Riemannian metrics having common geodesics with Berwald metrics

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Abstract

In Theorem 1, we generalize some results of Szabó [Sz1, Sz2] for Berwald metrics that are not necessarily strictly convex: we show that for every Berwald metric F there always exists a Riemannian metric affine equivalent to F. Further, we investigate geodesic equivalence of Berwald metrics. Theorem 2 gives a system of PDE that has a (nontrivial) solution if and only if the given essentially Berwald metric admits a Riemannian metric that is (nontrivially) geodesically equivalent to it. The system of PDE is linear and of Cauchy-Frobenius type, i.e., the derivatives of unknown functions are explicit expressions of the unknown functions.

As a corollary, we obtain that geodesic equivalence of an essentially Berwald metric and a Riemannian metric is always affine equivalence provided both metrics are complete.

1 Definitions and results

A *Finsler metric* on a smooth manifold M is a function $F : TM \to \mathbb{R}_{\geq 0}$ such that:

1. It is smooth on $TM \setminus TM_0$, where TM_0 denotes the zero section of TM.

2. For every $x \in M$, the restriction $F_{|T_xM}$ is a norm on T_xM, i.e., for every $\xi, \eta \in T_xM$ and for every nonnegative $\lambda \in \mathbb{R}$ we have

   (a) $F(\lambda \cdot \xi) = \lambda \cdot F(\xi)$,
   (b) $F(\xi + \eta) \leq F(\xi) + F(\eta)$,
   (c) $F(\xi) = 0 \implies \xi = 0$.

We always assume that $n := \dim(M) \geq 2$. We do not require that (the restriction of) the function F is strictly convex. In this point our definition is more general than the usual definition. In addition we do not assume that the metric is reversible, i.e., we do not assume that $F(-\xi) = F(\xi)$. Some standard references for Finsler geometry are [Al2, BCS, BBI, Sh1].

Example 1 (Riemannian metric). *For every Riemannian metric g on M, the function $F(x, \xi) := \sqrt{g_{(x)}(\xi, \xi)}$ is a Finsler metric.*

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A Finsler metric is *Berwald*, if there exists a symmetric affine connection Γ such that the parallel transport with respect to this connection preserves the function F. In this case, we call the connection Γ the *associated connection*.

Riemannian metrics are always Berwald. For them, the associated connection coincides with the Levi-Civita connection. We say that a Finsler metric is *essentially Berwald*, if it is Berwald, but not Riemannian. The simplest examples of essentially Berwald metrics are Minkowski metrics.

Example 2 (Minkowski metric). *Consider a smooth norm on $\mathbb{R}^n$, i.e., a smooth function $p : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying 2a, 2b, 2c. We canonically identify $T\mathbb{R}^n$ with $\mathbb{R}^n \times \mathbb{R}^n$ with coordinates $(\underbrace{x_1, ..., x_n}_{x \in \mathbb{R}^n}, \underbrace{\xi^1, ..., \xi^n}_{\xi \in T_x\mathbb{R}^n})$. Then, $F(x, \xi) := p(\xi)$ is a Finsler metric. We see that the metric is translation invariant. Hence, the standard flat connection preserves it, i.e., it is a Berwald metric. If the norm p does not satisfy the parallelogram equality, the Minkowski metric is essentially Berwald.*

Let F_1, F_2 be Finsler metrics on the same manifold. We say that F_1 is *geodesically equivalent* (or *projectively equivalent*) to F_2, if every F_1−geodesic, considered as **unparametrized** curve, is also an F_2−geodesic. We say that they are *affine equivalent*, if every F_1−geodesic, considered as **parametrized** curve, is also an F_2−geodesic. Of course, in the definition we can replace any of the Finsler metrics by a Riemannian or pseudo-Riemannian one, or by an affine connection.

