Geodesic equivalence via integrability

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Abstract

We suggest a construction that, given an orbital diffeomorphism between two Hamiltonian systems, produces integrals of them. We treat geodesic equivalence of metrics as the main example of it. In this case, the integrals commute; they are functionally independent if the eigenvalues of the tensor $g^{i\alpha}\bar{g}_{\alpha j}$ are all different; if the eigenvalues are all different at least at one point then they are all different at almost each point and the geodesic flows of the metrics are Liouville integrable. This gives us topological obstacles to geodesic equivalence.

1 Introduction

Let $g = (g_{ij})$ and $\bar{g} = (\bar{g}_{ij})$ be smooth Riemannian metrics on the same manifold M^n of dimension $n \geq 2$.

Definition 1. *The metrics g and $\bar{g}$ are* geodesically equivalent*, if they have the same geodesics (considered as unparameterized curves).*

This is rather classical material. The problem of describing geodesically equivalent metrics was stated by Beltrami [3]. Since the time of Beltrami, the main tool for investigation of geodesically equivalent metrics was the following system of PDE

$$2(n+1)\bar{g}_{ij,k} = 2\bar{g}_{ij}\Theta_{,k} + \bar{g}_{ik}\Theta_{,j} + \bar{g}_{kj}\Theta_{,i}. \tag{1}$$

which is the criterion for the metrics g, $\bar{g}$ to be geodesically equivalent. Here Θ denotes the function $\ln\left(\frac{det(\bar{g})}{det(g)}\right)$ and $T_{,l}$ is the covariant derivative of the tensor T with respect to the metric g.

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Although the system is non-linear and over-determined, it is possible to solve it near the points where the eigenvalues of the tensor $g^{i\alpha}\bar g_{\alpha j}$ do not bifurcate. This was done by Dini [7] for surfaces and Levi-Civita [10] for manifolds of arbitrary dimension. All attempts to solve this system near the points of bifurcation or globally were unsuccessful. Moreover, there were almost no examples of geodesically equivalent metrics on closed manifolds: the only known examples were the Beltrami's examples of metrics geodesically equivalent to the round metric on the sphere [3], some examples on the torus which immediately follow from Levi-Civita [10], and a series of examples on the sphere with both metrics being certain metrics of revolution obtained by Mikes [19].

All known global results on geodesically equivalent metrics require additional strong geometrical assumptions. For example, for Einstein or (hyper)Kahlerian metrics beautiful results were obtained by Lichnerowicz [11], Venzi [28], Mikes [20], Couty [5] and Hasegawa and Fujimura [8].

2 Results

In our paper we present a construction which, given a diffeomorphism between two Hamiltonian systems that takes the orbits and the isoenergy surfaces of the first Hamiltonian system to the orbits and the isoenergy surfaces of the second one, produces n integrals of the first system, where n is the number of the degrees of freedom of the system.

The construction is applied to geodesically equivalent metrics: for such, an orbital diffeomorphism Φ is given by $\Phi(x,\xi) = (x, \frac{\|\xi\|_g}{\|\xi\|_{\bar g}}\xi)$. Here $(x,\xi) \in TM^n$, x is a point of M^n and $\xi \in T_xM^n$.

We identify the tangent and the cotangent bundles of M^n by the metric g. This gives us the Poisson bracket $\{\ ,\ \}$ on TM^n. Recall that two functions $F_1, F_2 : TM^n \to R$ *commute,* if the Poisson bracket $\{F_1, F_2\}$ vanishes. By the geodesic flow of the metric g we mean the Hamiltonian system with the Hamiltonian $H \stackrel{\rm def}{=} \frac{1}{2}g(\xi,\xi)$. It is known that, for any geodesic $\gamma : R \to M^n$, the curve $(\gamma,\dot\gamma) : R \to TM^n$ is an orbit of the geodesic flow of g and vice versa. Recall that a function $F : TM^n \to R$ is *an integral* for the geodesic flow of g, if it is constant on any orbit of the geodesic flow. It is known that a function is an integral for a Hamiltonian system, if and only if it commutes with the Hamiltonian.

The main result of the paper is the following theorem.

Theorem 1. *Let g and $\bar g$ on M^n be geodesically equivalent. Denote by $G : TM^n \to TM^n$ the fiberwise linear mapping given by the tensor $G^i_j = g^{i\alpha}\bar g_{\alpha j}$. Consider the characteristic polynomial*

$$det(G - \mu Id) = c_0\mu^n + c_1\mu^{n-1} + ... + c_n.$$

The coefficients $c_1,..,c_n$ are smooth functions on the manifold M^n, and $c_0 \equiv$

$(-1)^n$. *Then the functions* $I_k \stackrel{\text{def}}{=} g(S_k\xi,\xi)$, $k=0,1,...,n-1$, *where*

$$S_k \stackrel{\text{def}}{=} \left(\frac{det(g)}{det(\bar g)}\right)^{\frac{k+2}{n+1}} \sum_{i=0}^{k} c_i G^{k-i+1},$$

are integrals for the geodesic flow of the metric g *and commute pairwise.*

Remark 1. *According to [10], in local coordinates, the integral*

$$I_0 = \left(\frac{det(g)}{det(\bar g)}\right)^{\frac{2}{n+1}} \bar g(\xi,\xi)$$

was known to Painlevé. The integral I_{n-1} *is the energy integral (multiplied by minus two). The integrals* $I_1, I_2, ..., I_{n-2}$ *are new.*

Corollary 1. *Let the geodesic flow of* g *be ergodic. Then the following two statements are equivalent:*

1. *The metrics* $g, \bar g$ *are geodesically equivalent.*
2. *The metrics* $g, \bar g$ *are homothetic:* $\bar g = C\ g$, *where* C *is a positive constant.*

It is known (see, for example, [2]) that, for closed manifolds, the geodesic flow of any metric of negative sectional curvature is ergodic. Therefore each metric of negative sectional curvature on a closed manifold does not admit non-trivial example of geodesically equivalent metric.

Definition 2. *Two metrics* $g, \bar g$ *on* M^n *are* strictly non-proportional *at a point* $x \in M^n$, *if all eigenvalues of the restriction of the mapping* G *to* T_xM^n *are different.*

Let us denote by $N(x)$ the number of different eigenvalues of the restriction of G to T_xM^n.

Corollary 2 *Suppose* M^n *is connected. Let metrics* g, $\bar g$ *on* M^n *be geodesically equivalent. Then at almost every point* $y \in M^n$ *the number* $N(y)$ *is equal to*

$$\max_{x\in M^n} N(x).$$

By Corollary 2, if two geodesically equivalent metrics on a connected manifold are strictly non-proportional at a point then they are strictly non-proportional at almost each point of the manifold. Then, as we will show in Section 9, the integrals I_k are functionally independent almost everywhere so that the geodesic flow is completely integrable. This gives us a topological condition that prevents a closed real-analytic manifold from possessing a pair of real-analytic geodesically equivalent metrics that are strictly non-proportional at least at one point. This obstacle immediately follows from the following theorem of Taimanov [25].

Theorem 2 ([25]). *If a real-analytic closed manifold M^n with a real-analytic metric satisfies at least one of the conditions:*

a) $\pi_1(M^n)$ *does not contain a commutative subgroup of finite index or*

b) $dim H_1(M^n; \mathbf{Q}) > dim M^n$,

then the geodesic flow on M^n is not analytically integrable.

Corollary 3. *Let M^n be a closed connected real-analytic manifold supplied with two real-analytic metrics $g, \bar{g}$ such that the metrics $g, \bar{g}$ are geodesically equivalent and strictly non-proportional at least at one point. Then the fundamental group $\pi_1(M^n)$ of the manifold M^n contains a commutative subgroup of finite index, and the dimension of the homology group $H_1(M^n; \mathbf{Q})$ is no greater than n.*

For the two-dimensional case, in view of the results of Kolokoltsov [14] and Kiyohara [12], we do not need the condition for metrics to be real-analytic:

Corollary 4. *Let M^2 be a closed surface of genus greater than one equipped with metrics g, $\bar{g}$. Then the following two statements are equivalent:*

1. *The metrics $g, \bar{g}$ are geodesically equivalent.*
2. *The metrics $g, \bar{g}$ are homothetic: $\bar{g} = C\ g$, where C is a positive constant.*

This corollary is a partial answer to the following question: for closed surfaces, which local structures constructed by a given metric, determine the metric. Corollary 4 shows that, for surfaces of negative Euler characteristic, *the projective class* of a metric (= the set of all metrics geodesically equivalent to the metric) uniquely defines the metric modulo multiplication by a constant. The projective class of a metric is a differential-geometrical object (see for example Chapter 4 of Kobayashi [13]), and, generally speaking, it defines the metric neither locally nor on closed surfaces of genus one or zero.

For metrics on surfaces, the existence of a geodesically equivalent metric is equivalent to the existence of an integral quadratic in velocities. The metrics are homothetic if and only if the integral is proportional to the Hamiltonian of the geodesic flow, see [15]. Quadratically integrable geodesic flows on surfaces were described in Kolokoltsov [14], Kiyohara [12], Igarashi et al [9], Bolsinov et al [4] and Babenko and Nekhoroshev [1]. These results allow one to describe completely all pairs of geodesically equivalent metrics on closed surfaces, and also obtain a slightly weaker description for pairs of geodesically equivalent metrics on geodesically complete (with respect to one of the metrics) surfaces.

