Comment. Math. Helv. 80 (2005), 541–570 **Commentarii Mathematici Helvetici**

Lichnerowicz–Obata conjecture in dimension two

Vladimir S. Matveev

Abstract. We prove that if a connected Lie group acts on a connected complete Riemannian surface of nonconstant curvature by diffeomorphisms that take (unparameterised) geodesics to geodesics, then it acts by isometries.

Mathematics Subject Classification (2000). 53A20, 54H17, 58F07, 53C05, 53C15, 53C22, 53C24, 58F17, 53A15.

Keywords. Geodesic and projective transformations, geodesically equivalent metrics, projectively related metrics, Dini's Theorem, Liouville metrics, integrable systems, quadratically integrable geodesic flows.

1. Introduction

1.1. History. Let (M^n, g) be a Riemannian manifold of dimension $n \geq 2$.

Definition 1. A Riemannian metric $\bar{g}$ on M^n is called *projectively equivalent* to g, if every geodesic of g (considered as unparameterised curve) is a geodesic of $\bar{g}$. A diffeomorphism $F\colon M^n \to M^n$ is called a *projective transformation*, if the pull-back F^*g of g is projectively equivalent to g.

The study of projective transformations of Riemannian manifolds is a very classical subject. The first examples are due to Beltrami [2]; the first big and very important paper is Lie [16]. In this paper, Lie formulated the problem of finding the largest continuous group of projective transformations for Riemannian surfaces, and found all metrics admitting sufficiently large groups of projective transformations.

For complete manifolds, the problem of finding the largest continuous group of projective transformations was formulated by Schouten in [32].

Since the time of Beltrami, it is known that the connected component of the group of projective transformations of the standard sphere (S^n, g_{round}) is $\mathrm{SL}(n+1, \mathbb{R})$: the element $A \in \mathrm{SL}(n+1,\mathbb{R})$ acts by the diffeomorphism

$$a\colon S^n \to S^n, \quad a\colon v \mapsto \frac{A(v)}{\|A(v)\|}.$$

Clearly, a is a diffeomorphism taking geodesics to geodesics. Indeed, the geodesics of g are great circles (the intersections of planes that go through the origin with the sphere). The mapping A is linear and, hence, takes planes to planes. Since the normalisation $w \mapsto \frac{w}{\|w\|}$ takes planes to their intersections with the sphere, the mapping a takes great circles to great circles.

The following conjecture is classical:

Conjecture 1. *Let a connected Lie group G act on a complete connected Riemannian manifold (M^n, g) of dimension $n \geq 2$ by projective transformations. Then it acts by affine transformations* (*i.e. the action of the group preserves the Levi-Civita connection*)*, or g has constant nonnegative curvature.*

In Europe and America, this statement is known as Lichnerowicz conjecture; in Japan, it is known as Obata conjecture. Unfortunately, we did not manage to find a paper in which either Lichnerowicz or Obata formulated this conjecture explicitly; actually in their time it was not usual to publish conjectures. As a well-known classical conjecture it was formulated later, in [31], [41], [42], [8].

Most results on the Lichnerowicz–Obata conjecture require additional geometric assumptions (mostly written as a tensor equation). For example, for dimensions greater than two, if the metrics are Einstein, Kähler or Ricci-flat, the conjecture was proved by Couty [5] and Akbar-Zadeh [1], scholars of Lichnerowicz's school. If the metric has constant scalar curvature, the conjecture was proved by Yamauchi [41]; later this result was generalised in [8], [43].

Probably the only important result that does not require additional geometric assumptions is due to Solodovnikov. He proved the conjecture under the following assumptions:

- The dimension of the manifold is greater than two.
- All objects (the metric, the manifold, the projective transformations) are real-analytic.

The statement itself is in [35], but the technique was mostly developed in [33], and certain statements of [33] were only proved in [34].

Both assumptions are very important for the methods of Solodovnikov, especially the first, since Solodovnikov's methods are based on a very accurate analysis of the behaviour of the curvature tensor under projective transformation and fail completely in dimension two.

1.2. Main result

Theorem 1 (Announced in [27], [28]). *Let a connected Lie group G act on a complete connected Riemannian surface (M^2, g) by projective transformations. Then it acts by isometries, or g has constant nonnegative curvature.*

Recall that the groups of projective transformations of surfaces of constant curvature are known (essentially since Beltrami, [2].) As we already explained, the group of projective transformations of the standard 2-sphere is $\mathrm{SL}(3,\mathbb{R}) \cup (-\,\mathrm{Sl}(3,\mathbb{R}))$. The group of projective transformations of the Euclidean plane coincides with the group of affine transformation and is $\mathrm{GL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$. The group of oriented projective transformations of hyperbolic plane coincides with the group of oriented isometries and is $\mathrm{SL}(2,\mathbb{R})$. Thus, Theorem 1 closes the theory of Lie groups of projective transformations of complete Riemannian surfaces.

I would like to thank Prof. Alekseevskii for the formulation of the problem, Prof. Bangert, Prof. Bolsinov, Prof. Hasegawa, Prof. Igarashi, Prof. Kiyohara, Prof. Kowalsky and Prof. Voss for useful discussions and DFG-Programm 1154 (Global Differential Geometry) and Ministerium für Wissenschaft, Forschung und Kunst Baden-Württemberg (Eliteförderprogramm Postdocs 2003) for financial support.

2. New and classical instruments of the proof

2.1. Projective transformations and quadratic in velocities integrals. The main new instrument of the proof is the following result.

Theorem 2 ([17] ,[18], [19], [21], [20], [36], [37]). *Let g, $\bar g$ be Riemannian metrics on M^2. Then they are projectively equivalent if and only if the function*

$$I : TM^2 \to \mathbb{R}, \quad I(\xi) := \bar g(\xi,\xi)\left(\frac{\det(g)}{\det(\bar g)}\right)^{2/3} \tag{1}$$

is an integral of the geodesic flow of g.

A vector field v on M^2 is called a *projective vector field* if its flow acts by projective transformations.

An infinitesimal version of Theorem 2 is

Corollary 1 ([27], [38]). *Let g be a Riemannian metric on M^2. If v is a projective vector field, then the function $I : TM \to \mathbb{R}$,*

$$I(\xi) := -(\mathcal{L}_v g)(\xi,\xi) + \frac{2}{3}\mathrm{trace}(g^{-1}\mathcal{L}_v g)\, g(\xi,\xi),$$

where $\mathcal{L}_v g$ is the Lie-derivative of g with respect to v, is an integral for the geodesic flow of g.

