero almost everywhere. Then, without loss of generality we can assume $v_1 \neq 0$, $v_2 \neq 0$. Take the new coordinate system $(x(x_{old}), y_{new}(y_{old}))$ given by

$$dx = \frac{1}{v_1} dx_{old} \quad dy = \frac{1}{v_2} dy_{old}. \quad (36)$$

In these new coordinate system the last two equations of (35) are $\dot{X} = 1 + X^2$, $\dot{Y} = 1 + Y^2$ implying

$$X(x) = \tan(x + \text{const}_1), \quad Y(y) = \tan(y_{new} + \text{const}_2). \quad (37)$$

Since the the formulas (36) defines x and y up to addition of a constant, we can (and will) assume without loss of generality that $\text{const}_1 = \text{const}_2 = 0$. Now consider the first and the second equations of (35). In the new coordinates they are $\dot{v}_1 = (\tan(x)/2 - 3/2 \lambda)v_1$, $\dot{v}_2 = (\tan(y)/2 - 3/2 \lambda)v_2$. Solving them we obtain

$$v_1 = \frac{C_1 e^{-3/2 \lambda x}}{\sqrt{\cos(x)}}, \quad v_2 = \frac{C_2 e^{-3/2 \lambda y}}{\sqrt{\cos(y)}}. \quad (38)$$

Finally, combining (36 – 38) we obtain that (after the appropriate coordinate change and the scaling) the metric g has the form

$$ds_g^2 = (X - Y) (dx_{old}^2 \pm dy_{old}^2) = (\tan(x) - \tan(y)) \left(\frac{C_1^2 e^{-3 \lambda x} dx^2}{\cos(x)} \pm \frac{C_2^2 e^{-3 \lambda y} dy^2}{\cos(y)} \right), \quad (39)$$

⁵which was clear in advance since the family $\hat{\mathcal{A}}$ determines the lines of the coordinates, see [30, 32]; hence the coordinate lines must be preserved by the flow of v

and the projective vector field v is $\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. It is easy to see that if $\lambda = 0$ and $C_1 = \pm C_2$, then the metric admits a Killing vector field. Indeed, because of scaling, it is sufficient to show this for $C_1 = C_2 = 1$. If the sign “ \pm ” in (39) is “ $-$ ”, then the vector field $\frac{\cos(x)}{\sin(\frac{1}{2}(x-y))} \frac{\partial}{\partial x} - \frac{\cos(y)}{\sin(\frac{1}{2}(x-y))} \frac{\partial}{\partial y}$ is a Killing one for g . If the sign “ \pm ” in (39) is “ $+$ ”, then the vector field $\frac{\cos(x)}{\cos(\frac{1}{2}(x-y))} \frac{\partial}{\partial x} + \frac{\cos(y)}{\cos(\frac{1}{2}(x-y))} \frac{\partial}{\partial y}$ is a Killing one for g .

We see that the metric and the projective vector field v are as in case 1b of Theorem 1.

3.2.2 Complex-Liouville Case

Assume the metric g and \bar{g} have the Complex-Liouville form (25). Arguing as above, we obtain that the conditions (18) are equivalent to a certain system of 6 PDE of the first order. Solving this system with respect to the first derivatives, and using the Cauchy-Riemann conditions for the holomorphic function h , we obtain that the system is equivalent to the system

$$\left. \begin{aligned} v_y^2 = v_x^1 &= \frac{1}{2} \Re(h) - \frac{3}{2} \lambda \\ -v_y^1 = v_x^2 &= \frac{1}{2} \Im(h) \\ \Re(h)_x = \Im(h)_y &= \frac{-v^1 \Im^2(h) + 2\Re(h)\Im(h)v^2 + v^1 + v^1 \Re^2(h)}{(v^2)^2 + (v^1)^2} \\ -\Re(h)_y = \Im(h)_x &= \frac{v^2 \Im^2(h) - v^2 - v^2 \Re^2(h) + 2v^1 \Re(h)\Im(h)}{(v^2)^2 + (v^1)^2} \end{aligned} \right\} \quad (40)$$

From the first two equations of (40) we see that the function $V := v_1 + i \cdot v_2$ is a holomorphic function of the variable $z := x + i \cdot y$. It is easy to check that the last two equations of (40) are equivalent to

$$V h_z = h^2 + 1, \quad (41)$$

and the first two equations of (40) are equivalent to

$$V_z = \frac{1}{2} h - \frac{3}{2} \lambda \quad (42)$$

(where h_z, V_z are the derivatives of h, V with respect to z). Thus, the equations (40) are direct analog of (35). After the coordinate change

$$dz_{\text{new}} = \frac{1}{V} dz_{\text{old}}, \quad (43)$$

the equation (41) reads $\frac{dh}{dz_{\text{new}}} = 1 + h^2$ implying

$$h(z_{\text{new}}) = \tan(z_{\text{new}} + \text{const}). \quad (44)$$

Since the the formulas (43) defines z_{new} up to addition of a constant, we can (and will) assume without loss of generality that $\text{const} = 0$. In this new coordinate the vector field v is $V \frac{\partial}{\partial z} + \bar{V} \frac{\partial}{\partial \bar{z}}$.

Now consider the equation (42). In the new coordinates it reads

$$\frac{dV}{dz_{\text{new}}} = (\tan(z_{\text{new}})/2 - \frac{3}{2} \lambda) V \quad \text{implying} \quad V = \frac{C e^{-3/2 \lambda z_{\text{new}}}}{\sqrt{\cos(z_{\text{new}})}}. \quad (45)$$

Finally, combining (43 – 45) we obtain that (after the appropriate scaling) the metrics g and \bar{g} have the form

$$\begin{aligned} ds_g^2 &= \frac{1}{4} (\tan(z_{\text{new}}) - \tan(\bar{z}_{\text{new}})) \left(\frac{\bar{C} e^{-3 \lambda \bar{z}_{\text{new}}} d\bar{z}_{\text{new}}^2}{\cos(\bar{z}_{\text{new}})} - \frac{C e^{-3 \lambda z_{\text{new}}} dz_{\text{new}}^2}{\cos(z_{\text{new}})} \right), \\ ds_{\bar{g}}^2 &= \frac{1}{4} (\cotan(\bar{z}_{\text{new}}) - \cotan(z_{\text{new}})) \left(\frac{\bar{C} e^{-3 \lambda \bar{z}_{\text{new}}} d\bar{z}_{\text{new}}^2}{\sin(\bar{z}_{\text{new}})} - \frac{C e^{-3 \lambda z_{\text{new}}} dz_{\text{new}}^2}{\sin(z_{\text{new}})} \right), \end{aligned}$$

and the projective vector field v is $\frac{\partial}{\partial z_{\text{new}}} + \frac{\partial}{\partial \bar{z}_{\text{new}}} = \frac{\partial}{\partial x_{\text{new}}}$. It is easy to see that if $\lambda = 0$ and $C \in \mathbb{R}$, then the metric admits a Killing vector field. Indeed, it is sufficient to consider $C = 1$. For this case, the following vector field is a Killing one: $\sin(x) \frac{\partial}{\partial x} + \frac{\cos(x) \cosh(y)}{\sinh(y)} \frac{\partial}{\partial y}$.

We see that the metric and the projective vector field v are as in case 2b of Theorem 1.

3.2.3 Jordan-block Case

Assume the metrics g and \bar{g} are given the matrices (33). Arguing as above, we obtain that the condition (18) is equivalent to a certain system of 6 PDE on the unknown functions v_1, v_2, Y . Solving the first 5 PDE with respect to the derivatives of the unknown function, and substituting the solution in the remaining equation, we obtain that the system is equivalent to

$$\left. \begin{aligned} \frac{\partial v_1}{\partial x} &= \frac{1}{2}y - \frac{3}{2}\lambda \\ \frac{\partial v_1}{\partial y} &= \frac{1}{2}Y + \frac{1}{2}x, \\ v_2 &= y^2 + 1 \\ Y' &= -\frac{1}{2} \frac{-5yY + 3\lambda Y - 5yx + 3\lambda x + 2v_2'Y + 2v_2'x + 2v_1}{v_2} \end{aligned} \right\} \quad (46)$$

We see that the first three equations of (46) are equivalent to

$$v_2 = y^2 + 1, \quad v_1 = \left(\frac{1}{2}y - \frac{3}{2}\lambda \right) x + \frac{1}{2}Y_1(y), \quad \text{where } Y_1' = Y(y)$$

Substituting these in the last equation of (46), we obtain the following linear ODE on Y_1 :

$$(1 + y^2)Y_1'' = \frac{1}{2}(yY_1' - 3\lambda Y_1' - Y_1).$$

The equation can be solved, the general solution is

$$Y_1 = (y - 3\lambda) \left(C_1 + C_2 \int_{y_0}^y e^{-\frac{3}{2}\lambda \arctan(\xi)} \frac{\sqrt[4]{\xi^2 + 1}}{(\xi - 3\lambda)^2} d\xi \right).$$

Then, the function $Y = Y_1'$ is

$$Y = C_1 + C_2 \int_{y_0}^y e^{-\frac{3}{2}\lambda \arctan(\xi)} \frac{\sqrt[4]{\xi^2 + 1}}{(\xi - 3\lambda)^2} d\xi + (y - 3\lambda) \left(C_2 e^{-\frac{3}{2}\lambda \arctan(y)} \frac{\sqrt[4]{y^2 + 1}}{(y - 3\lambda)^2} \right).$$

And the components v_1, v_2 of the projective vector field are

$$v_2 = y^2 + 1, \quad v_1 = \left(\frac{1}{2}y - \frac{3}{2}\lambda \right) x + \frac{y - 3\lambda}{2} \left(C_1 + C_2 \int_{y_0}^y e^{-\frac{3}{2}\lambda \arctan(\xi)} \frac{\sqrt[4]{\xi^2 + 1}}{(\xi - 3\lambda)^2} d\xi \right).$$

We see that, after an appropriate coordinate change and scaling, the metric and the projective vector field v are as in case 3b of Theorem 1.