Below we reformulate some beautiful results from the theory of quadratically integrable geodesic flows on surfaces in terms of geodesically equivalent metrics:

Corollary 5. *Let metrics $g, \bar{g}$ on the torus T^2 be geodesically equivalent. If they are proportional at a point $x \in T^2$, then $g = C\bar{g}$, where C is a positive constant.*

Recall that a vector field on M^n is *Killing* (with respect to a metric), if the flow of the field preserves the metric.

Corollary 6. *Let metrics g, $\bar{g}$ on the sphere S^2 be geodesically equivalent. Then there are three possibilities.*

1. *The metrics are proportional at exactly two points.*
2. *The metrics are proportional at exactly four points.*
3. *The metrics are completely proportional, i.e. $g = C\bar{g}$, where C is a positive constant.*

In the first case the metrics admit a non-trivial Killing vector field.

In particular, if a metric on the 2-sphere admits a Killing vector field then any geodesically equivalent metric also admits a Killing vector field, and any two geodesically equivalent metric on the sphere must have points where they are not strictly non-proportional. These facts appear to be multidimensional:

Corollary 7. *If metrics $g, \bar{g}$ on a manifold M^n are geodesically equivalent, and if the metric g admits a non-trivial Killing vector field, then the metric $\bar{g}$ also admits a non-trivial Killing vector field.*

Corollary 8. *Let M^n be closed connected. Let g, $\bar{g}$ on M^n be geodesically equivalent. Suppose they are strictly non-proportional at each point of the manifold. Then the manifold can be covered by the torus.*

For the two-dimensional case, Corollary 8 is evident and even does not require the assumption for metrics to be geodesically equivalent. For dimensions more than two it is not trivial.

Theorem 1 suggests that we look for examples of geodesically equivalent metrics in the class of integrable geodesic flows. Probably the most famous integrable geodesic flow is that of the restriction of the Euclidean metric to the standard ellipsoid

$$\left\{ (x^1, x^2, ..., x^n) \in R^n : \sum_{i=1}^{n} \frac{(x^i)^2}{a_i} = 1 \right\}.$$

Here $a_i > 0$, $i = 1, ..., n$. It appears that this metric admits a geodesically equivalent one. This is the first example of a metric on the sphere such that it admits a geodesically equivalent one but does not admit a non-trivial Killing vector field.

Theorem 3 (Independently obtained in [27] and [23, 24]). *The restriction of the metric $\sum_{i=1}^{n}(dx^i)^2$ to the ellipsoid*

$$\left\{ (x^1, x^2, ..., x^n) \in R^n : \sum_{i=1}^{n} \frac{(x^i)^2}{a_i} = 1 \right\}$$

is geodesically equivalent to the restriction of the metric

$$\frac{1}{\sum_{i=1}^{n}\left(\frac{x^i}{a_i}\right)^2}\left(\sum_{i=1}^{n}\frac{(dx^i)^2}{a_i}\right)$$

to the same ellipsoid.

The paper is organized as follows. Section 3 is technical. In Section 4, we present the main construction. There, Theorem 4 produces an explicit formula for a one-parameter family of integrals from a given orbital diffeomorphism between two Hamiltonian systems.

In Section 5, we apply Theorem 4 to the orbital diffeomorphism for the geodesic flows of geodesically equivalent metrics, and prove that the functions $I_0, ..., I_{n-1}$ from Theorem 1 are integrals of the geodesic flow of the metric g.

In Section 6, we formulate Levi-Civita Theorem about a local form of geodesically equivalent metrics near the stable points and Painlevé results about commutativity of some special class of integrals.

In Section 7, we prove Corollary 2 and therefore show that almost each point of the manifold is stable; then it is sufficient to show the commutativity of the integrals I_k near the stable points only which is done in Section 8.

In Section 9 we show that, under the assumptions of Corollary 8, we can find a Liouville torus which covers the manifold.

In Section 10, we observe that the integral I_0 allows one to transform any linear (in velocities) integral for the geodesic flow of g into a linear (in velocities) integral for the geodesic flow of $\bar{g}$. This proves Corollary 7.

In Section 11, we verify that, in the elliptic coordinates, the metrics from Theorem 3 have precisely the form from Levi-Civita Theorem and therefore are geodesically equivalent.

In a slightly different form, Theorem 1 has been announced in [16], see also [27]. The quantum version of Theorem 1 has been announced in [17]; the proof appeared in [18]. For two-dimensional manifolds, Theorem 1 has been proven in [15, 4].

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3 The integrals I_k in the basis of eigenvectors of G

Let g, $\bar{g}$ be Riemannian metrics on M^n. We do not assume that the metrics are geodesically equivalent. Let us fix $x \in M^n$. Consider the linear mapping $G : T_xM^n \to T_xM^n$ given by the tensor $g^{i\alpha}\bar{g}_{\alpha j}$. Denote by $\rho_1 \geq \rho_2 \geq ... \geq \rho_n$ the eigenvalues of G. Since the quadratic form g, $\bar{g}$ are positive definite, the eigenvalues of G are real. Therefore, there exists a basis in the space T_xM^n such that in this basis the metric g is given by the matrix $diag(1, 1, ..., 1)$ and the mapping G is given by the matrix $diag(\rho_1, \rho_2, ..., \rho_n)$.

Denote by $\phi_1 \leq \phi_2 \leq ... \leq \phi_n$ the numbers given by

$$\phi_i \stackrel{\text{def}}{=} \frac{1}{\rho_i}(\rho_1\rho_2...\rho_n)^{\frac{1}{n+1}}.$$

Remark 2. *It is easy to see that the numbers ϕ_i are precisely the eigenvalues of the so-called Sinjukov mapping $(det(G))^{\frac{1}{n+1}}G^{-1}$.*

Denote by σ_p the elementary symmetric polynomial of degree p of n variables

$$\phi_1, \phi_2, ..., \phi_n.$$

Denote by $\sigma_p(\check{\phi}_i)$ the elementary symmetric polynomial of degree p of $n-1$ variables

$$\phi_1, \phi_2, ..., \phi_{i-1}, \phi_{i+1}, ..., \phi_n.$$

Consider the characteristic polynomial

$$det(G - \mu Id) = c_0\mu^n + c_1\mu^{n-1} + ... + c_n$$

of the mapping G. Consider the mappings $S_0, S_1, ..., S_{n-1}$ given by the general formula

$$S_k \stackrel{\text{def}}{=} \left(\frac{1}{det(G)}\right)^{\frac{k+2}{n+1}} \sum_{i=0}^{k} c_i G^{k-i+1}.$$

Lemma 1. *The matrices of the mappings S_k are given by*

$$S_k = (-1)^{n-k}\text{diag}\left(\sigma_{n-k-1}(\check{\phi}_1), \sigma_{n-k-1}(\check{\phi}_2), ..., \sigma_{n-k-1}(\check{\phi}_n)\right).$$

Proof. It is easy to see that the coefficients c_k are given by

$$c_k = (-1)^{n-k}\frac{\sigma_{n-k}}{(\phi_1\phi_2...\phi_n)^{k+1}}.$$

In particular,

$$det(G) = c_n = \frac{1}{(\phi_1\phi_2...\phi_n)^{n+1}}.$$

Let us verify the lemma for $k=0$. We have

$$\begin{aligned} S_0 &= \left(\frac{1}{det(G)}\right)^{\frac{2}{n+1}} c_0 G \\ &= (-1)^n(\phi_1\phi_2...\phi_n)^2 \mathrm{diag}\left(\frac{1}{\phi_1(\phi_1\phi_2...\phi_n)}, \frac{1}{\phi_2(\phi_1\phi_2...\phi_n)}, ..., \frac{1}{\phi_n(\phi_1\phi_2...\phi_n)}\right) \\ &= (-1)^n \mathrm{diag}(\phi_2\phi_3...\phi_n, \phi_1\phi_3...\phi_n, ..., \phi_1\phi_2...\phi_{n-2}\phi_n, \phi_1\phi_2...\phi_{n-2}\phi_{n-1})) \\ &= (-1)^n \mathrm{diag}\left(\sigma_{n-1}(\check{\phi}_1), \sigma_{n-1}(\check{\phi}_2), ..., \sigma_{n-1}(\check{\phi}_n)\right). \end{aligned}$$

Suppose that the lemma is true for S_{k-1}. Then for S_k we have

$$\begin{aligned} S_k &= \left(\frac{1}{det(G)}\right)^{\frac{1}{n+1}} G \left(S_{k-1} + \left(\frac{1}{det(G)}\right)^{\frac{k+1}{n+1}} c_k \; Id\right) \\ &= \mathrm{diag}\left(\frac{1}{\phi_1}, \frac{1}{\phi_2}, ..., \frac{1}{\phi_n}\right) \mathrm{diag}\left((-1)^{n-k}(\sigma_{n-k} - \sigma_{n-k}(\check{\phi}_1)),\right. \\ &\quad \left.(-1)^{n-k}(\sigma_{n-k} - \sigma_{n-k}(\check{\phi}_2)), ..., (-1)^{n-k}(\sigma_{n-k} - \sigma_{n-k}(\check{\phi}_n))\right). \end{aligned}$$

Using that $(\sigma_l - \sigma_l(\check{\phi}_i)) = \phi_i\sigma_{l-1}(\check{\phi}_i)$, we obtain that S_k is equal to

$$(-1)^{n-k}\mathrm{diag}\left(\sigma_{n-k-1}(\check{\phi}_1), \sigma_{n-k-1}(\check{\phi}_2), ..., \sigma_{n-k-1}(\check{\phi}_n)\right).$$

Lemma 1 is proved.

Consider the function $F : R \times TM^n \to R$ given by

$$F_t(x,\xi) = t^{n-1}I_{n-1}(x,\xi) + ... + I_0(x,\xi).$$

For a fixed point $(x,\xi) \in TM$, the function F_t is a polynomial in t.