Proof. Consider the flow F_t of v. Let g_t be the pull-back $F_t^* g$ of g. By Theorem 2, for every $t \in [-\varepsilon, \varepsilon]$, the function $I_t(\xi) = g_t(\xi, \xi) \left(\frac{\det(g)}{\det(g_t)}\right)^{2/3}$ is an integral. Then the function $\left(\frac{d}{dt} I_t(\xi)\right)_{|t=0}$ is also an integral. We have:

$$
\begin{aligned}
\left(\frac{d}{dt} I_t(\xi)\right)_{|t=0} &= \frac{d}{dt}\left(g_t \left(\frac{\det(g)}{\det(g_t)}\right)^{2/3}\right)_{|t=0} (\xi, \xi) \\
&= \left(\frac{\det(g)}{\det(g_t)}\right)^{2/3}_{|t=0} \left(\frac{dg_t}{dt}\right)_{|t=0} (\xi, \xi) \\
&\quad - \frac{2}{3}\left(\frac{\det g)}{\det(g_t)}\right)^{-2/3}_{|t=0} \left(\frac{\det(g)}{\det(g_t)}\right)_{|t=0} \frac{d}{dt}\big(\det(g^{-1} g_t)\big)\, g(\xi, \xi) \\
&= -\,(\mathcal{L}_v g)\,(\xi, \xi) + \frac{2}{3}\operatorname{trace}(g^{-1}\mathcal{L}_v g)\, g(\xi, \xi).
\end{aligned}
$$

Thus, $I(\xi)$ is an integral of the geodesic flow of g. Corollary 1 is proved. □

The integral I is quadratic in velocities. The integrals quadratic in velocities form a linear space: If two integrals are quadratic in velocities, then every linear combination of them is an integral quadratic in velocities. We denote this linear space by $\mathcal{I}(M^2, g)$. Its dimension is at least one, since the energy-function $g(\xi, \xi)$ is an integral. If g has a projectively equivalent metric $\bar g$ which is non-proportional to g, then the dimension of $\mathcal{I}(M^2, g)$ is at least two.

The following lemma shows that the linear spaces $\mathcal{I}(M^2, g)$ and $\mathcal{I}(M^2, \bar g)$ are canonically isomorph, if the metrics are projectively equivalent.

Lemma 1. *Suppose $\bar I \colon TM^2 \to \mathbb{R}$, $\bar I(\xi) := \bar f(\xi, \xi)$, where $\bar f$ is a bilinear form, is an integral of the geodesic flow of $\bar g$. If g is projectively equivalent to $\bar g$, then the function $I(\xi) := \bar I(\xi)\left(\frac{\det(g)}{\det(\bar g)}\right)^{\frac{2}{3}}$ is an integral of the geodesic flow of g.*

Corollary 2. *Let the space $\mathcal{I}(M^2, g)$ be two-dimensional with the basis $\{I_1, I_2\}$. Consider the set $\{P \in M^2 \mid I_{1|T_P M^2} = \mathrm{const}\, I_{2|T_P M^2}\}$ of the points, where the functions I_1, I_2 are proportional. Then this set is invariant under projective transformations.*

Proof. Suppose the integrals I_1, I_2 are proportional at P. Let F be a projective transformation. Then, by Lemma 1, the functions

$$
\bar I_1 := I_1 \left(\frac{\det(\bar g)}{\det(g)}\right)^{\frac{2}{3}}, \quad \bar I_2 := I_2 \left(\frac{\det(\bar g)}{\det(g)}\right)^{\frac{2}{3}}
$$

are quadratic in velocities integrals for the geodesic flow of $\bar{g} := F^*g$. We see that they are proportional at P. Since the functions I_1, I_2 are linearly independent, the functions $\bar{I}_1, \bar{I}_2$ are linearly independent as well, so they form a basis of the space $(M^2, \bar{g})$. Then every two quadratic in velocities integrals for $\bar{g}$ are proportional at P. Finally, every two quadratic in velocities integrals for g are proportional at $F(P)$. Corollary 2 is proven. □

Proof of Lemma 1. Since the metrics g and $\bar{g}$ are projectively equivalent, the reparameterisation

$$r\colon TM^2 \to TM^2, \quad r(\xi) := \frac{\sqrt{g(\xi,\xi)}}{\sqrt{\bar{g}(\xi,\xi)}}\xi,$$

takes the orbits of the geodesic flow of g to the orbits of the geodesic flow of $\bar{g}$. Hence, the function

$$\bar{I}\left(\frac{\sqrt{g(\xi,\xi)}}{\sqrt{\bar{g}(\xi,\xi)}}\xi\right) = \frac{g(\xi,\xi)}{\bar{g}(\xi,\xi)}\bar{f}(\xi,\xi)$$

is an integral for the geodesic flow of g. Since the energy-function $g(\xi,\xi)$ and (by Theorem 2) the function $\bar{g}(\xi,\xi)\left(\frac{\det(g)}{\det(\bar{g})}\right)^{2/3}$ are integrals, the function $I(\xi) := \bar{I}(\xi)\left(\frac{\det(g)}{\det(\bar{g})}\right)^{\frac{2}{3}}$ is also an integral for the geodesic flow of g. Lemma 1 is proved. □

For future use we need the following result of Dini: Consider the (1, 1)-tensor $G^i_j := \bar{g}^{i\alpha}g_{\alpha j}$ and the functions $A, B\colon M^2 \to \mathbb{R}$,

$$A(P) := \text{The largest eigenvalue of } G \text{ at } P,$$
$$B(P) := \text{The smallest eigenvalue of } G \text{ at } P.$$

We define $X = \frac{A}{(AB)^{1/3}}$, $Y = \frac{B}{(AB)^{1/3}}$. We see that the functions X and Y are well defined, positive and at least continuous on the whole manifold; at the points, where the metrics are non-proportional, they are smooth. By construction, $X = Y$ at the points where the metrics are proportional.

Theorem 3 (Reformulation of Dini 1869 [6]). *Suppose g and $\bar{g}$ are projectively equivalent on M^2 and non-proportional in $P \in M^2$. Then there exists a coordinate system (x, y) in a neighbourhood of P, such that the functions X and Y are independent of the y- and x-coordinate, respectively, and such that the metrics have the following form:*

$$ds_g^2 = (X(x) - Y(y))(dx^2 + dy^2), \tag{2}$$

$$ds_{\bar{g}^2} = \left(\frac{1}{Y(y)} - \frac{1}{X(x)}\right)\left(\frac{dx^2}{X(x)} + \frac{dy^2}{Y(y)}\right). \tag{3}$$

Corollary 3 ([23], [24], [25], [26], [29]). *Let (M^2, g) be a complete connected 2-dimensional Riemannian manifold. If the metric $\bar{g}$ is projectively equivalent to g, then, for every $P, Q \in M^2$, the following inequality holds:*

$$X(P) \geq Y(Q).$$

Corollary 4 (Eisenhart [7], Weyl [39], [40]). *Let M^2 be complete and connected. If g, $\bar{g}$ are projectively and conformally equivalent on M^2, then they are proportional: $\bar{g} = \text{const}\, g$.*