3.3 The matrix of L_v is as (c) in (16)

Assume that in the basis $\{a, \bar{a}\}$ the matrix of $L_v : \hat{\mathcal{A}} \rightarrow \hat{\mathcal{A}}$ is given by $\begin{pmatrix} \lambda & \\ & 1 \end{pmatrix}$, where $\lambda \in (-\infty, -1] \cup [1, +\infty)$.

Without loss of generality, in view of Lemma 2, we can assume that $a = \det(g)^{-2/3} \cdot g$, $\bar{a} = \det(\bar{g})^{-2/3} \cdot \bar{g}$ for certain metrics g, \bar{g} from the projective class of (9). Then, as we explained in §2.3, the condition

$$L_v \begin{pmatrix} a \\ \bar{a} \end{pmatrix} = \begin{pmatrix} \lambda & \\ & 1 \end{pmatrix} \begin{pmatrix} a \\ \bar{a} \end{pmatrix}$$

is equivalent to the condition $L_v a = \lambda a$, $L_v \bar{a} = \bar{a}$, which is equivalent to the condition

$$L_v g - \frac{2}{3} \text{trace}_g(L_v g)g - \lambda g = 0, \quad L_v \bar{g} - \frac{2}{3} \text{trace}_{\bar{g}}(L_v \bar{g})\bar{g} - \bar{g} = 0,$$

which is equivalent to the condition

$$L_v g = -\frac{\lambda}{3}g, \quad L_v \bar{g} = -\frac{1}{3}\bar{g}. \quad (47)$$

As we explained in Appendix, in a neighborhood of almost every point the metrics g and \bar{g} have one of three normal forms. We will carefully consider all three cases.

Remark 7. We will also see that $\lambda \neq 1$. This will imply that if two nonproportional projectively equivalent metrics g and \bar{g} have $L_v g = \lambda \cdot g$ and $L_v \bar{g} = \lambda \cdot \bar{g}$ for a certain $v \neq 0$, then the metrics admit a Killing vector field, which will be used in the proof of Theorem 2.

3.3.1 Liouville Case

We assume that the metrics g and \bar{g} are given by (21). Then, the condition (47) is equivalent to a system of 6 PDE on v_1, v_2, X, Y .

Solving these equations with respect to derivatives, we obtain

$$\begin{aligned} \frac{\partial v_1}{\partial x} &= -\lambda - 1/2, & \frac{\partial v_1}{\partial y} &= 0, & X'v_1 &= -X(-1 + \lambda) \\ \frac{\partial v_2}{\partial y} &= -\lambda - 1/2, & \frac{\partial v_2}{\partial x} &= 0, & Y'v_2 &= -Y(-1 + \lambda) \end{aligned} \quad (48)$$

This system of equations can be easily solved (we recall that by the assumption $|\lambda| \geq 1$). If $\lambda = 1$, at least one of the functions X, Y is a constant implying the existence of a Killing vector field as we promised in Remark 7. For other λ , the solution is (up to the coordinate change $(x_{new}, y_{new}) = (x_{old} + \text{const}_1, y_{old} + \text{const}_2)$)

$$\begin{aligned} v_1 &= -\left(\frac{1}{2} + \lambda\right)x, & X &= C_1 x^{2\frac{\lambda-1}{1+2\lambda}} \\ v_2 &= -\left(\frac{1}{2} + \lambda\right)y, & Y &= C_2 y^{2\frac{\lambda-1}{1+2\lambda}}, \end{aligned}$$

and the corresponding g and v , after dividing v by $-\left(\frac{1}{2} + \lambda\right)$, are

$$\left(C_1 x^{2\frac{\lambda-1}{1+2\lambda}} - C_2 y^{2\frac{\lambda-1}{1+2\lambda}} \right) (dx^2 + \varepsilon dy^2), \quad x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}.$$

We see that after the coordinate change $(x_{old} = e^x, y_{old} = e^y)$, after an appropriate scaling, and after denoting $\frac{2(\lambda-1)}{2\lambda+1}$ by ν , the metric and the projective vector field v are as in case 1c of Theorem 1. Note that in the case $\nu = 2$, $C_1 = -\varepsilon C_2$ the metric g has a constant curvature (and, therefore, a Killing vector field). Since $\lambda \in (-\infty, -1] \cup (1, +\infty)$, we have $\nu \in (0, 4]$, $\nu \neq 1$. Since $\lambda \neq 1$, then $\nu \neq 0$.

3.3.2 Complex-Liouville Case

Assume that g, \bar{g} are as in (25). Arguing as above, we obtain that the equations (47) are equivalent to a system of 6 PDE which can be written as

$$\begin{aligned} v_1 + i \cdot v_2 &= -(\lambda + 1/2) \cdot z + \text{const} \\ \frac{\partial h}{\partial z} &= \frac{h}{z} \cdot \frac{2(\lambda - 1)}{1 + 2\lambda}. \end{aligned}$$

The system can be easily solved. If $\lambda = 1$, at the function h is a constant implying the existence of a Killing vector field as we promised in Remark 7.

If $\lambda \neq 1$, then, in view of the coordinate change $x_{new} = x_{old} + \text{const}_1$, $y_{new} = y_{old} + \text{const}_1$ we can think that $\text{const} = 0$. Then, the solution is $h = C \cdot z^{2\frac{\lambda-1}{1+2\lambda}}$. Then, the metrics g and \bar{g} are as in (26) with this function h , and the projective vector field v is $x \frac{\partial}{\partial x}$.

We see that after the coordinate change $z_{old} = e^z$, after an appropriate scaling, and after denoting $\frac{2(\lambda-1)}{2\lambda+1}$ by ν , the metric and the projective vector field v are as in case 2c of Theorem 1. Since $\lambda \in (-\infty, -1] \cup (1, +\infty)$, we have $\nu \in (0, 4]$, $\nu \neq 1$, as we assumed in Theorem 1.

3.3.3 Jordan-block Case

Assume the metrics g and \bar{g} are given by (33). Arguing as above, we obtain that the condition (47) is equivalent to a certain system of 6 PDE on the unknown functions v_1, v_2, Y . Solving this system with respect to the derivatives of the functions we see that $\frac{\partial v_2}{\partial x} = 0$, $\frac{\partial v_1}{\partial y} = 0$ implying that v_1 is a function of x and v_2 is a function of y only, and that the system is equivalent to

$$\left. \begin{aligned} v_1' &= & -\lambda - \frac{1}{2} \\ v_2 &= & -(\lambda - 1)y \\ Y'v_2 &= & -\frac{1}{2}(2v_2'Y + 2v_2'x + 4\lambda Y - Y + 4\lambda x - x + 2v_1) \end{aligned} \right\} \quad (49)$$

From the first equation of (49) we see that $v_1 = -\lambda x - \frac{x}{2} + C$. Without loss of generality we can think that $C = 0$. Substituting the expressions for v_1, v_2 in the last equation of (49), we obtain (we can assume $y > 0$ since it can be achieved by a coordinate change)

$$Y' = \frac{(2\lambda + 1)Y}{(2\lambda - 2)y}$$

Solving this equation, we obtain

$$Y = y^{\frac{2\lambda+1}{2\lambda-2}} C_1. \quad (50)$$

We see that after an appropriate scaling and after denoting $\frac{2(\lambda-1)}{2\lambda+1}$ by η , the metric and the projective vector field v are as in case 3c (for $\eta \neq \frac{1}{2}$) or as in case 3d (for $\eta = \frac{1}{2}$) of Theorem 1.

4 Proof of Theorem 2

4.1 In the cases 1a – 3c, it is sufficient to prove that \mathcal{A} is precisely two-dimensional. In the case 3d, it is sufficient to prove that \mathcal{A} is precisely three-dimensional.

Within this paragraph we assume that the metric g is one of the metrics from Theorem 1. We additionally assume that it admits no Killing vector field. Let us explain why, in order to prove Theorem 2, it is sufficient to show that the space \mathcal{A} of solutions of (12) is as in the title of this paragraph.

For every metric g from Theorem 1, consider its canonically projectively equivalent metric \bar{g} given by the appropriate formula from (5 – 7). By definition, the metrics g and \bar{g} have the same projective connection. Then, $a = g/\det(g)^{2/3}$ and $\bar{a} = \bar{g}/\det(\bar{g})^{2/3}$ lie in the space \mathcal{A} corresponding to the metric g .

Therefore, every their linear combination $\alpha \cdot a + \beta \cdot \bar{a}$ is also an element of \mathcal{A} . Comparing the definition of $G(g, \bar{g})$ with the formulas in Remark 2, we see that $G(g, \bar{g})$ is precisely the set of the metrics corresponding to the solutions of the form $\alpha \cdot a + \beta \cdot \bar{a}$. In particular, all metrics from $G(g, \bar{g})$ lie in the projective class of g .

Thus, in order to show that the projective class of the metrics g from cases 1a – 3c of Theorem 1 coincides with $G(g, \bar{g})$, it is sufficient to show that \mathcal{A} coincides with the set of linear combinations of a and \bar{a} , i.e., is two-dimensional.