In the proof of Lemma 2 we will show that all roots of the polynomial are real. Let us denote the roots of the polynomial by

$$t_1(x,\xi) \le t_2(x,\xi) \le ... \le t_{n-1}(x,\xi).$$

Lemma 2 *Let x be a point of M^n. Then for any $i \in \{1,2,...,n-1\}$ the following statements are true.*

1. *For any $\xi \in T_xM$,*
$$\phi_i(x) \le t_i(x,\xi) \le \phi_{i+1}(x).$$
In particular, if $\phi_i(x) = \phi_{i+1}(x)$ then $t_i(x,\xi) = \phi_i(x) = \phi_{i+1}(x)$.

2. *If $\phi_i(x) < \phi_{i+1}(x)$ then for any constant τ the Lebesgue measure of the set*
$$V_\tau \subset T_xM, \quad V_\tau \stackrel{\mathrm{def}}{=} \{\xi \in T_xM^n : t_i(x,\xi) = \tau\},$$
is zero.

Proof. In the proof we assume that the point $x \in M^n$ is fixed. For simplicity, we will write ϕ_i, $t_i(\xi)$ instead of $\phi_i(x)$, $t_i(x,\xi)$. There exists a basis in the space T_xM^n such that in this basis the metric g is given by the matrix $diag(1,1,...,1)$ and the mapping G is given by the matrix $diag(\rho_1,\rho_2,...,\rho_n)$.

Let us denote by P_i the polynomial

$$\begin{aligned} P_i(t) &= (t-\phi_1)(t-\phi_2)...(t-\phi_{i-1})(t-\phi_{i+1})...(t-\phi_n) \qquad (2)\\ &= \sum_{\alpha=0}^{n-1}(-1)^{n-\alpha-1}t^\alpha\sigma_{n-\alpha-1}(\check{\phi}_i). \end{aligned}$$

Then, for any $\xi = (\xi_1,\xi_2,...,\xi_n) \in T_xM^n$, the polynomial F_t has the following form:

$$\begin{aligned} F_t(x,\xi) &= \sum_{i=0}^{n-1}\sum_{\alpha=1}^{n}(-1)^{n-i}\xi_\alpha^2\sigma_{n-i-1}(\check{\phi}_\alpha)t^i \\ &= -\left(P_1(t)\xi_1^2 + P_2(t)\xi_2^2 + ... + P_n(t)\xi_n^2\right). \qquad (3) \end{aligned}$$

It is easy to see that the coefficients of the polynomial F_t depend continuously on the eigenvalues ϕ_i and on the components ξ_i. Then it is sufficient to prove the first statement of the lemma assuming that the eigenvalues ϕ_i are all different and that ξ_i are non-zero. For any $\alpha \neq i$, we evidently have $P_\alpha(\phi_i) \equiv 0$. Then

$$F_{\phi_i} = -\sum_{\alpha=1}^{n}P_\alpha(\phi_i)\xi_\alpha^2 = -P_i(\phi_i)\xi_i^2.$$

Hence F_{ϕ_i} and $F_{\phi_{i+1}}$ have different signs and therefore the open interval $]\phi_i,\phi_{i+1}[$ contains a root of the polynomial F_t. The degree of the polynomial F_t is equal $n-1$; we have $n-1$ disjoint intervals; any of these intervals contains at least one root so that all roots are real and the root number i lies between ϕ_i and ϕ_{i+1}. The first statement of the lemma is proved.

Let us prove the second statement of the lemma. Suppose $\phi_i < \phi_{i+1}$. Denote by k_i, k_{i+1} the multiplicities of the eigenvalues ρ_i, ρ_{i+1}, respectively. Consider the function

$$\hat{F}_t(x,\xi) : T_xM^n \to R, \quad \hat{F}_t(x,\xi) \stackrel{\text{def}}{=} \frac{F(x,\xi)}{(t-\phi_i)^{k_i-1}(t-\phi_{i+1})^{k_{i+1}-1}}.$$

Since each P_α from (2) has the factor $(t-\phi_i)^{k_i-1}(t-\phi_{i+1})^{k_{i+1}-1}$, the function $\hat{F}_t$ is polynomial in t. Take a constant $\tau \in [\phi_i,\phi_{i+1}]$. Clearly, if $t_i(x,\xi) = \tau$, then $\hat{F}_\tau(x,\xi) = 0$. Then the set V_τ is a subset of a non-degenerate quadric. Thus, its measure is zero. The lemma is proved.

4 Orbital diffeomorphisms and integrals

Let v and $\bar{v}$ be Hamiltonian systems on symplectic manifolds (M,ω) and $(\bar{M},\bar{\omega})$ with Hamiltonians H and $\bar{H}$ respectively. Consider the isoenergy surfaces

$$Q \stackrel{\text{def}}{=} \{x \in M : H(x) = h\}, \quad \bar{Q} \stackrel{\text{def}}{=} \left\{x \in \bar{M} : \bar{H}(x) = \bar{h}\right\},$$

where h and $\bar{h}$ are regular values of the functions H, $\bar{H}$ respectively. Let $U(Q) \subset M$ and $U(\bar{Q}) \subset \bar{M}$ be neighborhoods of the isoenergy surfaces Q and $\bar{Q}$.

Definition 3. *A diffeomorphism $\Phi : U(Q) \longrightarrow U(\bar{Q})$, $\Phi(Q) = \bar{Q}$, is said to be* orbital on Q, *if the restriction $\Phi|_Q$ takes the orbits of the system v to the orbits of the system $\bar{v}$.*

Denote the restriction $\Phi|_Q$ by ϕ. Since ϕ takes the orbits of v to the orbits of $\bar{v}$, it takes the vector field v to the vector field that is proportional to $\bar{v}$. Denote by $a_1 : Q \to R$ the coefficient of proportionality, i.e. $\phi_*(v) = a_1\bar{v}$. Since Φ takes Q to $\bar{Q}$, it takes the differential $\mathrm{d}H$ to a form that is proportional to $d\bar{H}$. Denote by $a_2 : Q \to R$ the coefficient of proportionality, i.e. $\phi_*\mathrm{d}H = a_2\mathrm{d}\bar{H}$. By a we denote the product $a_1 a_2$. We denote the Pfaffian of a skew-symmetric matrix X by $\mathrm{Pf}(X)$.

Theorem 4. *Let a diffeomorphism $\Phi : U(Q) \to U(\bar{Q})$, $\Phi(Q) = \bar{Q}$, be orbital on Q. Then for each value of the parameter t the polynomial*

$$\mathcal{P}^{n-1}(t) \stackrel{\mathrm{def}}{=} \frac{\mathrm{Pf}\,(\Phi^*\bar{\omega} - t\omega)}{\mathrm{Pf}\,(\omega)\,(t-a)} \tag{4}$$

is an integral of the system v on Q. In particular, all the coefficients of the polynomial $\mathcal{P}^{n-1}(t)$ are integrals.

Proof. Denote by σ, $\bar{\sigma}$ the restrictions of the forms $\omega, \bar{\omega}$ to Q, $\bar{Q}$ respectively. Consider the form $\phi^*\bar{\sigma}$ on Q.

Lemma 3 ([26]). *The flow v preserves the form $\phi^*\bar{\sigma}$.*

Proof of Lemma 3. The Lie derivative L_v of the form $\phi^*\bar{\sigma}$ along the vector field v satisfies

$$L_v\phi^*\bar{\sigma} = \mathrm{d}\,[\imath_v\phi^*\bar{\sigma}] + \imath_v\mathrm{d}\,[\phi^*\bar{\sigma}]\,.$$

On the right side both terms vanish. More precisely, for an arbitrary vector $u \in T_xQ$ at an arbitrary point $x \in Q$ we have

$$\begin{aligned} \imath_v\phi^*\bar{\sigma}(u) &= \bar{\sigma}(\phi_*(v), \phi_*(u)) &&= \\ &= \bar{\sigma}(a_1\bar{v}, \phi_*(u)) &&= \\ &= -a_1\mathrm{d}\bar{H}(\phi_*(u)) &&= 0. \end{aligned}$$

Since the form $\bar{\omega}$ is closed, the form $\bar{\sigma}$ is also closed and $\mathrm{d}\,[\phi^*\bar{\sigma}] = \phi^*(\mathrm{d}\bar{\sigma}) = 0$. Lemma 3 is proved.

It is obvious that the kernels of the forms σ and $\phi^*\bar{\sigma}$ coincide (in the space T_xQ at each point $x \in Q$) with the linear span of the vector v. Therefore these forms induce two non-degenerate tensor fields on the quotient bundle $TQ/\langle v\rangle$. We shall denote the corresponding forms on $TQ/\langle v\rangle$ also by the letters $\sigma, \phi^*\bar{\sigma}$.

Lemma 4. *The characteristic polynomial of the operator $(\sigma)^{-1}(\phi^*\bar{\sigma})$ on $TQ/\langle v\rangle$ is preserved by the flow v.*

Proof of Lemma 4. Since the flow v preserves the Hamiltonian H and the form ω, the flow v preserves the form σ. Since the flow v preserves both forms, it preserves the characteristic polynomial of the operator $(\sigma)^{-1}(\phi^*\bar{\sigma})$. Lemma 4 is proved.

