# Geometric explanation of the Beltrami Theorem

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## 1 The classical Beltrami Theorem

**Definition 1.** *Let $g$ be a (smooth) Riemannian metric on a (differentiable) manifold $M^n$. A Riemannian metric $\bar g$ on $M^n$ is called* **projectively equivalent** *to $g$, if every geodesic of $\bar g$, considered as an unparameterized curve, is a geodesic of $g$.*

In 1865, Beltrami [1] proved the following theorem for dimension 2. For dimensions $n > 2$, the theorem was proven by Lipschitz [7] and Schur [11].

**Theorem 1 (The Beltrami Theorem).** *Let the Riemannian metrics $g$ and $\bar g$ on $M^n$ be projectively equivalent. If $g$ has constant sectional curvature, then $\bar g$ has constant sectional curvature as well.*

Note that the metrics of constant curvature are always locally projectively equivalent. It was first understood by Lagrange [6]. We will explain his example of a diffeomorphism from the half-sphere to a plane that sends the geodesics of the round metric (i.e., the segments of the great circles) to the geodesics of the Euclidean metrics (i.e., the straight lines).

The half-sphere is

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 : \quad x^2 + y^2 + z^2 = 1, \ z < 0\},$$

and the plane is $E^2 := \{(x, y, z) \in \mathbb{R}^3 : \quad z = -1\}$. The mapping is the stereographic projection from the point $O = (0, 0, 0)$. Recall that the stereographic projection f takes a point $X$ of the half-sphere to the intersection of the straight line that goes through $O$ and $X$ with the plane, see the left-hand half of Figure 1.

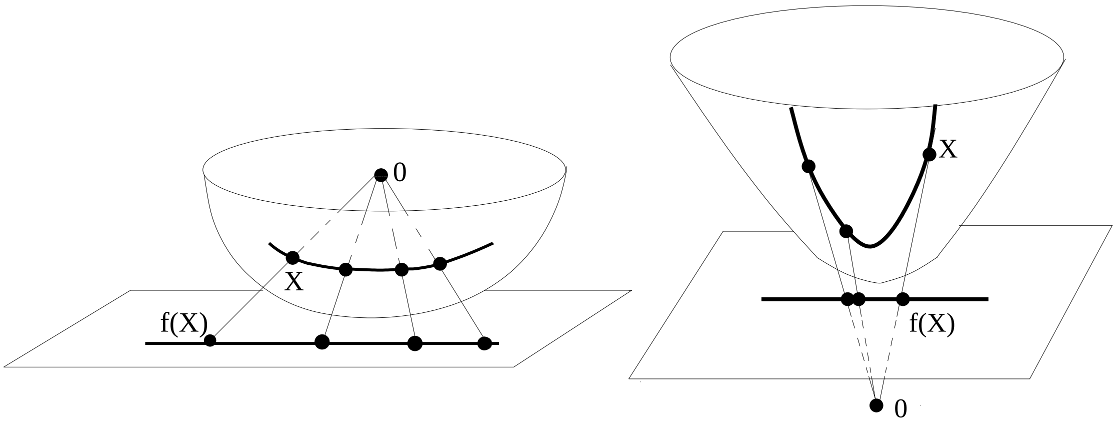


Figure 1: Surfaces of constant curvature are projectively equivalent

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It is easy to see that f takes geodesics of $S^2$ to geodesic of $E^2$. Indeed, the geodesics of $S^2$ are the intersections of planes that go through $O$ with $S^2$. Therefore, f takes them to the intersections of planes that go through $O$ with $E^2$, which are geodesics of $E^2$.

The same construction works for the hyperbolic plane $H^2$, which could be viewed as one sheet of the standard two-sheeted hyperboloid,

$$H^2 := \{(x, y, z) \in \mathbb{R}^3 : \quad x^2 + y^2 - z^2 = -1, \ z > 0\}.$$

The Lorenz metric $dx^2 + dy^2 - dz^2$ induces a Riemannian metric of constant negative curvature on it. The geodesics of $H^2$ are intersections of planes that go through $O = (0, 0, 0)$ with $H^2$. Consider the plane $E^2 := \{(x, y, z) \in \mathbb{R}^3 : \quad z = 1\}$. The stereographic projection f (which sends a point $X \in H^2$ to the intersection of the straight line that goes through $O$ and $X$ with $E^2$) takes the geodesics of $H^2$ to the straight lines on $E^2$, see the right-hand half of Figure 1.

These constructions of geodesic-preserving mapping between surfaces of constant curvature can be generalized for every dimension.