Now, let us consider the metric 3d. In this case, the space \mathcal{A} is at least three-dimensional. Indeed, the solutions $a = g/\det(g)^{2/3}$, $\bar{a} = \bar{g}/\det(\bar{g})^{2/3}$, and $\tilde{a} = \tilde{g}/\det(\tilde{g})^{2/3}$ are linearly independent. Clearly, the metrics corresponding to the linear combinations of these solutions are precisely the metrics from $G[g, \bar{g}, \tilde{g}]$. Hence, if the space of \mathcal{A} is precisely three-dimensional, the projective class coincides with $G[g, \bar{g}, \tilde{g}]$.

4.2 Schema of the proof

Theorem 1 gives us 10 explicit formulas for the metric g and, therefore, for the coefficients K_i of the equation (12). Our goal is to show that in the first 9 cases the space \mathcal{A} of the solutions of (12) is at most two-dimensional, and in the last case the space \mathcal{A} is at most three-dimensional.

There exists a highly computational method to do it: indeed, the system (12) is linear and of finite type. Then, the standard prolongation-projection method gives us an algorithm which calculates the dimension of the space of solutions.

Unfortunately, this method is too hard from the viewpoint of calculations, at least if one does the calculations straightforwardly: indeed, in order to do the algorithm, one needs to differentiate the entries

of the metric 7 times, and then calculate the rank of a 18×16 matrix. It is very boring to do it “by hands”.

It is still possible to do it with the help of computer algebra packages. Recently Kruglikov [20] and, independently, Bryant, Eastwood and Dunajski [10] used Mathematica[®] and Maple[®] (and also quite advanced theory) to construct curvature invariants such that if they do not vanish the dimension of \mathcal{A} is at most 2. But their invariants are still too complicated, and there is no hope to calculate them for our metrics without using a computer (though one can easily do it with computer).

In order to give a proof which is much easier from computational point of view, and which could be done by a human, we use the existence of the projective vector field to reduce the problem to more simple systems of PDE. We consider **three cases**.

The **first case** corresponds to the metrics 3a, 3b, 3c, 3d. In these cases, the general form of the metric is very simple and one actually can do the prolongation-projection algorithm “by hands” and without using the existence of the projective vector field, see §4.5. After few steps (we actually do short-cuts in the paper), we obtain the dimension of \mathcal{A} .

The **second and the third cases** corresponds to all other metrics. We assume that $\dim(\mathcal{A}) = 3$ and find a contradiction. (The case $\dim(\mathcal{A}) \geq 4$ is not possible because by Theorem 3 the metrics 1a – 2c admit no Killing vector field. We will not use Theorem 2 in the proof that the metrics 1a – 2c admit no Killing vector, so no logical loop appears). In order to do it, let us take a basis $\{a, \bar{a}, \hat{a}\}$ such that the matrix of L_v is one of (51), where A is a (2×2) - matrix given by (16)

$$\begin{pmatrix} \mu & & \\ & A & \\ & & \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix}. \quad (51)$$

The **second case** corresponds to the first matrix of (51). We will find a contradiction using the following trivial observation from linear algebra: if for a (3×3) matrix $m = (m_{ij})$ with $\det(m) \neq 0$

$$\left. \begin{aligned} m_{11}a_{11} + m_{12}a_{12} + m_{13}a_{22} &= 0 \\ m_{21}a_{11} + m_{22}a_{12} + m_{23}a_{22} &= 0 \\ m_{31}a_{11} + m_{32}a_{12} + m_{33}a_{22} &= 0, \end{aligned} \right\} \quad (52)$$

then $a_{ij} = 0$.

Let us explain how the assumptions of the **second case** allow to construct such equations on a_{ij} . Since the matrix of L_v is the first matrix of (51), we have

$$L_v \begin{pmatrix} a \\ \bar{a} \\ \hat{a} \end{pmatrix} = \begin{pmatrix} \mu & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} a \\ \bar{a} \\ \hat{a} \end{pmatrix}. \quad (53)$$

The last two equations of (53) are equations $L_v \begin{pmatrix} \bar{a} \\ \hat{a} \end{pmatrix} = A \begin{pmatrix} \bar{a} \\ \hat{a} \end{pmatrix}$, we solved them in the proof of Theorem 1, in a certain coordinate system (in a neighborhood of almost every point) the metric $g = \bar{a}/\det(\bar{a})^2$ is as in Theorem 1 after a possible scaling. We have 6 (explicit) possibilities 1a – 2c for the metric and therefore 6 (explicit) possibilities for the coefficients of the equation (12).

Let us now pass to the coordinate system such that the projective vector field is $\frac{\partial}{\partial x}$. In cases 2a – 2c, we are already in such coordinate system, in the cases 1a – 1c we use the coordinate change $x_{new} = \frac{x_{old} + y_{old}}{2}$, $y_{new} = \frac{x_{old} - y_{old}}{2}$. In this coordinate system, the coefficients K_0, \dots, K_3 of the projective connection are independent of x , direct calculations show that they are given by simple formulas, see the beginning of §4.3. We see that the first equation of (53) is $\frac{\partial a}{\partial x} = \mu \cdot a$ implying

$$a = e^{\mu x} \begin{pmatrix} a_{11}(y) & a_{12}(y) \\ a_{12}(y) & a_{22}(y) \end{pmatrix}. \quad (54)$$

Substituting (54) in (12), we obtain one homogeneous linear equation and 3 linear ODE on the three unknown functions $a_{ij}(y)$, see § 4.3 for the precise formulas. This linear equation (first equation of (60)) will play the role of the first equation of (52).

It is possible to explicitly solve the above mentioned 3 ODE with respect to derivatives, see (60). Differentiating the first equation of (60) with respect to y , and substituting the derivatives of a_{ij} from the other three equations of (60) inside, we obtain one more linear equation on a_{ij} . This equation will play the role of the second equation of (52). Repeating the procedure with this new equation, we obtain the third linear equation on a_{ij} , which will be the third equation of (52). Direct calculations show that the determinant of the correspondent (3×3) -matrix (m_{ij}) is not zero implying $a \equiv 0$. We obtain a contradiction with the assumption that $\{a, \bar{a}, \hat{a}\}$ is a basis.

This described procedure is not very complicated computationally (all formulas that appear have less than 50 terms, i.e., one can do all calculations “by hands”, and standard computer algebra packages, say Maple[®] or Mathematica[®], need less then 10 seconds for all the calculation.)

Let us also note that the proof for the cases 1a, 1b, 1c implies the proof for the cases 2a, 2b, 2c (so we need to do the calculation for the three cases 1a, 1b, 1c only). Indeed, the formulas for (the components of) the metrics 1a, 1b, 1c are real-analytic, and we can allow x to be a complex variable and y to be its conjugate, since it changes neither differentiation nor algebraic operations with the (components of the) metrics. After this change the metrics 1a, 1b, 1c become, up to a multiplication by a constant, the metrics 2a, 2b, 2c, and therefore our proof that the metrics 1a, 1b, 1c have two-dimensional \mathcal{A} , which uses only algebraic operation and differentiation, is also a proof for the cases 2a, 2b, 2c.

The **third case** corresponds to the second matrix of (51). We will find a contradiction using the following fact from linear algebra: if

$$\left. \begin{aligned} m_{11}a_{11} + m_{12}a_{12} + m_{13}a_{22} &= b_1 \\ m_{21}a_{11} + m_{22}a_{12} + m_{23}a_{22} &= b_2 \\ m_{31}a_{11} + m_{32}a_{12} + m_{33}a_{22} &= b_3, \end{aligned} \right\} \quad (55)$$

then the following two statements are contradictive:

$$\det \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} = 0, \quad \text{and} \quad \det \begin{pmatrix} b_1 & m_{12} & m_{13} \\ b_2 & m_{22} & m_{23} \\ b_3 & m_{32} & m_{33} \end{pmatrix} \neq 0.$$

The way to construct equations (55) are similar to that we use in the **second case**. Since the matrix of L_v is the second matrix of (51), the Lie derivatives of the basis elements a, \bar{a}, \hat{a} are given by 3 matrix equations

$$L_v \begin{pmatrix} a \\ \bar{a} \\ \hat{a} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ & 1 & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} a \\ \bar{a} \\ \hat{a} \end{pmatrix} = \begin{pmatrix} a & + & \bar{a} \\ & \bar{a} & + \\ & & \hat{a} \end{pmatrix}. \quad (56)$$

We see that the last two equations of (56) are the equations (19). We solved them in §3.1, see §§3.1.1, 3.1.2 there. Then, without loss of generality we can assume that the metric $g = \bar{a}/\det(\bar{a})^2$ and the projective field v are as in cases 1a, 1b of Theorem 1. We again pass to the coordinates such that the projective vector field v is $\frac{\partial}{\partial x}$: in the case 1b, we are already in this coordinates, in the case 1a, we will work in the coordinates $x_{new} = \frac{x_{old} + y_{old}}{2}$, $y_{new} = \frac{x_{old} - y_{old}}{2}$. In this coordinates, the coefficients K_0, \dots, K_3 of the projective connection are independent of y , and the components of the Lie-derivative $L_v a$ are the x -derivatives of the components of a . Then, the first equation of (56) is

$$\frac{\partial}{\partial x} a = a + \bar{a}. \quad (57)$$

This equation is actually a system of three equations, since a is a symmetric 2×2 -matrix. In this equation, \bar{a} is known: in view of Remark 2, it is given by $g/\det(g)^{2/3}$, and above we assumed that g is the metric 1a from Theorem 1. Direct calculations shows that a is given by (61). Then, (57) is a system of linear nonhomogeneous equations, its every solution is the sum of a partial solution P (for the metric 1a, a partial solution is (62)) and a solution of the equation $\frac{\partial a}{\partial x} = a$, i.e., has the following form:

$$e^x \cdot \begin{pmatrix} a_{11}(y) & a_{12}(y) \\ a_{12}(y) & a_{22}(y) \end{pmatrix} + P. \quad (58)$$

Substituting the ansatz (58) in the equations (12), we obtain one nonhomogeneous linear equation (which will play the role of the first equation of (55)), and three nonhomogeneous linear ODE of the first order on the components $a_{ij}(y)$. The ODE can be solved with respect to the derivatives of a_{ij} , see (63).

