There are no conformal Einstein rescalings of complete pseudo-Riemannian Einstein metrics

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Theorem 1. *Let g be a light-line-complete pseudo-Riemannian Einstein metric of indefinite signature (i.e., for no constant c the metric $c \cdot g$ is Riemannian) on a connected $(n > 2)-$dimensional manifold M. Assume that for the nowhere vanishing function ψ the metric $\psi^{-2}g$ is also Einstein. Then, ψ is a constant.*

Remark 1. Theorem 1 fails for Riemannian metrics (even if we replace light-line completeness by usual completeness) – Möbius transformations of the standard round sphere and the stereographic map of the punctured sphere to the Euclidean space are conformal nonhomothetic mappings. One can construct other examples on warped Riemannian manifolds, see [6, Theorem 21].

Remark 2. By Theorem 1, *light-line complete pseudo-Riemannian Einstein metrics of indefinite signature do not admit nonhomothetic conformal complete vector fields.* The Riemannian version of this result is due to Yano and Nagano [10]. Moreover, the assumption that the metric is Einstein can be omitted (by the price of considering only essential conformal vector fields): as it was proved by D. Alekseevksii [1], J. Ferrand [3] and R. Schoen [11], a Riemannian manifold admitting an essential complete vector field is the round sphere or the Euclidean space. It is still not known whether the last statement (sometimes called Lichnerowicz-Obata conjecture) can be extended to the pseudo-Riemannian case, see [8] for a counterexample in the C^1- smooth category, and [4, 5] for a good survey on this topic.

Remark 3. In the 4-dimensional lorenz case, Theorem 1 was known in folklore: more precisely, conformal Einstein rescalings of 4-dimensional Einstein metrics were described by Brinkmann [2], see also [7, Corollary 2.10]. The list of all such metrics and their conformal Einstein rescalings is pretty simple and one can directly verify our Theorem 1 by calculations.

Remark 4. A partial case of Theorem 1 is [7, Theorem 2.2], in which it is assumed that both metrics are complete. This extra-assumption is very natural in the context of [7] since the paper is dedicated to the classification of conformal vector fields; moreover, Theorem 2.2 is not the main result of the paper. It is not clear whether in the proof of [7, Theorem 2.2] the assumption that the second metric is complete could be omitted.

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Proof of Theorem 1. It is well-known (see for example [2, eq. (2.21)] or [6, Lemma 1]) that the Ricci curvatures R_{ij} and $\bar{R}_{ij}$ of two conformally equivalent metrics g and $\bar{g} = \psi^{-2} g = e^{-2\phi} g$ are related by

$$\bar{R}_{ij} = R_{ij} + (\Delta\phi - (n-2)\|\nabla\phi\|^2) g_{ij} + \frac{n-2}{\psi} \nabla_i \nabla_j \psi. \tag{1}$$

Consider a light-line geodesic $\gamma(t)$ of the metric g. Since the metric g is light-line-complete, $\gamma(t)$ is defined on the whole $\mathbb{R}$. "Light-line" means that $g(\dot{\gamma}(t), \dot{\gamma}(t)) = g_{ij}\dot{\gamma}^i(t)\dot{\gamma}^j(t) = 0$, where $\dot{\gamma}$ is the velocity vector of γ (it is well-known that if this property is fulfilled in one point then it is fulfilled at every point of the geodesic).

Now contract (1) with $\dot{\gamma}^i\dot{\gamma}^j$. Since the metrics are Einstein and conformally equivalent, $\bar{R}_{ij}$, R_{ij} and g_{ij} are proportional to g_{ij}, and therefore the only term which does not vanish is $\dot{\gamma}^i\dot{\gamma}^j \frac{n-2}{\psi} \nabla_i \nabla_j \psi$. Thus, $\dot{\gamma}^i\dot{\gamma}^j \nabla_i \nabla_j \psi = 0$.