Since both forms are skew-symmetric, each root of the characteristic polynomial of the operator $(\sigma)^{-1}(\Phi^*\bar{\sigma})$ has an even multiplicity. Then the characteristic polynomial is the square of a polynomial $\delta^{n-1}(t)$ of degree $n-1$. Hence the polynomial $\delta^{n-1}(t)$ is also preserved by the flow v. It is obvious that

$$\delta^{n-1}(t) = (-1)^{n-1}\frac{\mathrm{Pf}\,(\phi^*\bar{\sigma} - t\sigma)}{\mathrm{Pf}\,(\sigma)}. \tag{5}$$

The last step of the proof is to verify that

$$(t-a)\delta^{n-1} = \frac{\mathrm{Pf}\,(\Phi^*\bar{\omega} - t\omega)}{\mathrm{Pf}\,(\omega)} \stackrel{\mathrm{def}}{=} \Delta^n.$$

Take an arbitrary point $x \in Q$. Consider the form $\Phi^*\bar{\omega} - a\omega$ on T_xM. The form $\imath_v(\Phi^*\bar{\omega} - a\omega)$ equals zero. More precisely, for any vector $u \in T_xM$ we have

$$\begin{aligned}
\imath_v(\Phi^*\bar{\omega} - a\omega) &= \bar{\omega}(\Phi_*(v), \Phi_*(u)) - a\omega(v,u) = \\
&= \bar{\omega}(a_1 v, \Phi_*(u)) - a\omega(v,u) = \\
&= -a_1 \mathrm{d}\bar{H}(\Phi_*(u)) + a\mathrm{d}H = \\
&= -a\mathrm{d}H + a\mathrm{d}H = 0.
\end{aligned}$$

There exists a vector $A \in T_xM$ such that $\omega(A,v) \neq 0$ and the restriction of the form $\imath_A(\Phi^*\bar{\omega} - a\omega)$ to the space T_xM equals zero. More precisely, since the forms $\Phi^*\bar{\omega}$, ω are skew-symmetric, then the kernel $K_{\Phi^*\bar{\omega}-a\omega}$ of the form $\Phi^*\bar{\omega} - a\omega$ has an even dimension, and the kernel of the restriction of the form $\Phi^*\bar{\omega} - a\omega$ to T_xQ has an odd dimension. Thus the intersection $K_{\Phi^*\bar{\omega}-a\omega} \cap (T_xM \setminus T_xQ)$ is not empty. For each vector A from the intersection we obviously have $\omega(A,v) \neq 0$ and $\imath_A(\Phi^*\bar{\omega} - a\omega) = 0$. Without loss of generality we can assume $\omega(A,v) = 1$.

Consider a basis $(v, e_1, ..., e_{2n-2})$ for the space T_xQ. The set $(A, v, e_1, ..., e_{2n-2})$ is a basis for the space T_xM. In this basis we have

$$\begin{aligned}
\det(\Phi^*\bar{\omega} - t\omega) &= \det\left|\begin{array}{cc|c} 0 & a-t & (*) \\ -(a-t) & 0 & 0\cdots 0 \\ \hline -(*) & 0 & (\Phi^*\bar{\omega} - t\omega)_{\langle e_1,\ldots,e_{2n-2}\rangle} \end{array}\right| \\
&= (a-t)^2 \det((\Phi^*\bar{\omega} - t\omega)_{\langle e_1,\ldots,e_{2n-2}\rangle}) \\
&= (a-t)^2 \det(\phi^*\bar{\sigma} - t\sigma),
\end{aligned}$$

where $(\Phi^*\bar{\omega} - t\omega)_{\langle e_1,\ldots,e_{2n-2}\rangle}$ is the matrix of the form $\Phi^*\bar{\omega} - t\omega$ in the basis $(e_1, ...e_{2n-2})$. Finally, $\delta^{n-1} = \mathcal{P}^{n-1}$. Theorem 4 is proved.

5 Geodesic equivalence and corresponding integrals

Let g and $\bar g$ be Riemannian metrics on a manifold M^n. Let them be geodesically equivalent. Our goal is to prove that the functions I_k from Theorem 1 are integrals of the geodesic flow of g.
Define

$$U_g^r M^n \stackrel{\text{def}}{=} \{(x,\xi) \in TM^n : \quad ||\xi||_g = r\},$$

where $x \in M^n$, $\xi \in T_xM^n$ and $||\xi||_g \stackrel{\text{def}}{=} \sqrt{g(\xi,\xi)} = \sqrt{g_{ij}\xi^i\xi^j}$ is the norm of the vector ξ in the metric g.

By the geodesic flow of the metric g we mean the Hamiltonian system on TM^n (as a symplectic form we take $\omega_g \stackrel{\text{def}}{=} \mathrm{d}[g_{ij}\xi^j dx^i]$) with the Hamiltonian $H_g \stackrel{\text{def}}{=} \frac{1}{2}g_{ij}\xi^i\xi^j$.

Since the metrics $g, \bar g$ are geodesically equivalent, the mapping $\Phi : TM^n \to TM^n$, $\Phi(x,\xi) = \left(x, \xi\frac{||\xi||_g}{||\xi||_{\bar g}}\right)$, takes the orbits of the geodesic flow of the metric g to the orbits of the geodesic flow of the metric $\bar g$. This mapping is a diffeomorphism (for $r \neq 0$), takes $U_g^rM^n$ to $U_{\bar g}^rM^n$ and is orbital on $U_g^rM^n$. Obviously the surfaces U_g^r, $U_{\bar g}^r$ are regular isoenergy surfaces $\{H_g = \frac{r}{2}\}$, $\{H_{\bar g} = \frac{r}{2}\}$.

By Theorem 4, in order to obtain a family of first integrals we have to find the polynomial $\Delta^n(t)$ and divide it by $(t-a)$. In our case $H_g = H_{\bar g}\circ\Phi$. Therefore the function a from Theorem 4 equals to $\frac{||\xi||_{\bar g}}{||\xi||_g}$.
In coordinates we have

$$\omega_g = \mathrm{d}[g_{ij}\xi^j dx^i] \quad \text{and} \quad \omega_{\bar g} = \mathrm{d}[\bar g_{ij}\xi^j dx^i].$$

Therefore,

$$\begin{aligned}
\Phi^*\omega_{\bar g} &= \mathrm{d}\left[\frac{||\xi||_g}{||\xi||_{\bar g}}\bar g_{ij}\xi^j dx^i\right] = \\
&= \frac{\partial}{\partial x^k}\left[\frac{||\xi||_g}{||\xi||_{\bar g}}\bar g_{ij}\xi^j\right] dx^k \wedge dx^i - \\
&- \frac{\partial}{\partial \xi^k}\left[\frac{||\xi||_g}{||\xi||_{\bar g}}\bar g_{ij}\xi^j\right] dx^i \wedge d\xi^k.
\end{aligned}$$

It is easy to see that at a point $\xi \in T_xM^n$ the quantities

$$A_{ik} \stackrel{\text{def}}{=} -\frac{\partial}{\partial \xi^k}\left[\frac{||\xi||_g}{||\xi||_{\bar g}}\bar g_{ij}\xi^j\right]$$

form an element of $T_xM^n \otimes T_xM^n$. Since the metrics are positively definite, we can choose a basis in T_xM^n such that the matrices of the metrics g, $\bar g$ are diagonal matrices $diag(1,1,...,1)$ and $diag(\rho_1,\rho_2,...,\rho_n)$, respectively. Then

$$A_{ij} \stackrel{\text{def}}{=} -\rho_i\frac{\partial}{\partial \xi^j}\left(\xi^i\frac{\sqrt{{\xi^1}^2+...+{\xi^n}^2}}{\sqrt{\rho_1{\xi^1}^2+...+\rho_n{\xi^n}^2}}\right) =$$

$$
\begin{aligned}
&= \rho_i \delta^i_j \frac{||\xi||_g}{||\xi||_{\bar g}} - \rho_i \xi^i \left(\frac{\frac{||\xi||_{\bar g}}{||\xi||_g} - \rho_j \frac{||\xi||_g}{||\xi||_{\bar g}}}{||\xi||^2_{\bar g}} \xi^j \right) = \\
&= \mathrm{diag}(\mu_1, ..., \mu_n) - A \otimes B,
\end{aligned}
$$

where $\mu_i \stackrel{\mathrm{def}}{=} -\rho_i \frac{||\xi||_g}{||\xi||_{\bar g}}$, $A_i \stackrel{\mathrm{def}}{=} \rho_i \xi^i$ and

$$
B_i \stackrel{\mathrm{def}}{=} \frac{\frac{||\xi||_{\bar g}}{||\xi||_g} - \rho_i \frac{||\xi||_g}{||\xi||_{\bar g}}}{||\xi||^2_{\bar g}} \xi^i.
$$

We have

$$
\begin{aligned}
\det(\Phi^* \omega_{\bar g} - t\omega_g) &= \det \left| \begin{array}{c|c} (*) & (A_{ij} + t\delta_{ij}) \\ \hline -(A_{ij} + t\delta_{ij}) & 0 \end{array} \right| \\
&= \det(A_{ij} + t\delta_{ij})^2.
\end{aligned}
$$

Therefore,

$$
\Delta^n(t) = \det\left(\mathrm{diag}(t + \mu_1, ..., t + \mu_n) - a \otimes b\right). \tag{6}
$$

Lemma 5. *The following relation holds:*

$$
\begin{aligned}
\Delta^n(t) &= (t + \mu_1) \cdots (t + \mu_n) - (a_1 b_1)(t + \mu_2) \cdots (t + \mu_n) - \ldots \\
&- (t + \mu_1) \cdots (t + \mu_{n-1})(a_n b_n).
\end{aligned} \tag{7}
$$

The lemma follows from induction considerations.