Differentiating the above mentioned linear equation (which is the first equation of (63)) with respect to y , and substituting the derivatives of a_{ij} from the other equations of (63) inside, we obtain one more linear nonhomogeneous equation on a_{ij} . Repeating the procedure with the obtained equation, we obtain the third linear nonhomogeneous equation on a_{ij} . Thus, we have three nonhomogeneous linear relations on three functions a_{ij} as in (55). If we show that three nonhomogeneous linear relations are not compatible, then the dimension of \mathcal{A} is at most 2.

Clearly, the determinant of (m_{ij}) is zero, since $\hat{a} := \bar{g}/\det(\bar{g})^{2/3}$ is a solution of the system (12) and of the equation $\frac{\partial}{\partial x}\hat{a} = \hat{a}$, and, therefore, gives us a solution of the homogeneous part of the above

mentioned linear relations. Direct calculations show that $\det \begin{pmatrix} b_1 & m_{12} & m_{13} \\ b_2 & m_{22} & m_{23} \\ b_3 & m_{32} & m_{33} \end{pmatrix} \neq 0$, see (64). This gives us a contradiction which proves Theorem 2 for the metrics 1a, 1b from Theorem 2.

Let us also note that, similar to the **second case**, the proof for the case 1a implies the proof for the case 2a. Indeed, the formulas for (the components of) the metrics 1a are real-analytic, and we can allow x to be a complex variable and y to be its conjugate, since it changes neither differentiation nor algebraic operations with the (components of the) metric. After this change the metric 1a become, up to a multiplication by a constant, the metric 2a, and therefore our proof that the metrics 1a has two-dimensional \mathcal{A} , which uses only algebraic operation and differentiation, is also a proof for the case 2a.

4.3 Calculations related to proof of Theorem 2 for the metrics from cases 1a, 1b, 1c from Theorem 1 assuming that the matrix of L_v is as the first matrix of (51)

For the metrics 1a, 1b, 1c from Theorem 1, the projective connections in the new coordinates $x = \frac{x_{old}+y_{old}}{2}$, $y = \frac{x_{old}-y_{old}}{2}$ are respectively given by

$$\begin{aligned}
y'' &= \frac{(e^{6y}+c^2e^{-6y}+2c)}{8cy} + \frac{3(4yc-e^{6y}+c^2e^{-6y})}{8cy}y' \\
&+ \frac{(-2c+3e^{6y}+3c^2e^{-6y})}{8cy}(y')^2 - \frac{(12yc-c^2e^{-6y}+e^{6y})}{8cy}(y')^3 \quad (59) \\
y'' &= \frac{e^{6\lambda y}+c^2e^{-6\lambda y}+2c\cos(2y)}{4c\sin(2y)} + 3\frac{-e^{6\lambda y}+c^2e^{-6\lambda y}+2c\lambda\sin(2y)}{4c\sin(2y)}y' \\
&+ \frac{3e^{6\lambda y}+3c^2e^{-6\lambda y}-2c\cos(2y)}{4c\sin(2y)}(y')^2 - \frac{e^{6\lambda y}-c^2+6\lambda c\sin(2y)}{4c\sin(2y)}(y')^3 \\
y'' &= \frac{-\lambda-\lambda e^{4y}+\lambda e^{4y(\lambda-1)}c^2+\lambda e^{4\lambda y}c^2+\lambda e^{2y(2+\lambda)}c-\lambda e^{2y(-2+\lambda)}c}{(2e^{2\lambda y}-2)^2} \\
&+ \frac{-4e^{4\lambda y}c^2+8ce^{2\lambda y}-4+3\lambda e^{2y(2+\lambda)}c+\lambda e^{4\lambda y}c^2-2e^{2\lambda y}\lambda c+3\lambda e^{2y(-2+\lambda)}c-3\lambda e^{4y(\lambda-1)}c^2+\lambda-3\lambda e^{4y}}{(2e^{2\lambda y}-2)^2}y' \\
&+ \frac{3\lambda e^{4y(\lambda-1)}c^2+3\lambda e^{2y(2+\lambda)}c-\lambda e^{4\lambda y}c^2-3\lambda e^{2y(-2+\lambda)}c+\lambda-3\lambda e^{4y}}{(2e^{2\lambda y}-2)^2}(y')^2 \\
&+ \frac{4e^{4\lambda y}c^2-\lambda e^{4y(\lambda-1)}c^2-8ce^{2\lambda y}-\lambda-\lambda e^{4y}+4+\lambda e^{2y(2+\lambda)}c-\lambda e^{4\lambda y}c^2+2e^{2\lambda y}\lambda c+\lambda e^{2y(-2+\lambda)}c}{(2e^{2\lambda y}-2)^2}(y')^3
\end{aligned}$$

We see that all coefficients K_i of the projective connection are independent of x (which was clear in advance since the vector field $\frac{\partial}{\partial x}$ is projective). Substituting (54) in (12), we obtain

$$\begin{aligned}
a_{11}\mu - \frac{2}{3}K_1a_{11} + 2K_0a_{12} &= 0, \\
a'_{11} + 2a_{12}\mu - \frac{4}{3}K_2a_{11} + \frac{2}{3}K_1a_{12} + 2K_0a_{22} &= 0, \\
2a'_{12} + a_{22}\mu - 2K_3a_{11} - \frac{2}{3}K_2a_{12} + \frac{4}{3}K_1a_{22} &= 0, \\
a'_{22} - 2K_3a_{12} + \frac{2}{3}K_2a_{22} &= 0,
\end{aligned}$$

which is equivalent to the system

$$\left. \begin{aligned} 0 &= a_{11}\mu - \frac{2}{3}K_1a_{11} + 2K_0a_{12} \\ a'_{11} &= -2a_{12}\mu + \frac{4}{3}K_2a_{11} - \frac{2}{3}K_1a_{12} - 2K_0a_{22} \\ a'_{12} &= -\frac{1}{2}a_{22}\mu + K_3a_{11} + \frac{1}{3}K_2a_{12} - \frac{2}{3}K_1a_{22} \\ a'_{22} &= 2K_3a_{12} - \frac{2}{3}K_2a_{22}. \end{aligned} \right\} \quad (60)$$

The coefficients K_i of our connections are functions of y only. Differentiated the first equation of (60) by y and substituting the values of y -derivative of a_{ij} given by the last three equations, we obtain the following equation as a differential consequence of the equations (60): if a_{ij} satisfy (60), then they must satisfy the equation below.

$$0 = \frac{4\mu K_2 - 2K'_1 + 6K_0K_3 - \frac{8}{3}K_1K_2}{3}a_{11} + \frac{2\mu K_1 + \frac{4}{3}K_1^2 + 6K'_0 - 6\mu^2 + 2K_0K_2}{3}a_{12} - 3\mu K_0a_{22}$$

Differentiating this equation by y and substituting the values of y -derivative of a_{ij} from the last three equations of (60), we obtain another linear homogeneous equation on a_{ij} , whose coefficients are polynomial expressions in K_i and their derivatives.

Thus, we have three homogeneous linear equations on three unknown functions a_{ij} , which must be satisfied if a_{ij} satisfy (60). Determinant of the corresponding 3×3 - matrix is given by

$$\begin{aligned} &-2\mu^6 + \frac{14}{3}\mu^3K_1K_0K_2 + 10K'_0\mu^2K_0K_2 - \frac{32}{9}K'_0K_1^2K_0K_2 - \frac{40}{27}\mu K_1^3K_0K_2 + \frac{64}{729}K_1^6 + 6\mu^2K_0K_0'' - \\ &10\mu^4K_0K_2 - \frac{16}{3}\mu^2K_1^2K'_0 - \frac{14}{3}\mu^3K_1K'_0 - \frac{64}{81}K_1^4K_0K_2 + \frac{64}{27}K_1^3K_0^2K_3 - \frac{64}{81}K_1^3K_0K'_1 + \frac{16}{9}K_0^2K_2^2K_1^2 + \\ &\frac{8}{3}\mu K_1(K'_0)^2 + \frac{40}{27}\mu K_1^3K'_0 + \frac{8}{3}\mu K_1^2K_0^2K_3 - \frac{32}{9}\mu K_1^2K_0K'_1 + \frac{20}{3}\mu K_0^2K_2K'_1 - 8\mu K_0^3K_2K_3 + \frac{16}{3}\mu^2K_1^2K_0K_2 + \\ &4\mu^2K_1K_0K'_1 - \frac{32}{3}K_1K_0^3K_3 - 8\mu^2K_2^2K_0^2 - 12K_0^3\mu K_3 - 4K_1\mu K_0K_0'' + \frac{32}{3}K_1K'_0K_0^2K_3 - \frac{32}{9}K_1K'_0K_0K'_1 - \\ &\frac{4}{3}\mu K'_0K_1K_0K_2 - \frac{32}{3}K_1K_0^3K_2K_3 + 4K_1\mu K_0^2K'_2 + 4K_0^2\mu K'_1 + \frac{8}{3}\mu K_0^2K_2^2K_1 - 16\mu K'_0K_0^2K_3 - \frac{8}{3}\mu K'_0K_0K'_1 + \\ &\frac{32}{9}K_1K_0^2K_2K'_1 + \frac{16}{81}\mu K_1^5 + \frac{64}{81}K_1^4K'_0 - \frac{8}{9}\mu^2K_1^4 + \frac{10}{3}\mu^4K_1^2 + 10K'_0\mu^4 - \frac{28}{27}\mu^3K_1^3 - 8\mu^2K_0'^2 + \frac{16}{9}K_0^2K_1'^2 + \\ &16K_0^4K_3^2 + \frac{16}{9}K_1^2(K'_0)^2 - 14\mu^3K_0K_3 - 6\mu^2K_0^2K'_2 + \frac{14}{3}\mu^3K_0K'_1. \end{aligned}$$

Though the formula for the determinant looks ugly, for explicit K_i given at the beginning of this paragraph, one can calculate it (Maple[®] does it within few seconds, a human needs around one hour for it). Calculating this formula for the K_i corresponding to the projective connections corresponding to the metrics 1a, 1b, 1c (explicit formulas for the projective connections are at the beginning of the present paragraph), we obtain that the result is not zero implying the system (12) corresponding to the projective connections corresponding to the metrics g from cases 1a– 2c of theorem 1 does not admit three dimensional \mathcal{A} under the additional assumption that the matrix of L_v is the first matrix of (51).