Remark 1. *Geodesic equivalence (or affine equivalence) of Finsler metrics is not a priori a symmetric relation, as Example 3 below shows. The reason is that for certain Finsler metrics the uniqueness theorem for the geodesics does not hold: two different geodesics can have the same velocity vector, as in Example 3 below. Then, even under the assumption that all F_1−geodesics are F_2−geodesics, there may exist F_2−geodesics that are not F_1−geodesics.*

This phenomenon evidently does not happen, if the metrics are strictly convex (and of course in the Riemannian case); for such metrics, F_1 is geodesically equivalent to F_2 if and only if F_2 is geodesically equivalent to F_1. We will show in the beginning of Section 2.2.2 that under the assumption that the metric F is Berwald, if g is geodesically (or affine) equivalent to F, then g is geodesically (or affine, resp.) equivalent to the connection Γ associated to F.
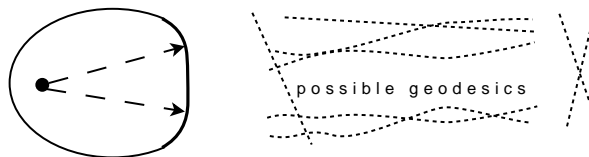

Figure 1: The unit sphere in the norm p and possible geodesics of the corresponding Minkowski metric

Example 3. *Consider the Minkowski metric $F(x, \xi) = p(\xi)$ such that the unit sphere $S_1 := \{\xi \in \mathbb{R}^n \mid p(\xi) = 1\}$ is as on Figure 1: the important feature of the picture is that the part of the unit sphere lying in the marked sector is a straight line segment. Then every curve such that its velocity vectors are in the sector is a geodesic. Beside such curves, the straight lines*

are also geodesics. We see that the standard flat metric is geodesically and affine equivalent to F, but the metric F is neither geodesically nor affine equivalent to the standard flat metric.

Geodesic equivalence of metrics is a classical subject. The first non-trivial examples of geodesically equivalent Riemannian metrics were discovered by Lagrange [La]. Geodesically equivalent Riemannian metrics were studied by Beltrami [Bel], Levi-Civita [LC], Painlevé [Pa] and other classics. One can find more historical details in the surveys [Am, Mi2] and in the introduction to the papers [Ma1, Ma4]. Geodesic equivalence of Riemannian and Finsler metrics is discussed in particular in Hilbert's 4th problem, see [Al1, Po]. Recent results on geodesic equivalence of Riemannian and Finsler metrics include [MBB, Sh2].

Our main results are

Theorem 1. *Let F be a Berwald metric. Then there exists a Riemannian metric which is affine equivalent to F.*

For strictly convex Finsler metrics, Theorem 1 is due to [Sz1]. Later, other proofs were suggested in [Sz2, To]. Our proof is similar to the proof in [Sz2]; the modification is based on the construction from [MRTZ].

Theorem 2. *Let F be an essentially Berwald metric on a connected manifold, and let Γ be its associated connection. Suppose a Riemannian or pseudo-Riemannian metric g is geodesically equivalent to F, but is not affine equivalent to F. Then there exists a constant μ, a symmetric $(2,0)-$tensor a^{ij}, and a nonzero vector field λ^i such that the following equations are fulfilled, where " $,$ " denotes the covariant derivative with respect to Γ:*

$$a^{ij}{}_{,k} = \lambda^i \delta^j_k + \lambda^j \delta^i_k \tag{1}$$

$$\lambda^i{}_{,j} = \mu \, \delta^i_j \tag{2}$$

We see that equations (1,2) are of Cauchy-Frobenius type, i.e., the derivatives of the unknown functions a^{ij}, λ^i are explicitly expressed as functions of the unknown functions and known data (connection Γ).

Remark 2. *If a Riemannian metric g is affine equivalent to F, equations (1,2) also have a nontrivial solution, namely $a^{ij} = g^{ij}$, $\lambda^i \equiv 0$, $\mu = 0$.*

Remark 3. *The converse of Theorem 2 is also true: the existence of a nondegenerate a^{ij} and of a nonzero λ^i satisfying equations (1,2) for a certain constant μ implies the existence of a Riemannian or a pseudo-Riemanninan metric geodesically equivalent to F, but not affine equivalent to g.*

Recently, a system of Cauchy-Frobenius type for metrics geodesically equivalent to Berwald Finsler metrics was obtained [MBB, Theorem 2]. Our system is much easier than one in [MBB]: first of all, it is linear in the unknown functions, second, it contains less equations, and, third, the equations are much simpler than those of [MBB] and, in particular, contain no curvature terms. One cannot obtain our equations from the equations of [MBB] by a change of unknown functions. In order to obtain our equations from those of [MBB], one should prolong the equations of [MBB] two times, and use the result of the prolongation to simplify the system.