To divide the polynomial by $(t - a)$ we shall use the Horner scheme. Suppose that $\Delta^n(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$ and $\delta^{n-1}(t) = t^{n-1} + b_{n-2}t^{n-2} + \cdots + b_0$. Then we have

$$
\begin{aligned}
&b_{n-1} = a_n = 1, \\
&b_{n-2} = a_{n-1} + a, \qquad (8) \\
&\cdots \\
&b_k = a_{k+1} + ab_{k+1}, \\
&\cdots \\
&0 = a_0 + ab_0. \qquad (9)
\end{aligned}
$$

It follows from Lemma 5 that

$$
\begin{aligned}
a_0 &= (\mu_1 ... \mu_n) - (A_1 B_1)(\mu_2 ... \mu_n) - \cdots - (\mu_1 ... \mu_{n-1}) A_n B_n = \\
&= (-1)^n \left(\frac{||\xi||_g}{||\xi||_{\bar g}} \right)^n (\rho_1 \cdots \rho_n).
\end{aligned}
$$

Combining with (9) we get

$$
b_0 = -\frac{a_0}{a} = (-1)^{n+1} \left(\frac{||\xi||_g}{||\xi||_{\bar g}} \right)^{n+1} (\rho_1 \cdots \rho_n).
$$

Since $\frac{1}{2}g_{ij}\xi^i\xi^j$ is an integral of the geodesic flow of the metric g, the function

$$I_0 = (\rho_1 \cdots \rho_n)^{-\frac{2}{n+1}} \bar{g}(\xi, \xi)$$

is also an integral of the geodesic flow of the metric g. Using Lemma 5 we have

$$\begin{aligned} a_{n-1} &= (\mu_1 + ... + \mu_n) - (A_1B_1 + ... + A_nB_n) = \\ &= \frac{||\xi||_g}{||\xi||^3_{\bar{g}}} \left\{ ({\rho_1}^2{\xi^1}^2 + ... + {\rho_n}^2 \xi^{n2}) - \right. \\ &- \left. (\rho_1 + ... + \rho_n)(\rho_1 {\xi^1}^2 + ... + \rho_n \xi^{n2}) \right\} - \frac{||\xi||_{\bar{g}}}{||\xi||_g}. \end{aligned}$$

Using (8), we obtain

$$\begin{aligned} b_{n-2} &= a_{n-2} + a = \\ &= \frac{||\xi||_g}{||\xi||^3_{\bar{g}}} \left\{ ({\rho_1}^2{\xi^1}^2 + ... + {\rho_n}^2 \xi^{n2}) - \right. \\ &- (\rho_1 + ... + \rho_n)(\rho_1 {\xi^1}^2 + ... + \rho_n \xi^{n2}) \} \end{aligned}$$

Therefore, the function

$$\begin{aligned} I_1 &\stackrel{\mathrm{def}}{=} (\rho_1 \cdots \rho_n)^{-\frac{3}{n+1}} \left\{ ({\rho_1}^2{\xi^1}^2 + ... + {\rho_n}^2 \xi^{n2}) - \right. \\ &- \left. (\rho_1 + ... + \rho_n)(\rho_1 {\xi^1}^2 + ... + \rho_n \xi^{n2}) \right\} \end{aligned}$$

is an integral. (It is easy to see that $\frac{||\xi||^2_g}{||\xi||^2_{\bar{g}}} = (\rho_1 \cdots \rho_n)^{-\frac{2}{n+1}} \frac{||\xi||^2_g}{I_0}$.)

Arguing as above, we see that the functions

$$\begin{aligned} I_k &\stackrel{\mathrm{def}}{=} (\rho_1 \cdots \rho_n)^{-\frac{k+2}{n+1}} \left\{ ({\rho_1}^{k+1}{\xi^1}^2 + ... + {\rho_n}^{k+1} \xi^{n2}) - \right. \\ &- (\rho_1 + ... + \rho_n)({\rho_1}^k {\xi^1}^2 + ... + {\rho_n}^k \xi^{n2}) + \cdots \\ &+ \left. (-1)^k \sigma_k(\rho_1, ..., \rho_n)(\rho_1 {\xi^1}^2 + ... + \rho_n \xi^{n2}) \right\}, \end{aligned}$$

are integrals of the geodesic flow of the metric g, where by σ_k we denote the elementary symmetric polynomial of degree k. It is obvious that $(-1)^k \sigma_k = c_k$ from Theorem 1, and therefore $I_k = g(S_k\xi, \xi)$. Thus I_k, $k = 0, ..., n-1$, are integrals of the geodesic flow of the metric g.

6 Levi-Civita Theorem

Let g, $\bar{g}$ be Riemannian metrics on M^n. Consider the fiberwise-linear mapping G given by the tensor $g^{i\alpha}\bar{g}_{\alpha j}$. At each point $x \in M^n$, consider the different eigenvalues $\rho_1(x) < \rho_2(x) < ... < \rho_m(x)$ of the restriction of the mapping G to T_xM^n. Let $k_i(x)$ be the multiplicity of the eigenvalue $\rho_i(x)$ so that $k_1(x) + ... + k_m(x) = n$. Consider the ordered set $K(x) \stackrel{\mathrm{def}}{=} \{k_1(x), k_2(x), ..., k_m(x)\}$.

Definition 4. *A point $x \in M^n$ is called* stable *(with respect to the metrics g, $\bar g$), if it has a neighborhood $U(x)$ such that $K(x) = K(y)$ for any $y \in U(x)$.*

The following theorem was proved in Levi-Civita [10].

Theorem 5 ([10]). *Let g, $\bar g$ be Riemannian metrics on M^n. Let a point $x \in M^n$ be stable; let $K(x)$ be equal to $\{k_1, k_2, ..., k_m\}$. The metrics are geodesically equivalent in some sufficiently small neighborhood $U(x)$ of the point x, if and only if there exists a coordinate system $\bar x = (\bar x_1, ..., \bar x_m)$ (in $U(x)$), where $\bar x_i = (x_i^1, ..., x_i^{k_i})$, $(1 \le i \le m)$, such that the quadratic forms of the metrics g and $\bar g$ have the following form:*

$$\begin{aligned} g(\dot{\bar x}, \dot{\bar x}) &= \Pi_1(\bar x)A_1(\bar x_1, \dot{\bar x}_1) + \Pi_2(\bar x)A_2(\bar x_2, \dot{\bar x}_2) + \cdots + \\ &+ \Pi_m(\bar x)A_m(\bar x_m, \dot{\bar x}_m), \qquad (10) \\ \bar g(\dot{\bar x}, \dot{\bar x}) &= \rho_1\Pi_1(\bar x)A_1(\bar x_1, \dot{\bar x}_1) + \rho_2\Pi_2(\bar x)A_2(\bar x_2, \dot{\bar x}_2) + \cdots + \\ &+ \rho_m\Pi_m(\bar x)A_m(\bar x_m, \dot{\bar x}_m), \qquad (11) \end{aligned}$$

where $A_i(\bar x_i, \dot{\bar x}_i)$ are positive-definite quadratic forms in the velocities $\dot{\bar x}_i$ with coefficients depending on $\bar x_i$,

$$\Pi_i \stackrel{\text{def}}{=} (\phi_i - \phi_1)\cdots(\phi_i - \phi_{i-1})(\phi_{i+1} - \phi_i)\cdots(\phi_m - \phi_i)$$

and $\phi_1, \phi_2, ..., \phi_m$, $0 < \phi_1 < \phi_2 < ... < \phi_m$, are smooth functions such that

$$\phi_i = \begin{cases} \phi_i(\bar x_i), & \textit{if} \quad k_i = 1 \\ \text{constant}, & \textit{otherwise.} \end{cases}$$

It is easy to see that the functions ρ_i as functions of ϕ_i and the functions ϕ_i as functions of ρ_i are given by

$$\begin{aligned} \rho_i &= \frac{1}{\phi_1 ... \phi_m}\frac{1}{\phi_i} \\ \phi_i &= \frac{1}{\rho_i}(\rho_1\rho_2...\rho_m)^{\frac{1}{m+1}} \end{aligned}$$

so that if $m = n$ then the numbers ϕ_i from Theorem 5 coincide with the numbers ϕ_i from Sections 3,7.

Levi-Civita observed that the following functions

$$\begin{aligned} L_1 &= \Pi_1 A_1 + \cdots + \Pi_m A_m, \ \text{ which is twice the energy integral,} \\ L_2 &= \sigma_1(\phi_2, ..., \phi_m)\Pi_1 A_1 + \cdots + \sigma_1(\phi_1, ..., \phi_{m-1})\Pi_m A_m, \\ L_3 &= \sigma_2(\phi_2, ..., \phi_m)\Pi_1 A_1 + \cdots + \sigma_2(\phi_1, ..., \phi_{m-1})\Pi_m A_m, \\ &\vdots \\ L_m &= (\phi_2 ... \phi_m)\Pi_1 A_1 + \cdots + (\phi_1 ... \phi_{m-1})\Pi_m A_m, \end{aligned}$$

are integrals of the geodesic flows of the metric g. Here σ_k denotes the elementary symmetric polynomial of degree k of the indicated variables. We will call these integrals *Levi-Civita integrals.*

From the results of Painlevé [22] it follows that Levi-Civita integrals commute. More precisely, let $D = (d^i_j)$ be an $m \times m$ matrix. Suppose that for any i, j the element d^i_j depends only on the variables $\bar{x}_j$. Denote by Δ the determinant of the matrix D and by Δ^i_j the minor of the element d^i_j. In the paper [22] it was shown that, for arbitrary functions $A_i(\bar{x}_i, \dot{\bar{x}}_i)$, quadratic in velocities $\dot{\bar{x}}_i$, the Lagrangian system with Lagrangian

$$T_1 = \Delta \left(\frac{A_1(\bar{x}_1, \dot{\bar{x}}_1)}{\Delta^1_1} + \frac{A_2(\bar{x}_2, \dot{\bar{x}}_2)}{\Delta^1_2} + \dots + \frac{A_m(\bar{x}_m, \dot{\bar{x}}_m)}{\Delta^1_m} \right)$$

admits $(m-1)$ integrals

$$T_i = \Delta \left(A_1(\bar{x}_1, \dot{\bar{x}}_1) \frac{\Delta^i_1}{(\Delta^1_1)^2} + A_2(\bar{x}_2, \dot{\bar{x}}_2) \frac{\Delta^i_2}{(\Delta^1_2)^2} + \dots + A_m(\bar{x}_m, \dot{\bar{x}}_m) \frac{\Delta^i_m}{(\Delta^1_m)^2} \right),$$

where $i = 2, \dots, m$, and if we identify the tangent and cotangent bundles the Lagrangian T_1 and consider the standard symplectic form on the cotangent bundle, then the integrals commute.