4.4 Calculations related to the proof of Theorem 2 for the metric 1a from Theorem 1 assuming that the matrix of L_v is as the second matrix of (51)

For the metric 1a, we consider the coordinate system $x_{new} = \frac{x_{old} + y_{old}}{2}$, $y_{new} = \frac{x_{old} - y_{old}}{2}$.

In this coordinate system, the projective vector field is $\frac{\partial}{\partial x}$, and the projective connection of g is (59). Direct calculations shows that the matrix of \bar{a} is given by

$$\left(\begin{array}{cc} \frac{e^{4x}(ce^{-3x-3y}(y-x) - e^{-3x+3y}(x+y))}{4\sqrt[3]{yc}^{\frac{2}{3}}} & \frac{e^{4x}(ce^{-3x-3y}(y-x) + e^{-3x+3y}(x+y))}{4\sqrt[3]{yc}^{\frac{2}{3}}} \\ \frac{e^{4x}(ce^{-3x-3y}(y-x) + e^{-3x+3y}(x+y))}{4\sqrt[3]{yc}^{\frac{2}{3}}} & \frac{e^{4x}(ce^{-3x-3y}(y-x) - e^{-3x+3y}(x+y))}{4\sqrt[3]{yc}^{\frac{2}{3}}} \end{array} \right). \quad (61)$$

Direct computations shows that the following matrix P is a partial solution of the equation (57).

$$\left(\begin{array}{cc} \frac{x(-2ye^{3y} - cxe^{-3y} + 2yc e^{-3y} - xe^{3y})e^x}{8\sqrt[3]{yc}^{\frac{2}{3}}} & \frac{x(2ye^{3y} - cxe^{-3y} + 2yc e^{-3y} + xe^{3y})e^x}{8\sqrt[3]{yc}^{\frac{2}{3}}} \\ \frac{x(2ye^{3y} - cxe^{-3y} + 2yc e^{-3y} + xe^{3y})e^x}{8\sqrt[3]{yc}^{\frac{2}{3}}} & \frac{x(-2ye^{3y} - cxe^{-3y} + 2yc e^{-3y} - xe^{3y})e^x}{8\sqrt[3]{yc}^{\frac{2}{3}}} \end{array} \right). \quad (62)$$

Thus, the solution of the equation (57) has the form (58). Substituting this ansatz in the equations (12) and solving the last three equations with respect to the first derivatives, we obtain the system

$$\left. \begin{aligned} 0 &= a_{11} - \frac{2}{3} K_1 a_{11} + 2 K_0 a_{12} &+ \frac{y^{\frac{2}{3}}(-e^{3y} + c e^{-3y})}{4c^{\frac{2}{3}}} \\ a'_{11} &= -2 a_{12} + \frac{4}{3} K_2 a_{11} - \frac{2}{3} K_1 a_{12} - 2 K_0 a_{22} &- \frac{y^{\frac{2}{3}}(e^{3y} + c e^{-3y})}{4c^{\frac{2}{3}}} \\ a'_{12} &= -\frac{1}{2} a_{22} + K_3 a_{11} + \frac{1}{3} K_2 a_{12} - \frac{2}{3} K_1 a_{22} &+ \frac{y^{\frac{2}{3}}(e^{3y} - c e^{-3y})}{4c^{\frac{2}{3}}} \\ a'_{22} &= 2 K_3 a_{12} - \frac{2}{3} K_2 a_{22}, \end{aligned} \right\} \quad (63)$$

where K_i are the coefficients of the projective connection (59).

The first equation of (63) plays the role of the first equation of (55). Differentiating the first equation of (63) by y and substituting the values of derivatives from the last three equations of (63) inside, we obtain the following nonhomogeneous linear equation on a_{ij} , which plays role of the second equation of (55).

$$\begin{aligned} &- \left(38 \sqrt[3]{c} \frac{5}{3} e^{18y} - 108 y^{4/3} c \frac{11}{3} e^6 y + 72 y^{4/3} c \frac{8}{3} e^{12y} - 38 \sqrt[3]{c} \frac{11}{3} e^6 y - 108 y^{4/3} c \frac{5}{3} e^{18y} - 9 \sqrt[3]{c} \frac{2}{3} e^{24y} + 9 c^{14/3} \sqrt[3]{y} \right) a_{11} \\ &- \left(72 y^{4/3} c \frac{11}{3} e^6 y + 20 \sqrt[3]{c} \frac{5}{3} e^{18y} - 72 y^{4/3} c \frac{5}{3} e^{18y} - 9 \sqrt[3]{c} \frac{2}{3} e^{24y} + 20 \sqrt[3]{c} \frac{11}{3} e^6 y - 9 c^{14/3} \sqrt[3]{y} + 58 \sqrt[3]{c} \frac{8}{3} e^{12y} \right) a_{12} \\ &- \left(72 y^{4/3} c \frac{8}{3} e^{12y} + 36 y^{4/3} c \frac{5}{3} e^{18y} + 36 y^{4/3} c \frac{11}{3} e^6 y \right) a_{22} \\ &= -25 c^3 e^9 y y^2 + 72 c^3 y^3 e^9 y + 72 y^3 c^{15} y c^2 + 9 e^{21} y y^2 c + 25 e^{15} y y^2 c^2 - 9 c^4 e^3 y y^2. \end{aligned}$$

Differentiating this equation by y and substituting the values of derivatives from the last three equations of (63) inside, we obtain the following nonhomogeneous linear equation on a_{ij} , which plays role of the third equation of (55).

$$\begin{aligned} &\left(-132 e^{24y} y c \frac{5}{3} + 103 e^{18y} y c \frac{8}{3} + 132 c \frac{17}{3} + 27 e^{30y} y c \frac{2}{3} + 2640 c \frac{14}{3} y e^6 y + 252 c \frac{17}{3} y + 1128 e^{12y} y c \frac{11}{3} - 103 c \frac{14}{3} e^6 y \right. \\ &- 27 c \frac{20}{3} e^{-6y} + 6048 y^2 c \frac{14}{3} e^6 y - 6912 y^2 c \frac{11}{3} e^{12y} + 1980 e^{24y} y c \frac{5}{3} + 14688 e^{18y} y y^2 c \frac{8}{3} - 4656 c \frac{8}{3} e^{18y} y \left. \right) a_{11} \\ &+ \left(-144 c \frac{17}{3} y + 8640 e^{18y} y y^2 c \frac{8}{3} - 78 c \frac{17}{3} - 2592 c \frac{14}{3} y e^6 y - 107 c \frac{14}{3} e^6 y - 107 e^{18y} y c \frac{8}{3} + 3456 y^2 c \frac{11}{3} e^{12y} + 27 e^{30y} y c \frac{2}{3} \right. \\ &- 5568 e^{12y} y c \frac{11}{3} + 27 c \frac{20}{3} e^{-6y} - 5184 y^2 c \frac{14}{3} e^6 y + 1872 e^{24y} y c \frac{5}{3} - 4 e^{12y} y c \frac{11}{3} - 1248 c \frac{8}{3} e^{18y} y - 78 e^{24y} y c \frac{5}{3} \left. \right) a_{12} \\ &- \left(6048 e^{18y} y y^2 c \frac{8}{3} + 336 c \frac{8}{3} e^{18y} y y + 6912 y^2 c \frac{11}{3} e^{12y} + 108 e^{24y} y c \frac{5}{3} + 864 y^2 c \frac{14}{3} e^6 y + 456 e^{12y} y c \frac{11}{3} + 108 c \frac{17}{3} y + 336 c \frac{14}{3} y e^6 y \right) a_{22} \\ &= 1872 e^{21y} y y \frac{8}{3} c^2 + 57 e^{21y} y y \frac{5}{3} c^2 - 27 c^6 y \frac{5}{3} e^{-3y} + 27 e^{27y} y y \frac{5}{3} c + 286 c^3 e^{15y} y y \frac{5}{3} + 286 c^4 e^9 y y \frac{5}{3} + 5184 c^4 e^9 y y \frac{11}{3} - 57 c^5 y \frac{5}{3} e^3 y \\ &+ 8640 e^{15y} y y \frac{11}{3} c^3 + 4944 e^{15y} y y \frac{8}{3} c^3 + 144 c^4 e^9 y y \frac{8}{3} 144 c^5 y \frac{8}{3} e^3 y \end{aligned}$$

Direct calculations show that $\det \begin{pmatrix} b_1 & m_{12} & m_{13} \\ b_2 & m_{22} & m_{23} \\ b_3 & m_{32} & m_{33} \end{pmatrix}$ is equal to

$$\frac{9}{64c^{\frac{8}{3}} y^{7/3}} \frac{18 e^{-9y} y^4 c^4 y - 18 c^2 e^3 y y + 32 c^3 e^{-3y} + 32 c^2 e^3 y - 18 e^9 y y c + 9 e^{-9y} y c^4 + 18 c^3 e^{-3y} y - c^5 e^{-15y} + 9 e^9 y c - e^{15y}}{64c^{\frac{8}{3}} y^{7/3}}. \quad (64)$$

We see that it is not zero, which gives us a contradiction which proves Theorem 2 for the metric from the case 1a of Theorem 1.