Proof. If the metrics are conformally equivalent, then $X(P) = Y(P)$ in every $P \in M^2$. If they are also projectively equivalent, then, by Corollary 3, the functions X, Y are constant. Thus, the metrics are proportional: $\bar{g} = \text{const}\, g$. Corollary 4 is proved. □

Proof of Corollary 3. Consider $TM^2 \setminus (TM^2)_0$, where $(TM^2)_0$ is the zero-section of the tangent space TM^2, and the function $I(\xi)/E(\xi)$, where $I(\xi)$ is the integral from Theorem 2 and $E(\xi)$ is the energy-integral $g(\xi, \xi)$. The function $I(\xi)/E(\xi)$ is well defined on $TM^2 \setminus (TM^2)_0$. Note, that for every $P \in M^2$ and $\xi \in T_P M^2$, $\xi \neq 0$, the following inequality holds:

$$Y(P) \leq I(\xi)/E(\xi) \leq X(P). \tag{4}$$

Indeed, take coordinates on $T_P M^2$ such that the metrics g, $\bar{g}$ have matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{X(P)} & 0 \\ 0 & \frac{1}{Y(P)} \end{pmatrix}.$$

In this coordinates, the function $I(\xi)/E(\xi)$ is given by

$$\begin{aligned} \frac{I(\xi)}{E(\xi)} = \frac{Y(P)\xi_1^2 + X(P)\xi_2^2}{\xi_1^2 + \xi_2^2} &= X(P) - \frac{(X(P) - Y(P))\xi_1^2}{\xi_1^2 + \xi_2^2} \\ &= Y(P) + \frac{(X(P) - Y(P))\xi_2^2}{\xi_1^2 + \xi_2^2}. \end{aligned}$$

Using that $\frac{X(P)-Y(P)}{\xi_1^2+\xi_2^2}$ is nonnegative, we obtain the inequality (4).

Since (M^2, g) is complete, for every two points $P, Q \in M^2$, there exists a connecting geodesic: $\gamma(0) = P$, $\gamma(1) = Q$. Since the functions $I(\xi)$ and $E(\xi)$ are integrals, the function $I(\xi)/E(\xi)$ is also an integral, so that $I(\dot\gamma(0))/E(\dot\gamma(0)) = I(\dot\gamma(1))/E(\dot\gamma(1))$. Combining with (4), we obtain:

$$Y(P) \leq I(\dot\gamma(0))/E(\dot\gamma(0)) = I(\dot\gamma(1))/E(\dot\gamma(1)) \leq X(Q).$$

Corollary 3 is proved. □

The integral $I(\xi)$ from Theorem 2 for the metrics from Theorem 3 has the form $I(\xi) = (X(x) - Y(y))(Y(y)\xi_1^2 + X(x)\xi_2^2)$. Then linear combinations of the integral and the energy function have the form

$$(X(x) - Y(y))\big((-\alpha Y(y) + \beta)\xi_1^2 + (-\alpha X(x) + \beta)\xi_2^2\big). \tag{5}$$

2.2. Birkhoff–Kolokoltsov meromorphic form. Let g be a Riemannian metrics on M^2. Without loss of generality, we can assume that M^2 is orientable. It is known that there exists a complex structure on M^2 such that locally the metric has the form $f(z)dzd\bar{z}$ (in every complex coordinate chart of this structure; of cause the function f depends on the choice of the chart). Here and up to the next section $\bar{z}$, $\bar{p}$, $\bar{a}$ etc. will denote the complex conjugation in a complex chart.

Consider the cotangent bundle T^*M^2. The tangent and the cotangent bundles will always be identified by g. We denote by p the corresponding (complex) coordinate on the fibres of T^*M^2. Consider the real-valued quadratic in the velocities function

$$F(z,\xi) = a(z)p^2 + b(z)p\bar{p} + \bar{a}(z)\bar{p}^2.$$

(a and b are not assumed to be holomorphic; moreover, since F is real-valued, b must be real-valued as well.) If we make a coordinate change $z = z(z_{\text{new}})$, the coefficients a and b will be changed as well; the next lemma controls how they do change. Consider

$$A := -\frac{1}{a(z)}dz \otimes dz.$$

For the energy integral

$$\frac{p\bar{p}}{f(z)}, \tag{6}$$

A is not defined.

Lemma 2 ([13]). *If F is an integral of the geodesic flow of g, and if it is not the energy integral* (6) *multiplied by a constant, then A is a meromorphic* $(2,0)$*-form without zeros.*

Explanation. After the holomorphic change $z = z(z_{\text{new}})$ of the coordinate, the momentum p changes as follows: $p_{\text{new}} = pz'$, where z' denotes the derivative $\frac{dz(z_{\text{new}})}{dz_{\text{new}}}$. Then the integral F in the new coordinates is

$$\begin{aligned}
F &= a(z)p^2 + b(z)p\bar{p} + \bar{a}(z)\bar{p}^2\\
&= a(z(z_{\text{new}}))\frac{p_{\text{new}}^2}{(z')^2} + b(z(z_{\text{new}}))\frac{p_{\text{new}}\bar{p}_{\text{new}}}{z'\bar{z}'} + \bar{a}(z(z_{\text{new}}))\frac{\bar{p}_{\text{new}}^2}{(\bar{z}')^2}\\
&= a_{\text{new}}(z_{\text{new}})p_{\text{new}}^2 + b_{\text{new}}(z_{\text{new}})p_{\text{new}}\bar{p}_{\text{new}} + \bar{a}_{\text{new}}(z_{\text{new}})\bar{p}_{\text{new}}^2,
\end{aligned}$$

where $a_{\mathrm{new}}(z_{\mathrm{new}}) = \frac{a(z(z_{\mathrm{new}}))}{(z')^2}$. We see that $-\frac{1}{a(z)}$ changes as a coefficient of a $(2,0)$-form. Thus A is a $(2,0)$-form.

The fact that $a(z)$ is holomorphic (in every coordinate chart) has been known at least to Birkhoff [3]; one obtains it immediately by considering the Poisson bracket of the energy integral (6) and F, see, for example, [13].

Since locally a is a holomorphic function, $\frac{1}{a}$ is a meromorphic function; in particular, if the integral is not proportional to the energy integral, the form A is a meromorphic form. By construction, the form A has no zeros.

Locally, in a neighbourhood of every point $P \in D^2$ which is not a pole of A, by local holomorphic change of the variable $w = w(z)$ we can always make the form A to look like $dw \otimes dw$. Indeed, under this assumption $a(P) \neq 0$, so the equation

$$-\frac{1}{a(z)} dz \otimes dz = dw \otimes dw \tag{7}$$

has a solution $w(z) = \int \frac{dz}{\sqrt{-a(z)}}$.