If we take $d^i_j = (\phi_j)^{m-i}$, then Δ and Δ^i_j are given by

$$\Delta^i_j = (-1)^{m-1} \sigma_{i-1}(\phi_1, \phi_2, \dots, \phi_{j-1}, \phi_{j+1}, \dots, \phi_m) \prod_{\alpha > \beta \geq 1, \alpha \neq j, \beta \neq j} (\phi_\alpha - \phi_\beta),$$

$$\Delta = (-1)^m \prod_{\alpha > \beta \geq 1} (\phi_\alpha - \phi_\beta).$$

Therefore,

$$\frac{\Delta \Delta^i_j}{(\Delta^1_j)^2} = \sigma_{i-1}(\phi_1, \phi_2, \dots, \phi_{j-1}, \phi_{j+1}, \dots, \phi_m) \Pi_j,$$

so $T_i = -L_i$ and thus the integrals L_i commute.

7 The eigenvalues of G behave regularly

Let g, $\bar{g}$ be Riemannian metrics on M^n. At each point $x \in M^n$, consider the linear mapping $G : T_x M^n \to T_x M^n$ given by the tensor $g^{i\alpha} \bar{g}_{\alpha j}$. Denote by $\rho_1(x) \geq \rho_2(x) \geq \dots \geq \rho_n(x)$ its eigenvalues. As in Section 3, we denote by $\phi_1(x) \leq \phi_2(x) \leq \dots \leq \phi_n(x)$ the numbers

$$\phi_i(x) \stackrel{\text{def}}{=} \frac{1}{\rho_i(x)} (\rho_1(x) \rho_2(x) \dots \rho_n(x))^{\frac{1}{n+1}}.$$

Lemma 6. *Suppose the metrics g, $\bar{g}$ on M^n are geodesically equivalent. Consider a geodesic segment $\gamma : [0 - \epsilon, 1 + \epsilon] \to M^n$ in the metric g, where ϵ is a small positive number. Then for any $i \in \{1, \dots, n-1\}$ the following statements are true:*

1. $\phi_i(\gamma(0)) \leq \phi_{i+1}(\gamma(1))$.

2. *If $\phi_i(\gamma(0)) < \phi_{i+1}(\gamma(0))$ then there exists a neighborhood $U(\gamma(1))$ of the point $\gamma(1)$ such that $\phi_i(z) < \phi_{i+1}(z)$ at almost every point $z \in U(\gamma(1))$.*

3. *If $\phi_i(\gamma(0)) = \phi_{i+1}(\gamma(1))$ then there exists $\tau \in [0,1]$ such that*

$$\phi_i(\gamma(\tau)) = \phi_{i+1}(\gamma(\tau)).$$

Remark 3. *Since any two points of a connected manifold can be joined by a polygonal line made up of geodesic segments, Corollary 2 immediately follows from the second statement of Lemma 6. We will use the first and the third statements in Section 9.*

Evidently, any point x such that $N(x) = \max_{y \in M^n} N(y)$ is stable.

Corollary 9. *For geodesically equivalent metrics, the set of stable points is everywhere dense.*

Proof of Lemma 6. Suppose the metrics g, $\bar{g}$ on M^n are geodesically equivalent. Consider the geodesic $\gamma : [0-\epsilon, 1+\epsilon] \to M^n$ in the metric g. As in Section 3, consider the polynomial in t

$$F_t(x,\xi) = t^{n-1} I_{n-1}(x,\xi) + ... + I_0(x,\xi)$$

and its roots

$$t_1(x,\xi) \le t_2(x,\xi) \le ... \le t_{n-1}(x,\xi).$$

By Theorem 1, the functions I_k are constant on the orbits of the geodesic flow of g. Then each root t_i is also constant on each orbit $(\gamma, \dot\gamma)$ of the geodesic flow of g so that

$$t_i(\gamma(0), \dot\gamma(0)) = t_i(\gamma(1), \dot\gamma(1)).$$

Using Lemma 2, we obtain

$$\phi_i(\gamma(0)) \le t_i(\gamma(0), \dot\gamma(0)), \quad \text{and} \quad t_i(\gamma(1), \dot\gamma(1)) \le \phi_{i+1}(\gamma(1)).$$

Therefore $\phi_i(\gamma(0)) \le \phi_{i+1}(\gamma(1))$ and the first statement of Lemma 6 is proved.

Let us prove the second statement of Lemma 6. There exists a sufficiently small neighborhood $U(\gamma(1))$ of the point $\gamma(1)$ such that the point $\gamma(0)$ can be joined with any point of $U(\gamma(1))$ by a geodesic lying in a small tubular neighborhood of the geodesic γ. We assume that any two points of the neighborhood $U(\gamma(1))$ can be joined by a geodesic; for example we can assume that $U(\gamma(1))$ is the ball of a small radius. Suppose $\phi_i(y) = \phi_{i+1}(y)$ for any point y of some subset $V \subset U(\gamma(1))$. Then by the first statement of Lemma 6, the value of ϕ_i is a constant (independent of $y \in V$). Indeed, joining any two points $y_0, y_1 \in V$ by a geodesic, we have

$$\phi_i(y_0) \le \phi_{i+1}(y_1) \ \text{ and } \ \phi_i(y_1) \le \phi_{i+1}(y_0).$$

Denote this constant by C. Let us prove that $\phi_i(\gamma(0)) = \phi_{i+1}(\gamma(0)) = C$. Let us join the point $\gamma(0)$ with every point of V by all possible geodesics. Consider the set $V_C \subset T_{\gamma(0)}M^n$ of the initial velocity vectors (at the point $\gamma(0)$) of these geodesics.

By the first statement of Lemma 2, for any geodesic γ_1 passing through any point of V, the value $t_i(\gamma_1, \dot\gamma_1)$ is equal to C. Then, by the second statement of Lemma 2, the measure of the set V_C is zero and therefore the measure of the set V is also zero. The second statement of Lemma 6 is proved.

Let us prove the third statement of Lemma 6. Let $\phi_i(\gamma(0)) = \phi_{i+1}(\gamma(1)) = \phi$ for some $i \in \{1, .., n-1\}$ (and for some constant ϕ). We will assume that $\phi_i(\gamma(0)) < \phi_{i+1}(\gamma(0))$. Let us show that the geodesic γ consists of the points where $\phi_i = \phi$ or $\phi_{i+1} = \phi$.

If t_i is a multiple root of the polynomial $F_t(\gamma(0), \dot\gamma(0))$, or if there exists a point $z \in U(\gamma(0)) \subset M^n$ such that $\phi_{i-1}(z) = \phi$ then the statement obviously follows from Lemma 2 and the first statement of Lemma 6. Suppose t_i is not a multiple root and $\phi_{i-1}(z) < \phi$ for any z from some neighborhood of $\gamma(0)$.

Consider the function $F_\phi : TM^n \to R$. Let at some point $(z, \nu) \in TM^n$, $\nu \neq 0$, the differential dF_ϕ is zero. Let us show that then $\phi_i = \phi$ or $\phi_{i+1} = \phi$.

Indeed, consider the coordinate system such that the metric g at the point z is given by the diagonal matrix $diag(1, 1, .., 1)$ and the mapping G is given by the diagonal matrix $diag(\rho_1, \rho_2, .., \rho_n)$. Then the restriction of the function F_ϕ to the tangent space T_zM^n is given by

$$-\sum_{\alpha=1}^{n} P_\alpha(\phi)\xi_\alpha^2,$$

where the polynomials P_i are given by (2). The partial derivatives $\frac{\partial F_\phi}{\partial \xi_\alpha}$ are

$$\frac{\partial F_\phi}{\partial \xi_\alpha} = -2P_\alpha(\phi)\xi_\alpha.$$

Then ϕ is equal to one of the numbers $\phi_1, ..., \phi_n$; by assumption it can be equal to either $\phi_i(z)$ or $\phi_{i+1}(z)$.

Now let us show that the differential dF_ϕ vanishes at every point $(\gamma(\tau), \dot\gamma(\tau))$. Evidently the differential of any integral is preserved by the geodesic flow so that it is sufficient to prove that the differential vanishes at the point $(\gamma(0), \dot\gamma(0))$.