4.5 Proof of Theorem 2 for the metrics 3a, 3b, 3c, and 3d

In all these cases the metric has the form $2(Y(y) + x)dx dy$. We first explain that the dimension of the space \mathcal{A} coincides with the dimension of the space of integrals quadratic in momenta for the Hamiltonian

$$H : T^*D \rightarrow \mathbb{R}, \quad H(x, y, p_x, p_y) := \frac{p_x p_y}{Y(y) + x}.$$

Indeed, as we explained in §2.1, for every solution a the function $(\det(g))^{2/3} a_{ij} \xi^i \xi^j$ is an integral of the geodesic flow of g , and vice versa. Since the mapping $a \mapsto (\det(g))^{2/3} a$ is linear and bijective, the dimensions of the space \mathcal{A} and of the space of integrals quadratic in momenta coincide.

Note that the space of the integrals quadratic in momenta is at least two-dimensional. Indeed, every linear combination of the Hamiltonian H and of the integral coming from the projectively equivalent metric (7) by formula (15) is an integral. In the notations below, these integrals will correspond to $a = \text{const}$, $c = 0$. Our goal is to show, that in the cases 3a, 3b, 3c all integrals have $a = \text{const}$, $c = 0$, and that in the case 3d there exists an additional linearly independent integral.

Suppose a function $f : T^*D \rightarrow \mathbb{R}$ of the form $a(x, y)p_x^2 + b(x, y)p_x p_y + c(x, y)p_y^2$ is an integral for the geodesic flow of g . Then, the condition $0 = \{H, f\}$, after multiplication by $-(Y+x)^2$, reads

$$\begin{aligned} 0 &= -(Y+x)^2 \cdot \left\{ \frac{p_x p_y}{Y+x}, a p_x^2 + b p_x p_y + c p_y^2 \right\} \\ &= p_x^3 (Y+x) a_y + p_x^2 p_y ((Y+x) a_x + (Y+x) b_y + 2a + Y' b) \\ &\quad + p_x p_y^2 ((Y+x) b_x + (Y+x) c_y + b + 2Y' c) + p_y^3 (Y+x) c_x, \end{aligned}$$

i.e., is equivalent to the following system of PDE:

$$\left. \begin{aligned} a_y &= 0 \\ (Y+x) a_x + (Y+x) b_y + 2a + Y' b &= 0 \\ (Y+x) b_x + (Y+x) c_y + b + 2Y' c &= 0 \\ c_x &= 0 \end{aligned} \right\} \quad (65)$$

We see that the first (the last, respectively) equation of (65) implies that the function a (c , respectively) is a function of the variable x (y , respectively) only.

Solving the second and the third equations with respect to the derivatives of b , we obtain

$$b_y = -\frac{2a + Y' b}{Y+x} - a', \quad b_x = -\frac{b + 2Y' c}{Y+x} - c'.$$

Substituting these expressions for the derivatives of b in the identity $\frac{\partial b_x}{\partial y} - \frac{\partial b_y}{\partial x} = 0$, we obtain

$$-a'' x + c'' x + c'' Y - 3a' + 3c' Y' - a'' Y + 2Y'' c = 0. \quad (66)$$

Taking the $\frac{\partial^3}{\partial^2 x \partial y}$ -derivative of this equation, we obtain $a'''' Y' = 0$. Since the function Y from cases 3a, 3b, 3c, 3d is not constant, we can assume $Y' \neq 0$. Then, $a = \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0$, where α_i are constants. Substituting this in (66), we obtain

$$-15\alpha_3 x^2 + (c'' - 8\alpha_2 - 6Y\alpha_3) x - 3\alpha_1 + 3c' Y' + c'' Y + 2Y'' c - 2Y\alpha_2 = 0. \quad (67)$$

Left-hand side of this equation is a polynomial in x whose coefficients depend on y only. They must be zero implying $\alpha_3 = 0$, $c = 4\alpha_2 y^2 + \beta_1 y + \beta_0$. Then, the equation (67) reads

$$6Y\alpha_2 - 3\alpha_1 + (3\beta_1 + 24\alpha_2 y) Y' + (2\beta_0 + 2\beta_1 y + 8\alpha_2 y^2) Y'' = 0. \quad (68)$$

If $\alpha_1 = \alpha_2 = \beta_1 = \beta_0 = 0$, then $c = 0$, $a = \text{const}$ implying that the integral is a linear combination of the Hamiltonian and the integral coming from the projectively equivalent metric (7) by formula (15). Otherwise (68) is a ODE on the function Y . Substituting the functions Y from the cases 3a, 3b, 3c of Theorem 1 we see that they are not solutions of this ODE. Thus, the metrics from the cases 3a, 3b, 3c from Theorem 1 have 2-dimensional \mathcal{A} . Substituting the functions Y from the case 3d from Theorem 1, we see that it is a solutions of this ODE, if and only if $\beta_1 = \alpha_2 = 0$, $4\beta_0 = 3\alpha_1$. We see that there is precisely one additional parameter (β_0) we can freely choose to construct the integral, i.e., the space of the integrals is at most three-dimensional. Direct calculations show that as the additional integral we can take the integral corresponding to the metric \tilde{g} by formula (15).

5 Proof of Theorem 3

The goal is to show that no metric from Theorem 1 has a Killing vector field. It is sufficient to do it for the cases 1a – 2c only, since in view of §4.5, in the cases 3a – 3d we know the space of quadratic integrals of the metrics 3a – 3d, so it is sufficient to check that no quadratic integral is degenerate at every point, which is an easy exercise. Moreover, in the case 3d the space of quadratic integrals is precisely 3-dimensional implying the metric admit no Killing vector field, see [20, Section 5].

We will use the following approach which was known to Darboux [11, §§688,689] and Eisenhart [15, pp. 323–325], see also [20] for an equivalent approach leading to similar calculations. For every g from the cases 1a – 2c of Theorem 1, let us consider the following functions on D^2 :

- scalar curvature $R := \sum_{i,j,k} R_{ijk}^i g^{jk}$, where R_{hjk}^i is the curvature tensor of g ,
- The square of the length of the derivative of the scalar curvature $L := \sum_{i,j} g^{ij} \frac{\partial R}{\partial x_i} \frac{\partial R}{\partial x_j}$,
- The laplacian of the scalar curvature $\Delta := \frac{1}{\sqrt{\det(g)}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{\det(g)} \frac{\partial R}{\partial x_j} \right)$.

If the metric admits a Killing vector field K , then in a coordinate system (x_1, x_2) such that $K = \frac{\partial}{\partial x_1}$, all these functions depend of x_2 only. Then, the differentials dR, dL are proportional, and the differentials $dR, d\Delta$ are proportional. Then, in every coordinate system (x, y) the following determinants are zero:

$$\det \begin{pmatrix} \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} \\ \frac{\partial L}{\partial x} & \frac{\partial L}{\partial y} \end{pmatrix}, \quad \det \begin{pmatrix} \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} \\ \frac{\partial \Delta}{\partial x} & \frac{\partial \Delta}{\partial y} \end{pmatrix}. \quad (69)$$

Calculating these determinants for all metrics from the cases 1a – 2c, we see that in every case they are not zero implying that the metrics admit no Killing vector field. Note that it is sufficient to calculate the determinants for the metrics from the cases 1a – 1c only, since the cases 2a – 2c, up to multiplication by a constant, can be obtained from the cases 1a – 1c by replacing x by z and y by \bar{z} .

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A Appendix: Dini theorem for pseudo-Riemannian metrics

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A.1 Introduction

Consider a Riemannian or a pseudo-Riemannian metric $g = (g_{ij})$ on a surface M^2 . We say that a metric \bar{g} on the same surface is *projectively equivalent* to g , if every geodesic of \bar{g} is a reparametrized geodesic of g . In 1865 Beltrami [4] asked⁹ to describe all pairs of projectively equivalent Riemannian metrics on surfaces. From the context it is clear that he considered this problem locally, in a neighbourhood of almost every point.

Theorem A below, which is the main result of this note (which is a short version of [8]), gives an answer to the following generalization of the question of Beltrami: we allow the metrics g and \bar{g} to be pseudo-Riemannian.

Theorem A. *Let g, \bar{g} be projectively equivalent metrics on M^2 , and $\bar{g} \neq \text{const} \cdot g$ for every $\text{const} \in \mathbb{R}$. Then, in the neighbourhood of almost every point there exist coordinates (x, y) such that the metrics are as in the following table.*

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⁹Italian original from [4]: La seconda ... generalizzazione ... del nostro problema, vale a dire: riportare i punti di una superficie sopra un'altra superficie in modo che alle linee geodetiche della prima corrispondano linee geodetiche della seconda.