Corollary 1. *Let F be an essentially Berwald metric on a connected closed (= compact without boundary) manifold. Then every Riemannian or pseudo-Riemannian metric geodesically equivalent to F is affine equivalent to F.*

Corollary 2. *Let F be a complete essentially Berwald metric on a connected manifold. Then every complete Riemannian or pseudo-Riemannian metric geodesically equivalent to F is affine equivalent to F.*

The assumptions in Theorem 2 and Corollaries are important: it is possible to construct counterexamples if the Berwald metric is not essentially Berwald (i.e., is a Riemannian metric), or if one of the metrics is not complete.

Corollary 3 (Hilbert's 4th problem for Berwald metrics). *Suppose an essentially Berwald metric F on a connected manifold is projectively flat, that is, there exists a flat Riemannian metric geodesically equivalent to F. Then F is isometric to a Minkowski metric.*

2   Proofs

2.1   Averaged metric and proof of Theorem 1

Given a Finsler Berwald metric F, we construct a Riemannian metric $g = g_F$ such that the associated connection Γ of F is the Levi-Civita connection of g implying that the metric g is affine equivalent to F. As we mentioned in the introduction, the construction is due to [MRTZ], and is similar to one from [Sz2].

Given a smooth norm p on $\mathbb{R}^{n \geq 2}$, we canonically construct a positive definite symmetric bilinear form $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. For the Finsler metric F, the role of p will be played by the restriction of F to $T_x M$. We will see that the constructed g smoothly depends on x, and hence it is a Riemannian metric.

Consider the sphere $S_1 = \{\xi \in \mathbb{R}^n \mid p(\xi) = 1\}$. Consider the (unique) volume form Ω on $\mathbb{R}^n$ such that the volume of the 1-ball $B_1 = \{\xi \in \mathbb{R}^n \mid p(\xi) \leq 1\}$ is equal to 1.

Denote by ω the volume form on S_1 whose value on the vectors $\eta_1, ..., \eta_{n-1}$ tangent to S_1 at the point $\xi \in S_1$ is given by $\omega(\eta_1, ..., \eta_{n-1}) := \Omega(\xi, \eta_1, \eta_2, ..., \eta_{n-1})$.

Now, for every point $\xi \in S_1$, consider the symmetric bilinear form $b_{(\xi)} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $b_{(\xi)}(\eta, \nu) = D^2_{(\xi)} p^2(\eta, \nu)$. In this formula, $D^2_{(\xi)} p^2$ is the second differential at the point ξ of the function p^2 on $\mathbb{R}^n$. The analytic expression for $b_{(\xi)}$ in the coordinates $(\xi^1, ..., \xi^n)$ is

$$b_{(\xi)}(\eta, \nu) = \sum_{i,j} \frac{\partial^2 p^2(\xi)}{\partial \xi^i \partial \xi^j} \eta^i \nu^j. \tag{3}$$

Since the norm p is convex, the bilinear form is nonnegative definite. Clearly, for every $\xi \in S_1$, we have

$$b_{(\xi)}(\xi, \xi) > 0 \tag{4}$$

4

(this is actually the reason why we take p^2 and not p in the definition of b).

Now consider the following symmetric bilinear 2−form g on $\mathbb{R}^n$: for $\eta, \nu \in \mathbb{R}^n$, we put

$$g(\eta, \nu) = \int_{S_1} b_{(\xi)}(\eta, \nu)\omega. \tag{5}$$

We assume that the orientation of S_1 is chosen in such a way that $\int_{S_1} \omega > 0$. Because of (4), g is positive definite.

Now let us extend this construction to every tangent space T_xM of the manifold, then $F_{|T_xM}$ plays the role of p. Since the construction depends smoothly on the point $x \in M$, we have that $g := g_F$ is a Riemannian metric on M. We show that if the metric F is Berwald with the associated connection Γ, then Γ is the Levi-Civita connection of g.