The proofs of Beltrami, Lipschitz and Schur are based on a complicated tensor analysis. The goal of the present paper is to give a geometric proof of the Beltrami Theorem using methods coming from mathematical physics. For dimension 2, it was done in [9].

Our proof has three steps. The last step (Corollary 1) is the observation that the spaces of Killing vector fields for projectively equivalent metrics have the same dimension. Therefore, since the dimension of the space of Killing vector fields for metrics of constant sectional curvature is maximal [5, 3] (we will explain this result after Corollary 1), a metric, projectively equivalent to a metric of constant sectional curvature has constant sectional curvature as well.

The proof of the observation is based on Theorem 2, which says that the existence of $\bar{g}$ projectively equivalent to $g$ allows one to construct an integral of a special form for the geodesic flow of $g$. By Noether's Theorem, it is known that Killing vector fields are in one-to-one correspondence with integrals linear in velocities. If two metrics are projectively equivalent, their flows are orbitally equivalent, so that an integral of the geodesic flow for $g$ immediately gives us an integral of the geodesic flow for $\bar{g}$. Using the integral from Theorem 2, we can make the integral for the geodesic flow of $\bar{g}$ to be linear in velocities. Thus, a Killing vector field for $g$ gives us a Killing vector field for $\bar{g}$.

We will also explain the geometric origin of the integral from Theorem 2: we will explain a construction that, given an orbit-preserving diffeomorphism between two Hamiltonian systems, produced integrals for them.

# 2 New proof of the Beltrami Theorem

**Theorem 2 ([8, 10]).** *Suppose Riemannian metrics $g$ and $\bar{g}$ on $M^n$ are projectively equivalent. Then, the function*

$$I : TM^n \to \mathbb{R}, \quad I(\xi) := \left(\frac{\det(g)}{\det(\bar{g})}\right)^{2/(n+1)} \bar{g}(\xi, \xi) \tag{1}$$

*is an integral of the geodesic flow for $g$.*

Recall that a function $I : TM^n \to \mathbb{R}$ is *an integral* for the geodesic flow, if for every geodesic $\gamma$ parameterized by natural parameter the function $I$ is constant on the orbit $(\gamma(t), \dot{\gamma}(t)) \subset TM^n$ of the geodesic flow.

Recall that a vector field $v$ on $M^n$ is called a *Killing vector field* for $g$, if the flow of $v$ acts by isometries.

Given $g$ and $\bar{g}$, consider the mapping $G : TM^n \to TM^n$ defined by the equality

$$g(\xi, \eta) = \bar{g}(G(\xi), \eta)$$

(for every $x \in M^n$ and every $\xi, \eta \in T_xM^n$). The mapping $G$ is linear on every fiber.

**Corollary 1 (Knebelman [4]).** *Suppose Riemannian metrics $g$ and $\bar{g}$ on $M^n$ are projectively equivalent. Then, if $v$ is a Killing vector field for $g$, then*

$$\bar{v} := (\det(G))^{-1/(n+1)}\, G(v)$$

*is a Killing vector field for $\bar{g}$.*

**Proof of the Beltrami Theorem assuming Corollary 1:** Suppose $g$ and $\bar{g}$ are projectively equivalent. Let $g$ have constant sectional curvature. Then, locally, the space of Killing vectors for $g$ is $\frac{n^2+n}{2}$-dimensional. Indeed, every space of constant sectional curvature is locally isometric to the Euclidean space, to the hyperbolic space or to the round sphere, and the space of Killing vector fields on each such space is $\frac{n^2+n}{2}$-dimensional. Then, by Corollary 1, the space of Killing vector fields for $\bar{g}$ has dimension $\frac{n^2+n}{2}$ as well. Then, $\bar{g}$ has constant sectional curvature.

The last statement is well-known (see, for example, [5] or §§27,71 of [3]), and it is easy to understand it geometrically. More precisely, assume the space of Killing vector fields for $\bar{g}$ has dimension $\frac{n^2+n}{2}$, where $n$ is the dimension of the manifold, and let us show that for every two planes $U_x^2 \in T_xM^n$ and $V_y^2 \in T_yM^n$ there exists an isometry that sends the first plane to the second one. (And, therefore, the metric has constant sectional curvature by definition). It is sufficient to show this for sufficiently close $x$ and $y$.