In this new coordinate w, the metric and the integral have the following very nice form (this is a folklore known at least to Birkhoff; a proof can be found in [13]):

Lemma 3. *Let I be a quadratic in velocities integral for the geodesic flow of the metric g on M^2. Suppose its form A from Lemma 2 is equal to $dz \otimes dz$. Then, in the coordinates $x := \Re(z)$, $y := \Im(z)$, the metric and the integral I have the following "Liouville" form:*

$$ds_g^2 = (X(x) - Y(y))(dx^2 + dy^2), \tag{8}$$

$$I = \frac{Y(y)p_x^2 + X(x)p_y^2}{X(x) - Y(y)},$$

where X and Y are functions of one variable.

If the integral I in Lemma 3 is constructed from a projectively equivalent metric g_1 by formula (1), the metric g_1 is precisely

$$ds_{g_1}^2 = \left(\frac{1}{Y(y)} - \frac{1}{X(x)}\right)\left(\frac{dx^2}{X(x)} + \frac{dy^2}{Y(y)}\right). \tag{9}$$

Thus, the notation in Lemma 3 is consistent with the notation in Theorem 3.

Remark 1. The form from Lemma 2 constructed for the linear combination $\alpha I + \beta E$ of I and the energy integral E given by (6) is equal to $\frac{1}{\alpha}A$, where A is the form constructed for I. In particular, it has the same structure of poles.

2.3. Igarashi–Kiyohara–Sugahara's description of quadratically integrable geodesic flows on the 2-disk. A complete description of complete Riemannian metrics on the two-dimensional disk such that

- the geodesic flow admits an integral quadratic in velocities,
- this integral is not a linear combination of the square of an integral linear in velocities and the energy integral

was obtained in [9]. We reformulate the results we need as Theorem 4.

Consider the triple (D^2, g, A), where D^2 is a 2-disk, g is a Riemannian metric on it, and A is a meromorphic (with respect to the complex structure constructed by the metric) (2, 0)-form. Two such triples (D_1^2, g_1, A_1) and (D_2^2, g_2, A_2) are said to be *isomorphic*, if there exists a diffeomorphism $H\colon D_1^2 \to D_2^2$ that takes g_1 to g_2 and A_1 to A_2.

Theorem 4 ([9]). *Let g be a complete Riemannian metric on D^2. Suppose F is a quadratic in velocities integral of the geodesic flow of g. Assume in addition that it is not a linear combination of the square of an integral linear in velocities and the energy integral. Then the triple (D^2, g, A), where A is the meromorphic $(2,0)$-form from Lemma 2, is isomorphic to one of the following model triples (for the appropriate parameters r_1, R_1, r_2, R_2 and f).*

Model 1a. *Parameters: $r_1 < R_1, r_2 < R_2 \in \mathbb{R} \cup -\infty \cup +\infty$; f is a positive function on the disk*

$$D^2 := \{z \in \mathbb{C} \mid r_1 < \Re(z) < R_1;\ r_2 < \Im(z) < R_2\}.$$

The metric g is given by $f(z)dzd\bar{z}$; the form A is equal to $dz \otimes dz$.

Model 2. *Parameters: $R_1, R_2 \in \mathbb{R}_+ \cup +\infty$; f is a positive function on the disk*

$$D^2 := \{z \in \mathbb{C} \mid R_1 > \Re(2\sqrt{z});\ R_2 > |\Im(2\sqrt{z})|\}.$$

(We assume that the domain where the function $\sqrt{\ }$ is defined is symmetric with respect to complex conjugation). The metric g is given by $f(z)dzd\bar{z}$; the form A is equal to $\frac{1}{z}dz \otimes dz$.

Model 3. *Parameters: $R_1 \in \mathbb{R}_+ \cup +\infty$; f is a positive function on the disk*

$$D^2 := \{z \in \mathbb{C} \mid R_1 > |\Im(\arcsin(z))|\}.$$

(We assume that the domain where the function arcsin *is defined is symmetric with respect to complex conjugation). The metric g is given by $f(z)dzd\bar{z}$; the form A is equal to $\frac{1}{z^2-1}dz \otimes dz$.*

The case when the integral is a linear combination of the square of an integral linear in velocities and of the energy integral is much more easy than the previous case, and, may be because of this, was not treated in [9]. We will need it, and therefore we must consider it here:

Theorem 5. *Let g be a complete metric on D^2. Suppose the function F is a linear combination of the square of a nonzero integral linear in velocities and the energy integral. Then the triple (D^2, g, A), where A is the meromorphic $(2,0)$-form from Lemma 2, is isomorphic to one of the following model triples (for the appropriate parameters r, R and f).*

Model 1b.: *Parameters: $r < R \in \mathbb{R}\cup+\infty\cup-\infty$; f is a positive function depending only on the variable $x := \Re(z)$ on the disk*

$$D^2 := \{z \in \mathbb{C} \mid r < \Re(z) < R; \}.$$

The metric g is given by $f(z)dzd\bar{z}$; the form A is equal to $dz \otimes dz$.

Model 4. *Parameters: $R \in \mathbb{R}\cup+\infty$; f is a positive function depending only on the absolute value of z on the disk*

$$D^2 := \{z \in \mathbb{C} \mid |\log(z)| < R\}.$$

(We assume that the domain where the function log *is defined is symmetric with respect to complex conjugation). The metric g is given by $f(z)dzd\bar{z}$; the form A is equal to $\frac{1}{z^2}dz \otimes dz$.*

Proof. Suppose the linear in momenta function $I = a(z)p_x + b(z)p_y$ is an integral for the geodesic flow of g. Then $v = (a, b)$ is a Killing vector field for g, and its flow preserves the metric. Suppose first that there exists no point where v vanishes. Consider the vector field w such that

(1) $g(v, v) = g(w, w)$; $g(v, w) = 0$.

(2) The pair of the vectors w, v is positively oriented.

These two conditions define the vector field w uniquely. Since w is defined using g and v only, and since g is preserved by the flow of v, the vector fields w and v commute. Then they define a coordinate system (x, y) on the disk. We treat w as the first vector field, so that the metric is independent of the variable y. By condition 1, the metric has the form $f(z)(dx^2 + dy^2)$. Then f depends on x only. Since the metric is complete, the lines $\{(x, y) \in D^2 \mid x = \text{const}\}$ are infinite in both directions. Indeed, the distance between two points of the line is less than the length of the line between these two points.

Then there exist $r < R \in \mathbb{R}\cup+\infty\cup-\infty$ such that the disk is actually the band

$$\{(x, y) \in \mathbb{C} \mid r < x < R\}.$$

By construction, $v = (0, 1)$. Then the square of the integral I is equal to p_y^2 and the corresponding form A is $dz \otimes dz$. Theorem 5 is proved under the assumption that there is no point where v is zero.

Now suppose there exist points where v vanishes. These points are stable points of the flow of v; since the flow acts by isometries, they are isolated. Let us show that actually there exists precisely one such point. Indeed, if v vanishes at the points $z_0 \neq z_1$, since our manifold is homeomorphic to the disk and is complete, there exist two geodesics γ_0, γ_1 such that the following holds:

- The geodesic γ_0 contains z_0; the geodesic γ_1 contains z_1.
- The geodesics intersect transversally at a point z where $v \neq 0$. (Otherwise every geodesic passing through z_0 should contain z_1, which implies that the manifold is compact.)