By Lemma 2, we have

$$\phi = \phi_i(\gamma(0)) \le t_i(\gamma(0), \dot\gamma(0)) = t_i(\gamma(1), \dot\gamma(1)) \le \phi_{i+1}(\gamma(1)) = \phi$$

so that ϕ is a root of the polynomial $F_t(\gamma(0), \dot\gamma(0))$ and therefore the orbit $(\gamma, \dot\gamma)$ of the geodesic flow of g lies in the topological space

$$Q \stackrel{\text{def}}{=} \{(z, \eta) \in TM^n : F_\phi(z, \eta) = 0\}.$$

In order to show that the differential dF_ϕ vanishes at the point $(\gamma(0), \dot\gamma(0)) \in TM^n$, we show that any neighbourhood $W \subset Q \subset TM^n$ of the point $(\gamma(0), \dot\gamma(0))$ in the topological space Q is not homeomorphic to a disk.

By assumptions, the eigenspace of G corresponding to the eigenvalue ϕ_i is one-dimensional in some small neighborhood $U \subset M^n$ of the point $\gamma(0)$. Then there exists a smooth vector field v on U such that $Gv = \rho_i v$ and $g(v,v) = 1$. In particular, the eigenvalue ϕ_i depends smoothly on the point of U and therefore the polynomial $P_i(t)$ from (2) depends smoothly on the point of U. Further we will write $P_i(t;z)$ instead of $P_i(t)$.

Consider a coordinate system in $T_{\gamma(0)}M^n$ such that the metric g is given by the matrix $diag(1,1,...,1)$ and the mapping G is given by the matrix

$$\mathrm{diag}(\rho_1(\gamma(0)), \rho_2(\gamma(0)), ..., \rho_n(\gamma(0))).$$

In this coordinates, the component number i of the vector v is equal ± 1 and the other components are zero; for any vector η, its component number i is equal to the scalar product $\pm g(\eta, v)$.

Consider the function $I : TM^n \to R$, $I(z,\eta) \stackrel{\mathrm{def}}{=} g(\eta, v)$. Evidently $I(\gamma(0), \dot\gamma(0)) = 0$ and the partial derivative $\frac{\partial I}{\partial \xi_i}$ at the point $(\gamma(0), \dot\gamma(0))$ is not zero. By implicit function theorem we have then that there exists some neighborhood V of the point $(\gamma(0), \dot\gamma(0))$ in the topological space

$$Q^* \stackrel{\mathrm{def}}{=} \{(z,\eta) \in TM^n : I(z,\eta) = 0\}$$

such that V is homeomorphic to the direct product $U' \times D^{n-1}$, where $U' \subset U$ is a neighborhood of the point $\gamma(0)$ and D^{n-1} denotes the disk of dimension $n-1$. Moreover, the restriction of the natural projection $\pi : TM^n \to M^n$ to V coincides with the natural projection : $U' \times D^{n-1} \to U'$.

For any point $(z,\nu) \in V \subset TU'$, consider the points

$$(z,\nu_+) = \left(z, \nu + v\sqrt{\frac{F_\phi(z,\nu)}{P_i(\phi;z)}}\right),$$

$$(z,\nu_-) = \left(z, \nu - v\sqrt{\frac{F_\phi(z,\nu)}{P_i(\phi;z)}}\right).$$

By assumptions, $P_i(\phi;z)$ is not zero and $\frac{F_\phi(z,\nu)}{P_i(\phi;z)}$ is greater than or equal to zero. It vanishes if and only if $\phi_i(z) = \phi$. It is easy to see that if, for some points $(z^1,\nu^1), (z^2,\nu^2) \in V$, at least one of the relations

$$(z^1,\nu^1_+) = (z^2,\nu^2_+), \quad (z^1,\nu^1_-) = (z^2,\nu^2_+),$$
$$(z^1,\nu^1_+) = (z^2,\nu^2_-), \quad (z^1,\nu^1_-) = (z^2,\nu^2_-),$$

holds then automatically $(z^1,\nu^1) = (z^2,\nu^2)$.

It is easy to verify that $F_\phi(z,\nu_+) = F_\phi(z,\nu_-) = 0$ and that any point (z,ξ) of some neighbourhood $W_1 \subset Q$ of the point $(\gamma(0), \dot\gamma(0))$ is either (z,ν^1_+) or (z,ν^1_-). Then some neighborhood of the point $(\gamma(0), \dot\gamma(0))$ in Q is homeomorphic to the direct product of two copies of the disk U' glued along the points z where $\phi_i(z) = \phi$ and the disk D^{n-1}. Then no neighborhood $W \subset Q$ of the point

$(\gamma(0), \dot\gamma(0))$ is homeomorphic to $(2n-1)$-dimensional disk. Thus, the differential dF_ϕ vanishes at each point of the orbit $(\gamma, \dot\gamma)$.

Finally, any point of the segment $[0,1]$ lies in one of the following sets:

$$\begin{aligned}\Gamma_0 &= \{\tau \in [0,1] : \phi_i(\gamma(\tau)) = \phi\},\\ \Gamma_1 &= \{\tau \in [0,1] : \phi_{i+1}(\gamma(\tau)) = \phi\}.\end{aligned}$$

The subsets Γ_0, Γ_1 are evidently closed and non-empty. Then they intersect; at each point τ of the intersection we have $\phi_i(\gamma(\tau)) = \phi_{i+1}(\gamma(\tau)) = \phi$. Lemma 6 is proved.

8 Commutativity of the integrals I_k

Let g, $\bar g$ be Riemannian metrics on M^n. Let them be geodesically equivalent. Our goal is to verify that the integrals $I_0, ..., I_{n-1}$ commute. In view of Lemma 6 and Corollary 9, almost each point of M^n is stable. Therefore it is sufficient to verify the commutativity near the stable points only. By Theorem 5, in some neighborhood of any stable points, the metrics are given by

$$\begin{aligned}g(\dot{\bar x}, \dot{\bar x}) &= \Pi_1(\bar x) A_1(\bar x_1, \dot{\bar x}_1) + \Pi_2(\bar x) A_2(\bar x_2, \dot{\bar x}_2) + \cdots + \\ &+ \Pi_m(\bar x) A_m(\bar x_m, \dot{\bar x}_m),\\ \bar g(\dot{\bar x}, \dot{\bar x}) &= \rho_1 \Pi_1(\bar x) A_1(\bar x_1, \dot{\bar x}_1) + \rho_2 \Pi_2(\bar x) A_2(\bar x_2, \dot{\bar x}_2) + \cdots \\ &+ \rho_m \Pi_m(\bar x) A_m(\bar x_m, \dot{\bar x}_m).\end{aligned}$$

We will show that the integrals I_k are linear combinations of the Levi-Civita integrals.

For each $t \in R$, consider the function $F_t : T^* D^n \to R$ given by

$$I_{n-1} t^{n-1} + I_{n-2} t^{n-2} + ... + I_0.$$

By definition, F_t is a linear combination of the integrals I_k.

Take different $t_0, t_1, ..., t_{n-1}$. It is easy to prove that each I_k is a linear combination of the functions $F_{t_0}, F_{t_1}, ..., F_{t_{n-1}}$. Indeed,

$$\begin{pmatrix} F_{t_0} \\ F_{t_1} \\ \vdots \\ F_{t_{n-1}} \end{pmatrix} = \begin{pmatrix} I_{n-1} t_0^{n-1} + I_{n-2} t_0^{n-2} + ... + I_0 \\ I_{n-1} t_1^{n-1} + I_{n-2} t_1^{n-2} + ... + I_0 \\ \vdots \\ I_{n-1} t_{n-1}^{n-1} + I_{n-2} t_{n-1}^{n-2} + ... + I_0 \end{pmatrix} = \begin{pmatrix} t_0^{n-1} & t_0^{n-2} & ... & 1 \\ t_1^{n-1} & t_1^{n-2} & ... & 1 \\ & & \vdots & \\ t_{n-1}^{n-1} & t_{n-1}^{n-2} & ... & 1 \end{pmatrix} \begin{pmatrix} I_{n-1} \\ I_{n-2} \\ \vdots \\ I_0 \end{pmatrix}.$$

The Vandermonde matrix

$$\begin{pmatrix} t_0^{n-1} & t_0^{n-2} & ... & 1 \\ t_1^{n-1} & t_1^{n-2} & ... & 1 \\ & & \vdots & \\ t_{n-1}^{n-1} & t_{n-1}^{n-2} & ... & 1 \end{pmatrix}$$

is non-degenerate. Therefore the functions I_k are linear combinations of the functions $F_{t_0}, F_{t_1}, ..., F_{t_{n-1}}$.

Now let us show that for any t the function F_t is a linear combination of Levi-Civita integrals $L_1, ..., L_m$ from Section 6.