Take an arbitrary point $x_0 \in M^n$ and consider the stabilizer of the point in the group of isometries. From one side, it is at most $\frac{n^2-n}{2}$-dimensional, since it acts by orthogonal transformations on $T_{x_0}M^n$, and the dimension of the group $O(n)$ is $\frac{n^2-n}{2}$. From the other side, it is at least $\frac{n^2-n}{2}$-dimensional, since we can choose a basis in the space of Killing vector fields such that the first $\frac{n^2-n}{2} = \frac{n^2+n}{2} - n$ basis vector fields vanish at $x_0$, and, therefore, lie in the Lie algebra of the stabilizer. Finally, the stabilizer is precisely $\frac{n^2-n}{2}$-dimensional, i.e., its connected component coincides with the connected component of the group of the orthogonal transformations of $T_{x_0}M^n$.

Since the dimension of $M^n$ is $n$, and since the dimension of the stabilizer is $\frac{n^2-n}{2} = \frac{n^2+n}{2} - n$, there exists a isometry that sends $x$ to $y$. We denote by $U_y \in T_yM^n$ the image of $U_x$ under this isometry. Since the stabilizer of $y$ contains the group of orientation-preserving orthogonal transformations of $T_yM^n$, there exists an isometry that preserves $y$ and takes $U_y$ to $V_y$. Thus, there exists an isometry that sends $U_x$ to $U_y$. Finally, the metric has constant sectional curvature.

**Proof of Corollary 1 assuming Theorem 2:** The initial Knebelman's proof of Corollary 1 uses a complicated tensor analysis. Note that the proof of Knebelman can be generalized for Killing tensors, as it was done in [2]. We present an alternative proof which is based on Theorem 2. This proof can also be generalized for Killing tensors.

Because of Noether's Theorem (see, for example, §71 of [3]), $v$ is a Killing vector field for $g$, if and only if the function

$$F : TM \to \mathbb{R}, \quad F(\xi) := g(v, \xi)$$

is an integral for the geodesic flow of $g$.

Suppose the metrics $g$ and $\bar{g}$ are projectively equivalent. Consider the mapping

$$m : TM^n \to TM^n, \quad m(\xi) := \frac{\sqrt{\bar{g}(\xi,\xi)}}{\sqrt{g(\xi,\xi)}} \cdot \xi. \tag{2}$$

If the vector $\xi$ has length one in $g$, then $m(\xi)$ is proportional to $\xi$ and has length one in $\bar{g}$. Therefore, since the metrics $g$ and $\bar{g}$ are projectively equivalent, the mapping $m$ takes the orbits of the geodesic flow of $\bar{g}$ to the orbits of the geodesic flow of $g$.

Then, if $F(\xi) = g(v, \xi)$ is an integral for the geodesic flow of $g$, the function

$$\bar{F}(\xi) := g(v, m(\xi)) = g\left(v, \frac{\sqrt{\bar{g}(\xi,\xi)}}{\sqrt{g(\xi,\xi)}} \cdot \xi\right)$$

is an integral of the geodesic flow for $\bar g$. Using definition and linearity of $G$, we obtain

$$\bar F(\xi) = \bar g\left(G(v), \frac{\sqrt{\bar g(\xi,\xi)}}{\sqrt{g(\xi,\xi)}}\cdot \xi\right) = \frac{\sqrt{\bar g(\xi,\xi)}}{\sqrt{\det(G)^{-2/(n+1)}g(\xi,\xi)}}\cdot \bar g(\det(G)^{-1/(n+1)}G(v),\xi).$$

The function $\sqrt{\bar g(\xi,\xi)}$ is the length of the velocity vector of the geodesic (in $\bar g$), and, therefore, is an integral of the geodesic flow for $\bar g$. The function $\sqrt{\det(G)^{-2/(n+1)}g(\xi,\xi)}$ is an integral of the geodesic flow for $\bar g$ by Theorem 2. Thus, the function $\bar g(\det(G)^{-1/(n+1)}G(v),\xi)$ is an integral of the geodesic flow for $\bar g$. Finally, $\bar v := \det(G)^{-1/(n+1)}G(v)$ is a Killing vector field for $g$. Corollary 1 is proven.

**Geometric sense of the integral from Theorem 2:** We will recall a construction that, given an orbit-preserving diffeomorphism between two Hamiltonian systems allows one to construct an integral of the system. The construction is essentially due to [13], see also [8]. Independently, it was discovered in [12].

Projectively equivalent metrics provide such diffeomorphism: the manifold is $TM^n$, the symplectic structures come from $T^*M^n$ via identification of $T^*M^n$ and $TM^n$ by $g$ and $\bar g$, and the orbit-preserving diffeomorphism is the mapping $m$ given by (2). Applying this construction to this diffeomorphism gives the integral (1).