Since the geodesics contain the point where v vanishes, the integral I is equal to zero on every velocity vector of the geodesics, and, therefore, the geodesics are orthogonal to v at every point. This gives us a contradiction at the point z. Thus, v vanishes precisely at one point. We will denote this point by z_0.

Since the flow of the field v acts by isometries and preserves the point z_0, it commutes with the exponent mapping $\exp_{z_0} \colon T_{z_0}D^2 \to D^2$. Since v is a Jacobi vector field for geodesics passing through z_0, there is no point conjugate to z_0 and, therefore, the exponential mapping is a bijection. The flow of v acts on $T_{z_0}D^2$ by orthogonal linear transformations, i.e., by rotations. Thus, in the standard polar coordinates (ρ, ϕ) on $T_{z_0}D^2$, the (pull-back of the) metric is $d\rho^2 + h(\rho)d\phi^2$, and the vector field v is proportional to $\frac{\partial}{\partial\phi}$. After the appropriate change of the variables, the metric has the form $f(|z|)dzd\bar{z}$ as needed, and the vector field v (after the appropriate scaling) is $(-y, x)$. Then the square of the integral I is $y^2p_x^2 - 2xyp_xp_y + x^2p_y^2$, and the form A is $\frac{1}{z^2}dz \otimes dz$. Theorem 5 is proved. $\square$

Remark 2. Clearly, not arbitrary parameters (R_i, r_i, f) can come from metrics with quadratically integrable geodesic flows: for example, the metric g must have the form (8) in the coordinates where A is $dz \otimes dz$. For all four models the equation (7) can be solved explicitly: for model 2 the solution is $w = 2\sqrt{z}$, for model 3 the solution is $w = \arcsin(z)$, for model 4 the solution is $w = \log(z)$. The coordinate nets of this "Liouville" coordinate w are shown in Figure 1. In the figure, the values of the parameters r_1, r_2, R_1, R_2, r, R are finite; if they are infinite, the disk D^2 coincides with the whole $\mathbb{C}$, and the coordinate system extends on the whole $\mathbb{C}$.

We will use and discuss this below; see [9], [4] for more information about it.

Corollary 5. *Suppose that the complete Riemannian metrics g and g_1 are projectively equivalent on the disk D^2. Let them be not proportional* (*i.e. let* $g \neq \text{const}\, g_1$ *for every* $\text{const} \in \mathbb{R}$). *Then the disk D^2 with the complex structure z such that the metric g has the form $f(z)dzd\bar{z}$ is complex-diffeomorph to the whole $\mathbb{C}$.*

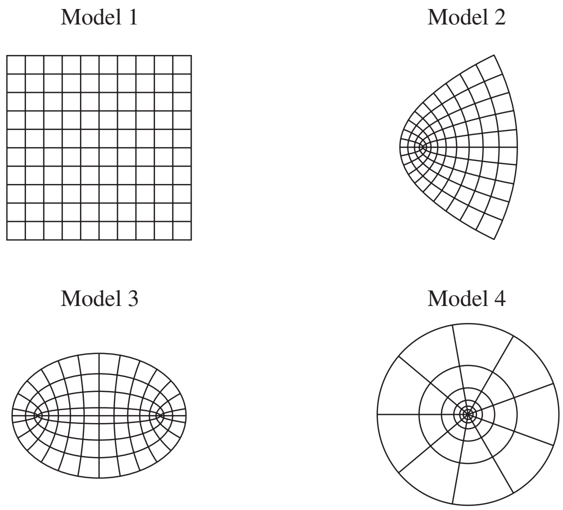

Figure 1. Disks D^2 and “Liouville” coordinates on them for models 1–4

Proof. Let g, g_1 be complete projectively equivalent non-proportional Riemannian metrics on D^2. Consider the integral (1) for the geodesic flow of g. It is quadratic in velocities and linear independent of the energy integral (6). Denote by A the form from Lemma 2. The triple (D^2, g, A) is isomorphic to one of the model triples from Theorems 4, 5. We will consider all four cases. Suppose (D^2, g, A) is as in model 1. We need to prove that R_1, R_2 are $+\infty$ and r_1, r_2 are $-\infty$. We will prove that R_1 is $+\infty$. By Lemma 3, in the coordinates $(x := \Re(z), y := \Im(z))$, the metrics g and g_1 have the form (8), (9), respectively. Take $t_1 \in]r_1, R_1[$ and consider the line $\{x + iy \in D^2 \mid y = 0,\ x > t_1\}$. The line must be infinite in both metrics. (Since the metrics are complete, and since the distance between points z_0, z_1 of the line is not greater than the length of the segment of the line with ends z_0, z_1.) Its length in g_1 is

$$\int_{t_1}^{R_1} \frac{\sqrt{X(t) - Y(0)}}{X(t)\sqrt{Y(0)}} dt.$$

Since $X(t) > Y(0)$ by Corollary 3, and since $Y(0) > 0$, we have

$$\frac{\sqrt{X(t) - Y(0)}}{X(t)\sqrt{Y(0)}} < \frac{1}{\sqrt{Y(0)X(t)}} < \frac{1}{Y(0)}.$$

Thus, if the integral is infinite, R_1 is $+\infty$. Similarly, one can prove that $r_1, r_2 = -\infty$ and $R_2 = +\infty$. Finally, D^2 is the whole $\mathbb{C}$.

Now suppose (D^2, g, A) is as in model 2. We need to prove that $R_1 = R_2 = +\infty$. Take $0 < t_1 < R_1$ and a small positive ε and consider the part of the disk

$$\{z \in D^2 \subset \mathbb{C} \mid \Re(2\sqrt{z}) > t_1,\ |\Im(2\sqrt{z})| < \varepsilon\}.$$

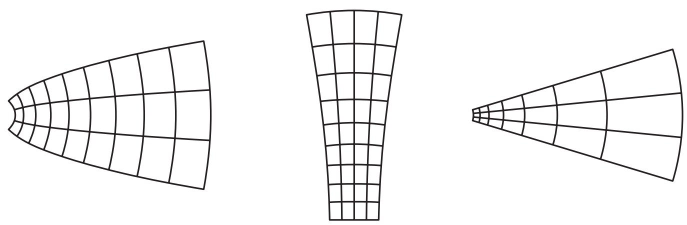

Figure 2. The sets $\{z \in D^2 \mid \Re(2\sqrt{z}) > t_1,\ |\Im(2\sqrt{z})| < \varepsilon\}$, $\{z \in D^2 \mid |\Re(\arcsin(z))| < \varepsilon,\ 0 < \Im(\arcsin(z)) < R_1\}$, $\{z \in D^2 \mid |\Im(\log(z))| < \varepsilon,\ 0 < \Re(\log(z)) < R\}$.