Indeed, consider a smooth curve γ connecting the points $\gamma(0), \gamma(1) \in M$. Let

$$\tau : T_{\gamma(0)}M \to T_{\gamma(1)}M$$

be the parallel transport of the vectors along the curve with respect to the connection Γ. τ is a linear map. Since the metric is Berwald, τ preserves the function F and, in particular, the one-sphere S_1. Since the forms Ω, ω were constructed by using the sphere S_1 and the linear structure of the space only, τ preserves the form ω. Since the function F is preserved as well, everything in formula (5) is preserved by the parallel transport which implies $\tau^* g = g$. Then $g_{ij,k} = 0$, therefore every (parametrized) geodesic of g is a geodesic of F. Theorem 1 is proved.

2.2 Proof of Theorem 2 and Corollaries 1, 2, 3

Within the whole section we assume that our underlying manifold is connected, orientable (otherwise we pass to an orientable cover), and has dimension at least two.

2.2.1 Holonomy group of a Berwald metric F

Lemma 1. *Let F be an essentially Berwald metric on a connected manifold M, and let g be a Riemannian metric affine equivalent to F (the existence of such metric is guaranteed by Theorem 1). Then, the metric g is symmetric of rank ≥ 2, or there exists one more Riemannian metric h such that it is not proportional to g, but is affine equivalent to g.*

Proof. We essentially repeat the argumentation of [Sz1, Sz2]. Take a fixed point $q \in M$. For every (smooth) loop $\gamma(t), t \in [0, 1]$ with the origin in q (i.e., $\gamma(0) = \gamma(1) = q$), we consider the parallel transport $\tau_\gamma : T_qM \to T_qM$ along the curve. It is well known (see for example, [Ber, Sim]), that the set

$$H_q := \{\tau_\gamma \mid \gamma : [0, 1] \to M \text{ is a smooth loop}, \gamma(0) = \gamma(1) = q\}$$

is a subgroup of the group of the orthogonal transformations of T_qM. Moreover, it is also known that at least one of the following conditions holds:

1. H_q acts transitively on the unit sphere $S_1 := \{\xi \in T_qM \mid g(\xi, \xi) = 1\}$,

2. the metric g is symmetric of rank ≥ 2,

3. there exists one more Riemannian metric h such that it is nonproportional to g, but is affine equivalent to g.

In the first case, since the holonomy group preserves both g and F, the ratio $F(\xi)^2/g(\xi, \xi)$ is the same for all $\xi \in T_qM, \xi \neq 0$, implying that the metric g is Riemannian. Lemma 1 is proved.

2.2.2 Metrics with degree of mobility ≥ 3

If the dimension of the manifold is 2, an essentially Berwald metric is a Minkowski metric, and Theorem 2 and Corollaries 1, 2, 3 are evident. Below, we assume that the dimension of the manifold is ≥ 3. Suppose the (Riemannian or pseudo-Riemannian) metric $\bar{g}$ is geodesically equivalent to F, but is not affine equivalent to F. Then the metric $\bar{g}$ is geodesically equivalent to the averaged metric $g = g_F$, but is not affine equivalent to g. If the uniqueness theorem for geodesics holds, the latter statement is trivial; for generic Finsler metrics, it probably requires additional explanation.

In order to explain why the metric $\bar{g}$ is geodesically equivalent to the averaged metric $g = g_F$, let us consider the set

$$N := \{(x, \xi) \in TM \setminus TM_0 \mid D^2 F^2_{|T_qM} \text{ nondegenerate}\}.$$

This set is evidently open. As from the following standard (see for example [Ku]) argument from differential geometry it turns out, its intersection with every $T_qM \setminus TM_0$ is not empty.