	<i>Liouville Case</i>	<i>Complex-Liouville Case</i>	<i>Jordan-block Case</i>
g	$(X(x) - Y(y))(dx^2 \pm dy^2)$	$2\Im(h)dxdy$	$(1 + xY'(y))dxdy$
\bar{g}	$\left(\frac{1}{Y(y)} - \frac{1}{X(x)}\right) \left(\frac{dx^2}{X(x)} \pm \frac{dy^2}{Y(y)}\right)$	$-\left(\frac{\Im(h)}{\Im(h)^2 + \Re(h)^2}\right)^2 dx^2$ $+ 2\frac{\Re(h)\Im(h)}{(\Im(h)^2 + \Re(h)^2)^2} dxdy$ $+ \left(\frac{\Im(h)}{\Im(h)^2 + \Re(h)^2}\right)^2 dy^2$	$\frac{1+xY'(y)}{Y(y)^4} (-2Y(y)dxdy$ $+ (1 + xY'(y))dy^2)$

where $h := \Re(h) + i \cdot \Im(h)$ is a holomorphic function of the variable $z := x + i \cdot y$.

Remark A. It is natural to consider the metrics from the Complex-Liouville Case as the complexification of the metrics from the Liouville Case: indeed, in the complex coordinates $z = x + i \cdot y$, $\bar{z} = x - i \cdot y$, the metrics have the form

$$\begin{aligned} ds_g^2 &= -\frac{1}{4}(\overline{h(z)} - h(z))(d\bar{z}^2 - dz^2), \\ ds_{\bar{g}}^2 &= -\frac{1}{4}\left(\frac{1}{h(z)} - \frac{1}{\overline{h(z)}}\right)\left(\frac{d\bar{z}^2}{\overline{h(z)}} - \frac{dz^2}{h(z)}\right) \end{aligned}$$

(this form is used in the proof of Theorem 1).

Remark B. In the Jordan-block Case, if $dY \neq 0$ (which is always the case at almost every point, if the restriction of g to any neighborhood does not admit a Killing vector field), after a local coordinate change, the metrics g and \bar{g} have the form

$$\begin{aligned} ds_g^2 &= (\tilde{Y}(y) + x) dxdy \\ ds_{\bar{g}}^2 &= -\frac{2(\tilde{Y}(y) + x)}{y^3} dxdy + \frac{(\tilde{Y}(y) + x)^2}{y^4} dy^2 \end{aligned}$$

(this form is used in the proof of Theorem 1).

We see that the metric g from Complex-Liouville and Jordan-block Cases always have signature $(+, -)$, and the metric g from the Liouville Case has signature $(+, +)$ or $(-, -)$, if the sign “ \pm ” is “ $+$ ”. In this case, the formulas from Theorem A are precisely the formulas obtained by Dini in [12].

We do not insist that we are the first to find these normal forms of projectively equivalent pseudo-Riemannian metrics. According to [2], a description of projectively equivalent metrics was obtained by P. Shirokov in [40]. Unfortunately, we were not able to find the reference [40] to check it. The result of Theorem A could be even more classical, see Remark D.

Given two projectively equivalent metrics, it is easy to understand what case they belong to. Indeed, the $(1, 1)$ -tensor $G_j^i := \sum_{\alpha=1}^2 \bar{g}_{j\alpha} g^{i\alpha}$, where $g^{i\alpha}$ is inverse to $g_{i\alpha}$, has two different real eigenvalues in the Liouville Case, two complex-conjugated eigenvalues in the Complex-Liouville Case, and is (conjugate to) a Jordan-block in the Jordan-block Case.

There exists an interesting and useful connection of projectively equivalent metrics with integrable systems.

Recall that a function $F : T^*M^2 \rightarrow \mathbb{R}$ is called *an integral* of the geodesic flow of g , if $\{H, F\} = 0$, where $H := \frac{1}{2}g^{ij}p_i p_j : T^*M^2 \rightarrow \mathbb{R}$ is the kinetic energy corresponding to the metric, and $\{, \}$ is the standard Poisson bracket on T^*M^2 . Geometrically, this condition means that the function is constant on the orbits of the Hamiltonian system with the Hamiltonian H . We say the integral F is *quadratic in momenta*, if for every local coordinate system (x, y) on M^2 it has the form

$$F(x, y, p_x, p_y) = a(x, y)p_x^2 + b(x, y)p_x p_y + c(x, y)p_y^2 \quad (70)$$

in the canonical coordinates (x, y, p_x, p_y) on T^*M^2 . Geometrically, the formula (70) means that the restriction of the integral to every cotangent space $T_p^*M^2 \cong \mathbb{R}^2$ is a homogeneous quadratic function. Of course, H itself is an integral quadratic in the momenta for g . We will say that the integral F is *nontrivial*, if $F \neq \text{const} \cdot H$ for all $\text{const} \in \mathbb{R}$.

Theorem B. *Suppose the metric g on M^2 admits a nontrivial integral quadratic in momenta. Then, in a neighbourhood of almost every point there exist coordinates (x, y) such that the metric and the integral are as in the following table*

	<i>Liouville Case</i>	<i>Complex-Liouville Case</i>	<i>Jordan-block Case</i>
g	$(X(x) - Y(y))(dx^2 \pm dy^2)$	$\Im(h)dxdy$	$(1 + xY'(y))dxdy$
F	$\frac{X(x)p_y^2 \pm Y(y)p_x^2}{X(x) - Y(y)}$	$p_x^2 - p_y^2 + 2\frac{\Re(h)}{\Im(h)}p_xp_y$	$p_x^2 - 2\frac{Y(y)}{1+xY'(y)}p_xp_y$

where $h := \Re(h) + i \cdot \Im(h)$ is a holomorphic function of the variable $z := x + iy$.

Indeed, as it was shown in [24, 25], and as it was essentially known to Darboux [11, §§600–608], if two metrics g and \bar{g} are projectively equivalent, then

$$I : TM^2 \rightarrow \mathbb{R}, \quad I(\xi) := \bar{g}(\xi, \xi) \left(\frac{\det(g)}{\det(\bar{g})} \right)^{2/3} \quad (71)$$

is an integral of the geodesic flow of g . Moreover, it was shown in [9, §2.4], see also [27], the above statement is proven to be true¹⁰ in the other direction: if the function (70) is an integral for the geodesic flow of g , then the metrics g and \bar{g} are projectively equivalent. Thus, Theorem A and Theorem B are equivalent. In this paper, we will actually prove Theorem B obtaining Theorem A as its consequence.

Remark C. The corresponding natural Hamiltonian problem on the hyperbolic plane has been recently treated in [38] following the approach used by Rosquist and Uggla [39].

Remark D. The formulas that will appear in the proof are very close to that in §593 of [11]. Darboux worked over complex numbers and therefore did not care about whether the metrics are Riemannian or pseudo-Riemannian. For example, Liouville and Complex-Liouville Case are the same for him. Moreover, in §594, Darboux gets the formulas that are very close to that of Jordan-block Case, though he was interested in the Riemannian case only, and, hence, treated this “imaginary” case as not interesting.

A.2 Proof of Theorem B (and, hence, of Theorem A)

If the metric g has signature $(+, +)$ or $(-, -)$, Theorem A and, hence, Theorem B, were obtained by Dini in [12]. Below we assume that the metric g has signature $(+, -)$.

A.2.1 Admissible coordinate systems and Birkhoff-Kolokoltsov forms

Let g be a pseudo-Riemannian metric on M^2 of signature $(+, -)$. Consider (and fix) two linear independent vector fields V_1, V_2 on M^2 such that

- $g(V_1, V_1) = g(V_2, V_2) = 0$ and
- $g(V_1, V_2) > 0$.

Such vector fields always exist locally (and, since our result is local, this is sufficient for our proof).

We will say that a local coordinate system (x, y) is *admissible*, if the vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are proportional to V_1, V_2 with positive coefficient of proportionality:

$$V_1(x, y) = \lambda_1(x, y) \frac{\partial}{\partial x}, \quad V_2(x, y) = \lambda_2(x, y) \frac{\partial}{\partial y}, \quad \text{where } \lambda_i > 0, \quad i = 1, 2.$$

Obviously,

- admissible coordinates exist in a sufficiently small neighborhood of every point,
- the metric g in admissible coordinates has the form

$$ds^2 = f(x, y)dxdy, \quad \text{where } f > 0, \quad (72)$$

¹⁰with a good will, one also can attribute this result to Darboux

- two admissible coordinate systems in one neighbourhood are connected by

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} x_{new}(x_{old}) \\ y_{new}(y_{old}) \end{pmatrix}, \quad \text{where } \frac{dx_{new}}{dx_{old}} > 0, \frac{dy_{new}}{dy_{old}} > 0. \quad (73)$$

Lemma A. Let (x, y) be an admissible coordinate system for g . Let F given by (70) be an integral for g . Then, $B_1 := \frac{1}{\sqrt{|a(x,y)|}}dx$ ($B_2 := \frac{1}{\sqrt{|c(x,y)|}}dy$, respectively) is a 1-form, which is defined at points such that $a \neq 0$ ($c \neq 0$, respectively). Moreover, the coefficient a (c , respectively) depends only on x (y , respectively), which in particular imply that the forms B_1, B_2 are closed.

Remark E. The forms B_1, B_2 are not the direct analog of the ‘‘Birkhoff’’ 2-form introduced by Kolokoltsov in [19]. In a certain sense, they are the real analog of the different branches of the square root of the Birkhoff form.