In this part of the disk, the equation (7) can be explicitly solved; the solution is $w = 2\sqrt{z}$. After the substitution $z = \frac{w^2}{4}$, the part of the disk becomes the rectangle

$$\{w \in \mathbb{C} \mid t_1 < \Re(w) < R_1,\ |\Im(w)| < \varepsilon\}.$$

By Lemma 3, the metrics g, g_1 in the coordinates $x := \Re(w)$, $y := \Im(w)$ have the form (8), (9), respectively. If the metric g_1 is complete, the length of the line

$$\{w \in \mathbb{C} \mid y = 0,\ x \in]t_1, R_1[\}$$

must be infinite. Hence $\int_{t_1}^{R_1} \frac{\sqrt{X(t)-Y(0)}}{X(t)\sqrt{Y(0)}}dt$ is infinite, so that R_1 is $+\infty$. Similarly, one can prove that R_2 is infinite. Thus D^2 is the whole $\mathbb{C}$.

Now suppose (D^2, g, A) is as in model 3. We need to prove that $R_1 = +\infty$. Take a small positive ε and consider the part of the disk given by

$$\{z \in D^2 \subset \mathbb{C} \mid |\Re(\arcsin(z))| < \varepsilon,\ 0 < \Im(\arcsin(z)) < R_1\}.$$

The equation (7) can be explicitly solved on this part of the disk; the solution is $w = \arcsin(z)$. By Lemma 3, the metrics g, g_1 in the coordinates $x := \Re(w)$, $y := \Im(w)$ have the form (8), (9), respectively. If the metric g is complete, the length of the line

$$\{w \in \mathbb{C} \mid x = 0,\ y \in]0, R_1[\}$$

must be infinite. Hence $\int_0^{R_1} \sqrt{X(0) - Y(t)}\, dt$ is infinite, so that R_1 is infinite. Thus D^2 is the whole $\mathbb{C}$.

Now suppose (D^2, g, A) is as in model 4. We need to prove that $R = +\infty$. Take a small positive ε and consider the part

$$\{z \in D^2 \subset \mathbb{C} \mid |\Im(\log(z))| < \varepsilon,\ 0 < \Re(\log(z)) < R\}$$

of the disk. The equation (7) can be explicitly solved in this part of the disk; the solution is $w = \log(z)$). By Lemma 3, the metrics g, g_1 in the coordinates $x := \Re(w)$, $y := \Im(w)$ have the form (8), (9), respectively; moreover, the function Y is constant. If the metric g_1 is complete, the length of the line

$$\{w \in \mathbb{C} \mid y = 0,\ x \in]0, R[\}$$

must be infinite. Hence $\int_0^R \frac{\sqrt{X(t)-Y}}{X(t)\sqrt{Y}} dt$ is infinite, so that R is infinite. Therefore D^2 is the whole $\mathbb{C}$. Corollary 5 is proved. □

Corollary 6. *Let a Riemannian metric g on $\mathbb{C}$ given by $ds_g^2 = f(z)dzd\bar{z}$ be complete. Suppose its geodesic flow has an integral I that is quadratic in velocities and functionally independent of the energy integral* (6)*. Denote by A the* 2*-form from Lemma* 2*. Then the following statements hold:*

(1) *The form A is $\frac{1}{\alpha z^2+\beta z+\gamma} dz \otimes dz$, where α, β and γ are complex constants. At least one of these constants is not zero.*

(2) *If the coefficient α is equal to* 0*, and the coefficient β is different from* 0*, the metric is preserved by the symmetry with respect to the straight line*

$$\left\{z \in \mathbb{C} \mid \frac{z}{\beta} + \frac{\gamma}{\beta^2} \in \mathbb{R}\right\}.$$

(3) *If the polynomial $\alpha z^2 + \beta z + \gamma$ has two simple roots, the metric g is preserved by the symmetry with respect to the straight line connecting the roots.*

(4) *If the polynomial $\alpha z^2 + \beta z + \gamma$ has one double root, the integral is a linear combination of the energy integral* (6) *and the square of an integral linear in velocities, which vanishes at the tangent plane to the root. The rotations around the root preserve the metric.*

Proof. By Theorems 4 and 5, there exists a complex coordinate w on $\mathbb{C}$ such that A is either $\frac{1}{C} dw \otimes dw$ or $\frac{1}{w} dw \otimes dw$ or $\frac{1}{w^2-1} dw \otimes dw$ or $\frac{1}{w^2} dw \otimes dw$. Since every bijective holomorphic mapping from $\mathbb{C}$ to $\mathbb{C}$ is linear, the change $w = w(z)$ of the coordinate is linear. After a linear change of the coordinates, the forms A listed above have the form $\frac{1}{\alpha z^2+\beta z+\gamma} dz \otimes dz$, where α, β and γ are complex constants. The first statement of Corollary 6 is proved.

In order to prove the second statement, it is sufficient to show that the symmetry $z \mapsto \bar{z}$ is an isometry of the metric whose triple (D^2, g, A) is isomorphic to model 2. Indeed, after the following change of coordinate

$$w = \frac{z}{\beta} + \frac{\gamma}{\beta^2},$$

the form A is $\frac{1}{w} dw \otimes dw$ and the line $\{z \in \mathbb{C} \mid \frac{z}{\beta} + \frac{\gamma}{\beta^2} \in \mathbb{R}\}$ becomes the line $\{w \in \mathbb{C} \mid \Im(w) = 0\}$.

Clearly, the form $A = \frac{1}{z} dz \otimes dz$ has precisely one pole. In the neighbourhood of every other point, the equation (7) can be explicitly solved: the solution is $w = 2\sqrt{z}$. It can not be solved globally: the global solution is defined on the double branch cover. But still the coordinate lines of the local coordinate system $(\Re(2\sqrt{z}), \Im(2\sqrt{z}))$ do not depend on the choice of branch of the square root and can be defined globally. By direct calculations one can see that these lines look like those in the picture.

We see that if a coordinate line of $\Re(w)$ intersects with a coordinate line of $\Im(w)$ at a point z, then the same coordinate lines intersect at the point $\bar{z}$ as well. By Lemma 3, the metric g has the form

$$\frac{1}{\sqrt{z\bar{z}}} \left(X(\Re(w)) - Y(\Im(w))\right) dz d\bar{z}.$$

Since the function X is constant along the coordinate lines of $\Im(w)$ and the function Y is constant along the coordinate lines of $\Re(w)$, X at the point z is equal to X at the point $\bar{z}$ and Y at the point z is equal to Y at the point $\bar{z}$. Thus the symmetry $z \mapsto \bar{z}$ is an isometry. Statement 2 is proved.

The proof of statement 3 is similar: the solution of the equation (7) for the model triple 3 is $w = \arcsin(z)$, and the coordinate lines where the metric has the form (8) are as in the Figure 1.