Clearly, at every point of the geodesic we have $\dot{\gamma}^i\dot{\gamma}^j \nabla_i \nabla_j \psi = \frac{d^2}{dt^2}\psi(\gamma(t))$. Thus, $\frac{d^2}{dt^2}\psi(\gamma(t)) = 0$ implying $\psi(\gamma(t)) = \text{const}_1 \cdot t + \text{const}$. Since by assumptions the function ψ is defined on the whole $\mathbb{R}$ and is equal to zero at no point, we have $\text{const}_1 = 0$ implying $\psi \equiv \text{const}$ along every light-line geodesic.

Now, every two points of a connected manifold can be connected by a finite sequence of light-line geodesics. Indeed, consider $\mathbb{R}^n$ with the standard pseudo-Euclidean metric g_0 of the same signature $(r, n-r)$, $1 \le r < n$ as the metric g. The union of all light-line geodesics passing through points a (resp. b) are the standard cones $C_a := \{(x_1, ..., x_n) \in \mathbb{R}^n \mid (x_1 - a_1)^2 + ... + (x_r - a_r)^2 - (x_{r+1} - a_{r+1})^2 - ... - (x_n - a_n)^2\}$ and, resp., $C_b := \{(x_1, ..., x_n) \in \mathbb{R}^n \mid (x_1 - b_1)^2 + ... + (x_r - b_r)^2 - (x_{r+1} - b_{r+1})^2 - ... - (x_n - b_n)^2\}$. These two cones always have points of transversal intersection. Thus, two arbitrary points of $\mathbb{R}^n$ can be connected by a sequence of two light-line geodesics of g_0. Since the restriction of the metric g to a small neighborhood $U \subseteq M^n$ can be viewed as a small perturbation of the metric g_0 in $\mathbb{R}^n$, two points in U can be connected by a sequence of two light-line geodesics. Then, the set of points of M that can be connected with a fixed point $p \in M^n$ by a finite sequence of light-line geodesics is open and closed implying it coincides with M.

Since every two points of M can be connected by a sequence of light-line geodesics, and since as we proved above the function ψ is constant along every light-line geodesic, we have that ψ is constant on the whole manifold as we claimed, □

Theorem 2. *Let g be a pseudo-Riemannian Einstein metric of indefinite signature on a connected closed (i.e., compact with no boundary) $(n > 2)-$dimensional manifold M. Assume that for the nowhere vanishing function ψ the metric $\psi^{-2}g$ is also Einstein. Then, ψ is a constant.*

Remark 5. Theorem 2 is not new and is in [9, Theorem 5]. Moreover, Wolfgang Kühnel explained us how one can obtain the proof combining the results of PhD thesis of Kerckhove, equation (1) due to [2], and also [7, Proposition 3.8(1)]. Our proof of Theorem 2 is much easier than the proofs of Mikes–Radulovich and Kühnel. Actually, the initial version of our paper did not contain Theorem 2 at all, but after J. Mikes sent us his paper we immediately saw that the proof of their Theorem 5 can be essentially simplified by using the trick from the proof of our Theorem 1.

Proof of Theorem 2. Since M is closed, there exists $p_0 \in M$ such that the value of ψ is maximal (we denote this value by ψ_{max}). We take a light-line geodesic γ such that $\gamma(0) = p_0$. As we explained in the proof of Theorem 1, the function $\psi(\gamma(t))$ is equal to $\text{const} \cdot t + \psi_{max}$. Since the value of ϕ at the point p_0 is maximal, $\text{const} = 0$ implying $\psi(\gamma(t)) \equiv \psi_{max}$. Then, for every point p_1 of geodesic γ the value of ψ is maximal. We can therefore repeat the argumentation and show that for every light-line geodesic γ_1 such that $\gamma_1(0) = p_1$ we have $\psi(\gamma_1(t)) \equiv \psi_{max}$ and so on. Since every two points of M can be connected by a sequence of light-line geodesics, we have that ψ is constant on the whole manifold, $\square$

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When we obtained the proof, we asked all experts we know whether the proof is new, and are grateful to those who answered, in particular to M. Eastwood, Ch. Frances, R. Gover, G. Hall, F. Leitner, and P. Nurowski.

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