Proof of Lemma A. The first part of the statement, namely that the $\frac{1}{\sqrt{|a|}}dx$ ($\frac{1}{\sqrt{|c|}}dy$, respectively) transforms as a 1-form under admissible coordinate changes is evident: indeed, after the coordinate change (73), the momenta transform as follows: $p_{x_{old}} = p_{x_{new}} \frac{dx_{new}}{dx_{old}}$, $p_{y_{old}} = p_{y_{new}} \frac{dy_{new}}{dy_{old}}$. Then, the integral F in the new coordinates has the form

$$\underbrace{a \left(\frac{dx_{new}}{dx_{old}} \right)^2}_{a_{new}} p_{x_{new}}^2 + \underbrace{b \frac{dx_{new}}{dx_{old}} \frac{dy_{new}}{dy_{old}}}_{b_{new}} p_{x_{new}} p_{y_{new}} + \underbrace{c \left(\frac{dy_{new}}{dy_{old}} \right)^2}_{c_{new}} p_{y_{new}}^2.$$

Then, the formal expression $\frac{1}{\sqrt{|a|}}dx_{old}$ ($\frac{1}{\sqrt{|c|}}dy_{old}$, respectively) transforms in

$$\frac{1}{\sqrt{|a|}} \frac{dx_{old}}{dx_{new}} dx_{new} \quad \left(\frac{1}{\sqrt{|c|}} \frac{dy_{old}}{dy_{new}} dy_{new}, \text{ respectively} \right),$$

which is precisely the transformation law of 1-forms.

Let us prove that the forms are closed. If g is given by (72), its Hamiltonian H is given by $\frac{p_x p_y}{2f}$, and the condition $0 = \{H, F\}$ reads

$$\begin{aligned} 0 &= \left\{ \frac{p_x p_y}{2f}, ap_x^2 + bp_x p_y + cp_y^2 \right\} \\ &= \frac{1}{f} (p_x^3 (fa_y) + p_x^2 p_y (fa_x + fb_y + 2f_x a + f_y b) + p_y p_x^2 (fb_x + fc_y + f_x b + 2f_y) + p_y^3 (c_x f)), \end{aligned}$$

i.e., is equivalent to the following system of PDE:

$$\begin{cases} a_y = 0 \\ fa_x + fb_y + 2f_x a + f_y b = 0 \\ fb_x + fc_y + f_x b + 2f_y c = 0 \\ c_x = 0 \end{cases} \quad (74)$$

Thus, $a = a(x)$, $c = c(y)$, which is equivalent to $B_1 := \frac{1}{\sqrt{|a|}}dx$ and $B_2 := \frac{1}{\sqrt{|c|}}dy$ are closed forms (assuming $a \neq 0$ and $c \neq 0$). \square

Remark 1. For further use let us formulate one more consequence of the equations (74): if $a \equiv c \equiv 0$ in a neighborhood of a point, then $bf = \text{const}$ implying $F \equiv \text{const} \cdot H$ in the neighborhood.

Assume $a \neq 0$ ($c \neq 0$, respectively) at a point P_0 . For every point P_1 in a small neighbourhood U of P_0 consider

$$x_{new} := \int_{\substack{\gamma: [0, 1] \rightarrow U \\ \gamma(0) = P_0 \\ \gamma(1) = P_1}} B_1, \quad \left(y_{new} := \int_{\substack{\gamma: [0, 1] \rightarrow U \\ \gamma(0) = P_0, \\ \gamma(1) = P_1}} B_2, \text{ respectively} \right). \quad (75)$$

Locally, in the admissible coordinates, the functions x_{new} and y_{new} are given by

$$x_{new}(x) = \int_{x_0}^x \frac{1}{\sqrt{|a(t)|}} dt, \quad y_{new}(y) = \int_{y_0}^y \frac{1}{\sqrt{|c(t)|}} dt.$$

The new coordinates (x_{new}, y_{new}) (or (x_{new}, y_{old}) if $c_{old} \equiv 0$, or (x_{old}, y_{new}) if $a_{old} \equiv 0$) are admissible. In these coordinates, the forms B_1 and B_2 are given by $\text{sign}(a_{old})dx_{new}$, $\text{sign}(c_{old})dy_{new}$ (we assume $\text{sign}(0) = 0$).

A.2.2 Proof of Theorem B

We assume that g of signature $(+, -)$ on M^2 admits a nontrivial quadratic integral F given by (70). Consider the matrix $F^{ij} = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$. It can be viewed as a $(2, 0)$ -tensor: if we change the coordinate system and rewrite the function F in the new coordinates, the matrix changes according to the tensor rule. Then,

$$\sum_{\alpha=1}^2 g_{j\alpha} F^{i\alpha}$$

is a $(1, 1)$ -tensor. In a neighborhood U of almost every point the Jordan normal form of this $(1, 1)$ -tensor is one of the following matrices:

$$\text{Case 1 } \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \text{Case 2 } \begin{pmatrix} \lambda + i\mu & 0 \\ 0 & \lambda - i\mu \end{pmatrix}, \quad \text{Case 3 } \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

where $\lambda, \mu : U \rightarrow \mathbb{R}$. Moreover, in view of Remark 1, there exists a neighborhood of almost every point such that $\lambda \neq \mu$ in Case 1 and $\mu \neq 0$ in Case 2. In the admissible coordinates, up to multiplication of F by -1 , and renaming $V_1 \leftrightarrow V_2$, Case 1 is equivalent to the condition $a > 0$, $c > 0$, Case 2 is equivalent to the condition $a > 0$, $c < 0$, and Case 3 is equivalent to the condition $c \equiv 0$.

We now consider all three cases.

A.2.3 Case 1: $a > 0$, $c > 0$.

Consider the coordinates (75). In this coordinates, $a = 1$, $c = 1$, and equations (74) are:

$$\begin{cases} (fb)_y + 2f_x = 0, \\ (fb)_x + 2f_y = 0. \end{cases}$$

This system can be solved. Indeed, it is equivalent to

$$\begin{cases} (fb + 2f)_x + (fb + 2f)_y = 0, \\ (fb - 2f)_x - (fb - 2f)_y = 0, \end{cases}$$

which, after the change of coordinates $x_{new} = x + y$, $y_{new} = x - y$, has the form

$$\begin{cases} (fb + 2f)_x = 0, \\ (fb - 2f)_y = 0, \end{cases}$$

implying $fb + 2f = Y(y)$, $fb - 2f = X(x)$. Thus, $f = \frac{Y(y) - X(x)}{4}$, $b = 2\frac{X(x) + Y(y)}{Y(y) - X(x)}$.

Finally, in the new coordinates, the metric and the integral have (up to a possible multiplication by a constant) the form

$$(X - Y)(dx^2 - dy^2) \\ 2 \left(p_x^2 - \frac{X(x) + Y(y)}{X(x) - Y(y)} (p_x^2 - p_y^2) + p_y^2 \right) = 4 \frac{p_y^2 X(x) - p_x^2 Y(y)}{X(x) - Y(y)}.$$

Theorem B is proved under the assumptions of Case 1.

A.2.4 Case 2: $a > 0, c < 0$.

Consider the coordinates (75). In this coordinates, $a = 1, c = -1$, and the equations (74) are:

$$\begin{cases} (fb)_y + 2f_x = 0, \\ (fb)_x - 2f_y = 0. \end{cases}$$

We see that these conditions are the Cauchy-Riemann conditions for the complex-valued function $fb + 2i \cdot f$. Thus, for an appropriate holomorphic function $h = h(x + i \cdot y)$, we have $fb = \Re(h)$, $2f = \Im(h)$. Finally, in a certain coordinate system the metric and the integral are (up to multiplication by constants):

$$2\Im(h)dxdy \quad \text{and} \quad p_x^2 - p_y^2 + 2\frac{\Re(h)}{\Im(h)}p_xp_y.$$

Theorem B is proved under the assumptions of Case 2.

A.2.5 Case 3: $a > 0, c = 0$.

Consider admissible coordinates x, y , such that x is the coordinate from (75). In these coordinates, $a = 1, c = 0$, and the equations (74) are:

$$\begin{cases} (fb)_y + 2f_x = 0 \\ (fb)_x = 0 \end{cases}.$$

This system can be solved. Indeed, the second equation implies $fb = -Y(y)$. Substituting this in the first equation we obtain $Y' = 2f_x$ implying

$$f = \frac{x}{2}Y'(y) + \widehat{Y}(y) \quad \text{and} \quad b = -\frac{Y(y)}{\frac{x}{2}Y'(y) + \widehat{Y}(y)}.$$

Finally, the metric and the integral are

$$\left(\widehat{Y}(y) + \frac{x}{2}Y'(y)\right)dxdy \quad \text{and} \quad p_x^2 - \frac{Y(y)}{\widehat{Y}(y) + \frac{x}{2}Y'(y)}p_xp_y \quad (76)$$

Moreover, by the change $y_{new} = \beta(y_{old})$ the metric and the integral (76) will be transformed to:

$$\left(\widehat{Y}(y)\beta' + \frac{x}{2}Y'(y)\right)dxdy \quad \text{and} \quad p_x^2 + \frac{Y(y)}{\widehat{Y}(y)\beta' + \frac{x}{2}Y'(y)}p_xp_y$$

Thus, by putting $\beta(y) = \int_{y_0}^y \frac{1}{\widehat{Y}(t)}dt$, we can make the metric and the integral to be

$$\left(1 + \frac{x}{2}Y'(y)\right)dxdy \quad \text{and} \quad p_x^2 - \frac{Y(y)}{1 + \frac{x}{2}Y'(y)}p_xp_y.$$

Moreover, after the coordinate change $x_{new} = \frac{x_{old}}{2}$ and dividing/multiplication of the metric/integral by 2, the metric and the integral have the form from Theorem B

$$(1 + xY'(y))dxdy \quad \text{and} \quad p_x^2 - 2\frac{Y(y)}{1 + xY'(y)}p_xp_y$$

Theorem B is proved.

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