Let $v$ and $\bar v$ be Hamiltonian systems on symplectic manifolds $(N^{2n},\omega)$ and $(\bar N^{2n},\bar\omega)$ with Hamiltonians $H$ and $\bar H$, respectively. Consider the isoenergy surfaces

$$Q^{2n-1} \stackrel{\rm def}{=} \left\{p\in N^{2n} : H(x)=h\right\},\quad \bar Q^{2n-1} \stackrel{\rm def}{=} \left\{p\in \bar N^{2n} : \bar H(x)=\bar h\right\},$$

where $h$ and $\bar h$ are regular values of the functions $H$, $\bar H$, respectively. A diffeomorphism $\phi : Q^{2n-1} \longrightarrow \bar Q^{2n-1}$ is *orbital*, if it takes the orbits (considered as unparameterized curves) of the system $v$ to the orbits of the system $\bar v$.

Given an orbital diffeomorphism, we can invariantly construct an integral for the system $v$. Denote by $\sigma$, $\bar\sigma$ the restrictions of the forms $\omega,\bar\omega$ to $Q^{2n-1}$, $\bar Q^{2n-1}$, respectively. Consider the pull-back $\phi^*\bar\sigma$ which is a skew-symmetric two-form on $Q^{2n-1}$.

**Lemma 1.** *The flow of $v$ preserves the forms $\sigma$ and $\phi^*\bar\sigma$.*

**Proof:** Since $v$ preserves $Q^{2n-1}$ and $\omega$, it preserves $\sigma$. Let us prove that $v$ preserves $\phi^*\bar\sigma$. We need to prove that the Lie derivative $L_v\phi^*\bar\sigma$ of the form $\phi^*\bar\sigma$ along the vector field $v$ vanishes.

By Cartan's formula, the Lie derivative is given by

$$L_v\phi^*\bar\sigma = \mathrm{d}\left[\imath_v\phi^*\bar\sigma\right] + \imath_v \mathrm{d}\left[\phi^*\bar\sigma\right].$$

On the right-hand side both terms vanish. Indeed, the term $\imath_v\mathrm{d}\left[\phi^*\bar\sigma\right]$ vanishes since the form $\bar\omega$ is closed and, therefore, the form $\phi^*\bar\sigma$ is also closed. Let us show that the term $\mathrm{d}\left[\imath_v\phi^*\bar\sigma\right]$ vanishes as well. We will even show that $\imath_v\phi^*\bar\sigma$ vanishes.

By definition of the vector field $\bar v$, for any vector $u$ we have $\omega(\bar v, u) = d\bar H(u)$. Then, the kernel of the form $\bar\sigma$ coincides (in the space $T_q\bar Q^{2n-1}$ at each point $q\in\bar Q^{2n-1}$) with the linear span of the vector $\bar v$. Since the mapping $\phi$ takes orbits to orbits, the differential $d\phi$ takes the vector field $v$ to a vector field which is proportional to $\bar v$. Therefore, $v$ lies in the kernel of $\phi^*\bar\sigma$, and $\imath_v\phi^*\bar\sigma$ vanishes. Finally, the Lie derivative vanishes, and the form is preserved by $v$. Lemma 1 is proved.

Since both forms are preserved by the flow, a function constructed invariantly by using these forms must automatically be an integral. The construction could be as follows. Take the eigenvalues of one form with respect to the other, (i.e., for every point $p\in Q^{2n-1}$ take all $\lambda\in\mathbb{C}$ such that $\mathbf{rank}(\sigma-\lambda\phi^*\bar\sigma) < 2n-2$). None of them are zero or infinite. The multiplicity of such $\lambda$ is $n-1-\frac{1}{2}\mathbf{rank}(\sigma-\lambda\phi^*\bar\sigma)$. Take the product of all such $\lambda$'s with multiplicities. It is a smooth real-valued function on $Q^{2n-1}$. Since its construction is coordinate-independent, and since the forms are preserved by $v$, the function is preserved by $v$ as well, i.e., it is an integral of the Hamiltonian system $v$.

By direct calculation (which are not very complicated and can be found, for example, in [10]), one can see that this integral constructed for the orbital diffeomorphism (2) of the geodesic flows for projectively equivalent metrics is a power of the function $I$ from Theorem 2. Thus, $I$ is an integral.


### Acknowledgements

I would like to thank E. Ghys for giving me an idea to write this paper, V. Bangert, S. Benenti, A. Bolsinov, A. Di Scala, A. Fomenko, P. Topalov and K. Voss for useful discussions, P. de la Harpe for grammatical and stylistic comments and DFG-programm 1154 (Global Differential Geometry) and Ministerium für Wissenschaft, Forschung und Kunst Baden-Württemberg (Eliteförderprogramm Postdocs 2003) for partial financial support.


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