The fiberwise linear mapping G is evidently given by the matrix

$$G = \mathrm{diag}(\underbrace{\rho_1, ..., \rho_1}_{k_1}, ..., \underbrace{\rho_m, ..., \rho_m}_{k_m}).$$

Then, by Lemma 1, the fiberwise linear mapping S_k is given by the matrix

$$S_k = (-1)^{n-k}\mathrm{diag}(\underbrace{\sigma_{n-k-1}(\check{\phi}_1), ..., \sigma_{n-k-1}(\check{\phi}_1)}_{k_1}, ..., \underbrace{\sigma_{n-k-1}(\check{\phi}_m), ..., \sigma_{n-k-1}(\check{\phi}_m)}_{k_m}),$$

where $\sigma_p(\check{\phi}_l)$ denotes the elementary symmetric polynomial of degree p of $n-1$ variables

$$\underbrace{\phi_1, ..., \phi_1}_{k_1}, ..., \underbrace{\phi_l, ..., \phi_l}_{k_l - 1}, ..., \underbrace{\phi_m, ..., \phi_m}_{k_m}.$$

Then the integrals I_k are given by

$$(-1)^{n-k} I_k = \sigma_{n-k-1}(\check{\phi}_1)\Pi_1 A_1 + ... + \sigma_{n-k-1}(\check{\phi}_m)\Pi_m A_m.$$

Using that

$$\sum_{k=0}^{n-1} t^k \sigma_{n-k-1}(\check{\phi}_i)(-1)^{n-k-1} = (t-\phi_1)^{k_1}(t-\phi_2)^{k_2}...(t-\phi_i)^{k_i-1}...(t-\phi_m)^{k_m}$$

we have

$$\begin{aligned} F_t &= \sum_{k=0}^{n-1}\left[(-1)^{n-k} t^k \sum_{i=1}^{m} \sigma_{n-k-1}(\check{\phi}_i)\Pi_i A_i.\right] \\ &= -\sum_{i=1}^{m}\left[(t-\phi_1)^{k_1}(t-\phi_2)^{k_2}...(t-\phi_i)^{k_i-1}...(t-\phi_m)^{k_m} A_i \Pi_i\right] \\ &= (t-\phi_1)^{k_1-1}(t-\phi_2)^{k_2-1}...(t-\phi_m)^{k_m-1}\sum_{k=0}^{m-1} t^k (-1)^{m-k} L_{m-k}. \end{aligned}$$

If $k_i > 1$ then ϕ_i is constant. Then for any $i \in \{0, 1, ..., m\}$, $t \in R$, the function $(t-\phi_i)^{k_i-1}$ is constant also. Hence each function F_t is a linear combination of Levi-Civita integrals $L_1, L_2, ..., L_m$. Therefore, each function I_k is a linear combinations of Levi-Civita integrals.

Finally, since the integrals $I_0, ..., I_{n-1}$ are linear combinations of Levi-Civita integrals with constant coefficients, and since Levi-Civita integrals commute, the integrals $I_0, ..., I_{n-1}$ also commute. Theorem 1 is proved.

9 Strictly non proportional geodesically equivalent metrics on closed manifolds

Let M^n be closed connected. Let g, $\bar{g}$ be geodesically equivalent Riemannian metrics on M^n. Let them be strictly non proportional at each point of M^n. Our goal is to prove Corollary 8; that is, we must prove that M^n is covered by the torus.

By Lemma 6, there exist numbers $\tau_1, ..., \tau_{n-1}$ such that at each point of $x \in M^n$ we have

$$\phi_1(x) < \tau_1 < \phi_2(x) < ... < \tau_{n-1} < \phi_n(x). \tag{12}$$

Consider the polynomial

$$-(t-\tau_1)(t-\tau_2)...(t-\tau_{n-1}) = C_{n-1}t^{n-1} + C_{n-2}t^{n-2} + ... + C_0.$$

Consider the subset

$$L^n = \{(x,\xi) \in TM^n : I_0(x,\xi) = C_0, I_1(x,\xi) = C_1, ..., I_{n-1}(x,\xi) = C_{n-1}\}.$$

Let us show that at each point of this subset the differentials

$$dI_0, dI_1, ..., dI_{n-1}$$

are linearly independent. It is sufficient to show that for each point $(x,\xi) \in L^n$ the determinant of the matrix

$$W_{ij} = \frac{\partial I_i}{\partial \xi_j}$$

is not zero. Let us fix the point $x \in M^n$ and consider the functions

$$F_{\phi_1(x)}, F_{\phi_2(x)}, ..., F_{\phi_n(x)}.$$

Since the functions $F_{\phi_1(x)}, F_{\phi_2(x)}, ..., F_{\phi_n(x)}$ are linear combinations of the functions $I_0, ..., I_{n-1}$, it is sufficient to show that the determinant of the matrix

$$\tilde{W}_{ij} = \frac{\partial F_{\phi_i}}{\partial \xi_j}$$

is not zero.

Consider a coordinate system in a neighborhood of x such that at the point x the metric g is given by the matrix $diag(1,1,...,1)$ and the mapping G is given by the matrix $diag(\rho_1(x), \rho_2(x), ..., \rho_n(x))$. In this coordinates, the restriction of the function F_t to the tangent space T_xM^n is given by (3) and therefore the matrix $\tilde{W}$ equals

$$\mathrm{diag}(-2P_1(\phi_1)\xi_1, -2P_2(\phi_2)\xi_2, ..., -2P_n(\phi_n)\xi_n).$$

By (12), each component $P_i(\phi_i)\xi_i$ is not zero and therefore the determinant of the matrix W is not zero.

By Arnold-Liouville theorem, the fiber L^n is homeomorphic to the n-torus. By implicit function theorem, the restriction of the projection $\pi : TM^n \to M^n$, $\pi(x, \xi) = x$, to the torus L^n has no singular points and therefore is a covering of the manifold M^n by the torus. Corollary 8 is proved.

Remark 4. *We see that if two metrics are strictly non proportional at a point then the differentials of the function I_k are linear independent at almost every point of the tangent space at the point. Therefore, if two geodesically equivalent metrics are strictly non proportional at least at one point of a connected manifold then the corresponding geodesic flows are Liouville integrable.*

10 Killing vector fields for geodesically equivalent metrics

Let g, $\bar{g}$ be Riemannian metrics on M^n. Let them be geodesically equivalent. Our goal is to prove Corollary 7; that is, given a (non-trivial) Killing vector field for the metric g we must produce a (non-trivial) Killing vector field for the metric $\bar{g}$.

Because of Noether's theorem, if a metric admits a (non-trivial) Killing vector field, then the geodesic flow of the metric admits a (non-trivial) integral, linear in velocities, and vice versa.

Suppose the function

$$F_1 = \sum_{i=1}^{n} a_i(x)\xi^i$$

is constant on the orbits of the geodesic flow of the metric $\bar{g}$. Then the function

$$\Phi^* F_1 = \frac{||\xi||_g}{||\xi||_{\bar{g}}} \sum_{i=1}^{n} a_i(x)\xi^i$$

is constant on the orbits of the geodesic flow of the metric g. Since the function $I_0 = \left(\frac{\det(g)}{\det(\bar{g})}\right)^{\frac{2}{n+1}} \bar{g}(\xi, \xi)$ is an integral of the geodesic flow of the metric g, and since the function $||\xi||_g = \sqrt{g(\xi, \xi)}$ is also an integral of the geodesic flow of the metric g, then the function

$$\frac{\sqrt{g(\xi, \xi)}}{\sqrt{I_0}} \Phi^* F_1 = \left(\frac{\det(g)}{\det(\bar{g})}\right)^{\frac{1}{n+1}} \sum_{i=1}^{n} a_i(x)\xi^i,$$

linear in velocities, is also an integral of the geodesic flow of the metric g. Corollary 7 is proved.

11 Geodesically equivalent metrics on the ellipsoid

Proof of Theorem 3. We show that in the elliptic coordinates the restrictions of the metrics

$$ds^2 \stackrel{\text{def}}{=} \sum_{i=1}^{n} (dx^i)^2 \quad \text{and} \quad dr^2 \stackrel{\text{def}}{=} \frac{1}{\sum_{i=1}^{n} \left(\frac{x^i}{a_i}\right)^2} \left(\sum_{i=1}^{n} \frac{(dx^i)^2}{a_i} \right)$$

to the ellipsoid $\sum_{i=1}^{n} \frac{(x^i)^2}{a_i} = 1$ have precisely the form from Levi-Civita Theorem, and therefore are geodesically equivalent. More precisely, consider the elliptic coordinates $\nu^1, ..., \nu^n$. Without loss of generality we can assume that $a^1 < a^2 < ... < a^n$. Then the relation between the elliptic coordinates $\bar{\nu}$ and the Cartesian coordinates $\bar{x}$ is given by

$$x^i = \sqrt{\frac{\prod_{j=1}^{n} (a^i - \nu^j)}{\prod_{j=1, j \neq i}^{n} (a^i - a^j)}}.$$

Recall that the elliptic coordinates are non-degenerate almost everywhere, and the set

$$\{\nu^1 = 0, a_1 < \nu^2 < a_2, a_2 < \nu^3 < a_3, ..., a_{n-1} < \nu^n < a^n\}$$

is the part of the ellipsoid, lying in the quadrant $\{x^1 > 0, x^2 > 0, ..., x^n > 0\}$. Since for any i the symmetry $x^i \to -x^i$ takes the ellipsoid to the ellipsoid and preserves the metrics ds^2 and dr^2, it is sufficient to verify the statement of the theorem only in the quadrant $\{x^1 > 0, x^2 > 0, ..., x^n > 0\}$.

In the elliptic coordinates the restriction of the metric ds^2 to the ellipsoid has the following form

$$\sum_{i=1}^{n} \Pi_i A_i (d\nu^i)^2,$$

where $\Pi_i \stackrel{\text{def}}{=} \prod_{j=1, j \neq i}^{n} (\nu^i - \nu^j)$, and $A_i \stackrel{\text{def}}{=} \frac{\nu^i}{\prod_{j=1}^{n} (a^j - \nu^i)}$. The restriction of the metric dr^2 to the ellipsoid is

$$(a^1 a^2 ... a^n) \sum_{i=1}^{n} \rho_i \Pi_i A_i (d\nu^i)^2,$$

where $\rho_i \stackrel{\text{def}}{=} \frac{1}{\nu^i (\nu^1 \nu^2 ... \nu^n)}$. We see that the metrics ds^2, dr^2 have the form (10,11) and therefore are geodesically equivalent. Theorem 3 is proved.

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