We need to show that for a smooth norm $p := F_{|T_qM}$ on $\mathbb{R}^n = T_qM$ there exists a point such that $D^2 p^2$ is nondegenerate at this point. We fix an Euclidean metric in $\mathbb{R}^n$ and consider the sphere in $\mathbb{R}^n$ (with respect to the chosen Euclidean metric in T_qM) of large radius such that the Finsler sphere $S_1 := \{\xi \in T_qM \mid F(\xi) = 1\}$ lies inside, see the left-hand side of Figure 2. Then, we make the radius smaller until the first point of the intersection of the sphere with S_1, see the right-hand side of Figure 2. Clearly, at the point of the intersection, the second differential of p^2 is nondegenerate as we claimed.
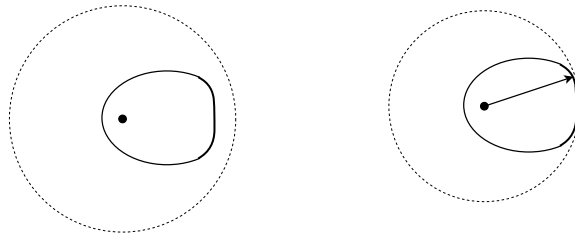

Figure 2: For a smooth norm p, there always exists a point such that the second differential of p^2 is nondegenerate

It is well known that for $(x, \xi) \in N$ the uniqueness theorem of geodesics holds: locally, there exists a unique F−geodesic γ such that $\gamma(0) = x$ and $\dot{\gamma}(0) = \xi$. Moreover, the geodesic γ is also the geodesic of the associated connection Γ. Then, every $\bar{g}$−geodesic such that $(\gamma(0), \dot{\gamma}(0)) \in N$ is also a Γ−geodesic. Since the set $N \cap T_q M$ is open for every q, the connection $\bar{\Gamma}$ of $\bar{g}$ satifies the Levi-Civita condition

$$\Gamma^i_{jk} - \bar{\Gamma}^i_{jk} - \tfrac{1}{n+1} \left(\delta^i_k \left(\Gamma^\alpha_{j\alpha} - \bar{\Gamma}^\alpha_{j\alpha} \right) + \delta^i_j \left(\Gamma^\alpha_{k\alpha} - \bar{\Gamma}^\alpha_{k\alpha} \right) \right) = 0$$

at every point (in the proof from [LC] it is sufficient to assume that only the geodesics whose velocity vectors are from certain open set $N \subseteq TM$; $N \cap T_q M \neq \varnothing$ are common for both metrics) implying that Γ and $\bar{g}$ are geodesically equivalent, and hence g and $\bar{g}$ are also geodesically equivalent.

Thus, the metric $\bar{g}$ is geodesically equivalent to the averaged metric g as well, but not affine equivalent to g. By Lemma 1, the metric g is symmetric, or there exists a Riemannian metric h affine equivalent to g but not proportional to g. We show that if the metric g is symmetric, the assumptions of Theorem 1 imply that it is flat from which it follows that there exists a metric $h = h_{ij}$ affine equivalent to g but not proportional to g at least on the universal cover of M, which is sufficient for our goals.

By a result of Sinjukov [Si1], every symmetric metric geodesically equivalent to g is affine equivalent to g, unless the metric has constant curvature. In the latter case, the metric must be flat, otherwise the holonomy group discussed in the previous section acts transitively on the unit sphere, and the Finsler metric F is actually Riemannian.

Thus, at least on the universal cover of the manifold there exists a Riemannian metric h affine equivalent to g but not proportional to g.

We consider the symmetric (1,1)-tensor $a_{ij} := \left| \frac{\det(\bar{g})}{\det(g)} \right|^{1/(n+1)} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}$, where $\bar{g}^{ij}$ is the tensor, dual to $\bar{g}_{ij}$ so that $g_{i\alpha} \bar{g}^{\alpha j} = \delta^j_i$, the function $\lambda := \frac{1}{2} a_{\alpha\beta} g^{\alpha\beta}$, and its differential $\lambda_i := (d\lambda)_i := \lambda_{,i}$. By the result of Sinjukov [Si2], see also [BM] and [EM], if the metric $\bar{g}$ is geodesically equivalent to g, the tensor a_{ij} and the (0,1) tensor λ_i satisfy the equation

$$a_{ij,k} \quad = \quad \lambda_i g_{jk} + \lambda_j g_{ik}. \tag{6}$$

Moreover, if the metrics g and $\bar{g}$ are not affine equivalent, λ_i is not identically zero.