Statement 4 follows from Theorem 5. Corollary 6 is proved. □

3. An answer to Kiyohara's question

Let (M^2, g) be a complete connected Riemannian surface of nonconstant curvature. Consider the space $\mathcal{I}(M^2, g)$ of integrals quadratic in velocities for the geodesic flow of g.

The following question was stated by Prof. Kiyohara (motivated by [10], [9]): can the dimension of the space $\mathcal{I}(M^2, g)$ be greater than two?

Theorem 6 ([14], [10]). *Under the assumption that the surface is closed, the answer is negative.*

Without this assumption, the answer is positive: here are two examples: consider the complex plane $\mathbb{C}$ with the standard coordinates $x + iy$ and the following metrics (where γ is a positive constant).

Example 1. $(x^2 + y^2 + \gamma)(dx^2 + dy^2)$,

Example 2. $(x^2 + \frac{1}{4}y^2 + \gamma)(dx^2 + dy^2)$.

The metrics are complete; their curvature is not constant; as we will show in the end of Section 4, the space of quadratic integrals has dimension 4 for Example 1 and dimension 3 for Example 2.

The next theorem shows that these are essentially all possible examples, if we assume that the surface is complex-diffeomorph to $\mathbb{C}$:

Theorem 7. *Consider a Riemannian metric g of the form $f(z)dzd\bar{z}$ on $\mathbb{C}$. Let the dimension of the space of quadratic (in velocities) integrals for the geodesic flow of g be greater than two. Then either g has constant curvature, or there exists a diffeomorphism $F\colon \mathbb{C} \to \mathbb{C}$ taking the metric g to a metric proportional to the metric from Example* 1 *or to the metric from Example* 2*.*

Proof. Consider two quadratic in velocities integrals I_1, I_2 such that I_1, I_2 and the energy integral (6) are linear independent. Denote by A_1 (respectively, by A_2) the $(2,0)$-form from Lemma 2 constructed for the integral I_1 (respectively, I_2).

Every triple $(\mathbb{C}, g, A_i)$ must be isomorphic to one of the four model triples. Below we will consider all ten possible (different) cases, and show that the metric is either as in Examples 1, 2 or has constant curvature.

Case (1,1). Suppose both triples are isomorphic to model 1. Then, after the appropriate change of coordinates, the form A_1 is $dz \otimes dz$ and the form A_2 is $C^2 dz \otimes dz$, where $C = \alpha + \beta i$ is a complex constant. Then the metric and the integrals have the model form from Lemma 3 in the coordinate systems $(x := \Re(z), y := \Im(z))$ and $(\alpha x - \beta y = \Re(Cz), \alpha y + \beta x = \Im(Cz))$. We have

$$ds_g^2 = (X_1(x) - Y_1(y))(dx^2 + dy^2) = (X_2(\alpha x - \beta y) - Y_2(\alpha y + \beta x))(dx^2 + dy^2).$$

If α or β is zero, the integrals I_1, I_2, E are linear dependent, which contradicts the assumptions. Assume $\alpha \neq 0 \neq \beta$. In view of

$$(X_1(x) - Y_1(y)) = (X_2(\alpha x - \beta y) - Y_2(\alpha y + \beta x)),$$

and since $\frac{\partial^2 (X_1(x) - Y_1(y))}{\partial x \partial y} = 0$, we have

$$\frac{\partial^2}{\partial x \partial y}(X_2(\alpha x - \beta y) - Y_2(\alpha y + \beta x)) = -\alpha\beta(X_2''(\alpha x - \beta y) + Y_2''(\alpha y + \beta x)) = 0.$$

Thus $X_2''(\alpha x - \beta y) = -Y_2''(\alpha y + \beta x)) = \text{const}$. Finally, $X_2 = -Y_2 = \text{const}_1$ or X_2 and $-Y_1$ are quadratic polynomials with the same coefficient near the quadratic term, or X_2 and $-Y_2$ are linear polynomials. In the first case, the metric has zero curvature. In the second case, after the appropriate change of variables, the metric has the form $(x^2 + y^2 + \gamma)(dx^2 + dy^2)$ as in Example 1. In the third case, the metric is not always positive definite. Theorem 7 is proved under the assumption that both triples are as in model 1.

Case (1,4). Let the first triple $(\mathbb{C}, g, A_1)$ be isomorphic to the model 1 and the second triple $(\mathbb{C}, g, A_2)$ be isomorphic to the model 4. Then (in the appropriate coordinate system) the metric is invariant with respect to rotations $z \mapsto e^{i\phi}z$. Then the push-forward of the integral I_1 is an integral. Clearly, the form from Lemma 2 for the push-forward of I_1 is $e^{-i\phi}dz \otimes dz$. Then, by Remark 1, for small ϕ, the push-forward of the integral I_1 is linear independent of the integral I_1 and of the energy integral. Thus, we reduced case (1,4) to case (1,1).

Case (1,2). Let the first triple $(\mathbb{C}, g, A_1)$ be isomorphic to the model 1 and the second triple $(\mathbb{C}, g, A_2)$ be isomorphic to the model 2. Then, by Lemma 3, the metric g is $(X(x)-Y(y))(dx^2+dy^2)$, and the form A_2 is $\frac{1}{\beta z+\gamma}dz\otimes dz$. If β is not a real multiple of 1, i or $1 \pm i$, by Remark 1, the isometry from Corollary 6 sends the integral I_1 to the integral that is not a linear combination of the integral I_2 and the energy integral. Clearly, the form from Lemma 2 for this integral has no pole, so we reduced this case to case (1,1).

Suppose β is $1+i$. Consider the integral $tI_1 + I_2$. Its form from Lemma 2 is

$$\frac{1}{(1+i)z + t + \gamma}.$$

Then, by Corollary 6, for every $T \in \mathbb{R}$, the symmetry with respect to the line

$$\left\{ z \in \mathbb{C} \;\Big|\; \frac{z}{1+i} + \frac{\gamma + T}{2i} \in \mathbb{R} \right\}$$

is an isometry of g. This one-parameter family of symmetries gives us a Killing vector field $v = (1+i)$. Thus, for every $z = x + iy$ and for every real constant c, $X(x+c) - Y(y+c) = X(x) - Y(y)$. Hence X and Y are linear functions or constants. If they are constants, the metric is flat. If they are linear functions, the metric is not positive-defined.

If β is a real multiple of $1-i$, the proof is similar.