Recall that the *degree of mobility* of the metric g is the dimension of the space of solutions of equation (6) considered as equation on the unknown a_{ij} and λ_i. In our case, the degree of mobility is at least 3. Indeed, $\bar{a}_{ij} := g_{ij}, \bar{\lambda}_i := 0$ and $\hat{a}_{ij} := h_{ij}, \hat{\lambda}_i := 0$ are also solutions, but by the assumptions they are linearly independent of the solution a_{ij}, λ_i.

Metrics with degree of mobility ≥ 3 on manifolds of dimensions ≥ 3 were studied, in particular, in [KM], see also references therein. The last part of the present paper will essentially use the results of [KM], so we recommend the reader to have [KM] at hand.

By results of [KM, Lemma 3], under the above assumptions, for every solution a_{ij}, λ_i of equation (6), in a neighbourhood of almost every point there exists a constant B and a function μ such that the following equations hold:

7

$$\lambda_{i,j} = \mu\, g_{ij} + Ba_{ij} \qquad (7)$$

$$\mu_{,i} = 2B\lambda_i. \qquad (8)$$

Indeed, equation (7) is equation (30) of [KM], and equation (8) is in [KM, Remark 8] (where the function μ is denoted by ρ).

Our next goal is to show that in our case $B = 0$ (and, therefore, equations (7) are fulfilled at every point of the manifold, and the function μ is actually a constant by (8)). This will also imply that (6, 7) coincide with (1, 2) after raising indices with the help of g.

In order to do this, let us consider the solution $A_{ij} := a_{ij} + h_{ij}$, $\Lambda_i := \lambda_i + 0 = \lambda_i$, which is the sum of the solutions a_{ij}, λ_i and $h_{ij}, 0$. The data A_{ij}, λ_i satisfy equation (6). As we explained above, they therefore also satisfy equation (7) in a neighbourhood of almost every point, i.e., in a neighbourhood of almost every point there exist a function $\tilde{\mu}$ and a constant $\tilde{B}$ such that

$$\lambda_{i,j} = \tilde{\mu}g_{ij} + \tilde{B}(a_{ij} + h_{ij}). \qquad (9)$$

Subtracting equation (7) from (9), we obtain

$$(\mu - \tilde{\mu})g_{ij} = (\tilde{B} - B)a_{ij} + \tilde{B}h_{ij}. \qquad (10)$$

We see that the right-hand side of equation (10) is a linear combination of two solution a_{ij} and h_{ij} and is therefore also a solution of (6) (with an appropriate λ_i). As it was proved in [BKM, Lemma 1] (the result is essentially due to Weyl [We]), the function $\mu - \tilde{\mu}$ must be a constant. Since g, a, and h are linearly independent, all coefficients in the linear combination (10) are zero implying $B = 0$.

Thus, equations (6, 7) coincide with equations (1,2) after raising the indexes. Theorem 2 is proved.

Proof of Corollaries 1,2. As we explained above, we can assume that the dimension of the manifold is ≥ 3 and the degree of mobility is ≥ 3. Under these assumptions, Corollary 1 follows from [KM, Theorem 2] (if g is Riemannian, the result is due to [Ma4, Theorem 16]; in view of Theorem 2, the result follows from [Mi1, Theorem 5]), and Corollary 1 follows from [Ma3, Theorem 2] (if g is Riemannian, the result is due to [KM, Theorem 1]).

Proof of Corollary 3. Suppose that a flat Riemannian metric $\bar{g}$ is geodesically equivalent to an essentially Berwald metric F. Consider the averaged metric $g = g_F$ constructed in Section 2.1. It is affine equivalent to F, and, therefore, as we explained in Section 2.2.2, is geodesically equivalent to $\bar{g}$.

By the classical Beltrami Theorem (see for example [Ma2], or the original papers [Bel] and [Sc]), the metric g has constant curvature. If the curvature of g is not zero, the holonony group of g acts transitively on the unit sphere implying the metric F is actually Riemannian. Thus, the metric g is flat. Then, there exists a coordinate system such that $\Gamma \equiv 0$. In this coordinate system, parallel transport along a curve does not depend on the curve and is the usual parallel transation $x \mapsto x + T$. Since the parallel transport preserves F, we have that F is translation-invariant implying it is Minkowski metric as we claimed.

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