Now suppose $\beta = 1$ or $\beta = i$. Without loss of generality, we can assume $\gamma = 0$. Then the metric has the form

$$(X_1(x) - Y_1(y))(dzd\bar{z}) = \sqrt{z\bar{z}}\,\big(X_1\big(x\big) - Y_1\big(y\big)\big)\,dwd\bar{w},$$

where $w = 2\sqrt{z}$, since dz and dw are connected by the relation

$$dw = \frac{1}{\sqrt{z}}dz. \tag{10}$$

Since by Lemma 3 the metric in the coordinates $\Re(w)$, $\Im(w)$ has the model form, the imaginary part of

$$\frac{\partial^2 (X_1(x) - Y_1(y))|z|}{\partial w^2}$$

must be zero. Employing (10) gives us the following equation:

$$X_1''y + Y_1''y + 3Y_1' = 0. \tag{11}$$

This equation can be solved. The solution is

$$X_1(x) = C_1x^2 + C_2x + C_3, \quad Y_1(y) = -\frac{1}{4}C_1y^2 + C_4\frac{1}{y^2} + C_5.$$

Since the (entries of the) metric must be bounded, $C_4 = 0$. If $C_1 = 0$, the metric is not always positive defined. Thus, after the appropriate scaling and linear change of the coordinate x, the metric is

$$\left(x^2 + \frac{y^2}{4} + \gamma\right)(dx^2 + dy^2) \tag{12}$$

as in Example 2. Theorem 7 is proved under the assumptions of case (1,2).

Case (1,3). Let the first triple $(\mathbb{C}, g, A_1)$ be isomorphic to the model 1 and the second triple $(\mathbb{C}, g, A_2)$ be isomorphic to the model 3. Then in the appropriate coordinate system the form A_1 is $\frac{1}{C}dz \otimes dz$ and the form A_2 is $\frac{1}{z^2-1}dz \otimes dz$. Then the form from Lemma 2 corresponding to the integral $\alpha I_1 + I_2$ is $\frac{1}{z^2-1+\alpha C}dz \otimes dz$. If C is real, the form from Lemma 2 for the linear combination $\frac{1}{C}I_1 + I_2$ is as in model 4, so we reduced case (1,3) to case (1,4).

Suppose C is not real. By Corollary 6, for every real α, the metric g is preserved by the symmetry with respect to the line connecting $\sqrt{-1+\alpha C}$ and $-\sqrt{-1+\alpha C}$. Then the rotations $z \mapsto e^{i\phi}z$ are isometries. We reduced again case (1,3) to case (1,4).

Case (2,2). Suppose $A_1 = \frac{1}{z}dz \otimes dz$ and $A_2 = \frac{1}{\beta z+\gamma}dz \otimes dz$. If β is real, the form from Lemma 2 for the integral $\beta I_1 - I_2$ has no pole, so that we reduced case (2,2) to the case (1,2). If β is not real, the form from Lemma 2 for the integral $tI_1 + I_2$ is $\frac{1}{(\beta+t)z+\gamma}dz \otimes dz$. We see that the line of the symmetry from Corollary 6 smoothly depends on t, and is not constant. Then the symmetries from Corollary 6 generate a one-parametric family of isometries of g. By Noether's Theorem, a family of isometries generates an integral linear in velocities. We consider the square of the integral. By Theorem 5, either there exists a point such that the integral vanishes at the tangent space to the point, or there is no such point. In the second case, we reduced case (2,2) to case (1,2).

In the first case, if the point where the vector field vanishes does not lie on the line $\Im(z) = 0$, the symmetry of the integral with respect to the line is also an integral, so that we constructed two linear independent Killing vector fields. Then the curvature is constant.

The only remaining possibility is when the point where the vector field vanishes lies on the line $\Im(z) = 0$. If the point does not coincide with the point 0, the symmetry with respect to the point gives us another integral with the form from Lemma 2 equal to $\frac{1}{C-z}dz\otimes dz$. Then a linear combination of the integral and the integral I_1 gives us an integral such that the form from Lemma 2 has no pole. Thus we reduced case (2,2) to case (1,2).

Now suppose the point where the Killing vector field vanishes coincides with 0. Then, by Theorem 5, the metric has the form

$$f(\sqrt{z\bar{z}})dzd\bar{z}=\frac{1}{4}w\bar{w}f\left(\frac{w\bar{w}}{4}\right)dwd\bar{w}.$$

Since by Lemma 3 the metric has the form (8) in the coordinates $\Re(w)$, $\Im(w)$, we obtain $\frac{1}{4}w\bar{w}f(\frac{w\bar{w}}{4})=C_1w\bar{w}+C_2$, which implies that the metric is $\left(4C_1+\frac{C_2}{\sqrt{z\bar{z}}}\right)dzd\bar{z}$. We see that the metric is flat or degenerate. Theorem 7 is proved under the assumptions of case (2,2).

Case (2,4). In this case, the metric is invariant with respect to rotations, so we can construct one more integral corresponding to the model 2 such that it is linear independent of I_1 and E. Thus we reduced case (2,3) to case (2,2).

Case (2,3). Suppose $A_1=\frac{1}{z}dz\otimes dz$ and $A_2=\frac{1}{\alpha z^2+\beta z+\gamma}dz\otimes dz$. The form from Lemma 2 for the linear combination I_2+tI_1 of the integrals is $\frac{1}{\alpha z^2+(\beta+t)z+\gamma}dz\otimes dz$. Consider the symmetries from Corollary 6 for the linear combination I_2+tI_1 of the integrals. We know that they are symmetries with respect to the line connecting the roots of the polynomial $\alpha z^2+(\beta+t)z+\gamma$. Analysing these symmetries, we see that they do not generate one more integral corresponding to the model 2 such that it is linear independent of I_1 and E, if and only if α, β and γ are real. In this case, a linear combination of the integral has a double root, so the metric is invariant with respect the rotations. Thus we reduced case (2,3) to cases (2,2), (2,4).

Case (3,4). Suppose $A_1=\frac{1}{z^2}dz\otimes dz$ and $A_2=\frac{1}{\alpha z^2+\beta z+\gamma}dz\otimes dz$. Then the metric is invariant with respect to rotations around 0. Consider the line connecting the roots of $\alpha z^2+\beta z+\gamma$. If the point 0 does not lie on the line, the reflection of 0 with respect to this line does not coincide with 0. The group of rotations around the image of 0 acts by isometries. Thus, we constructed a two-parameter group of isometries, which is possible only if the curvature is constant.

Now suppose 0 lies on the line connecting the roots of $\alpha z^2+\beta z+\gamma$. Then a linear combination of the square of the linear integral and of integral I is as in model 2 or as in model 1; so the we reduced case (3,3) to cases (2,3), (1,3).

Case (3,3). Suppose $A_1=\frac{1}{z^2-1}dz\otimes dz$ and $A_2=\frac{1}{\alpha z^2+\beta z+\gamma}dz\otimes dz$. Consider the symmetries from Corollary 6 for the linear combination I_2+tI_1 of the integrals. It is easy to check that if not all these symmetries coincide, they generate an at least one-parametric family of isometries of g. By Noether's Theorem, this family generates a linear in velocities integral of the geodesic flow. The square of the integral is as in model 1b (so we reduced case (3,3) to case (1,3)), or as in model 4 (so we reduced case (3,3) to case (3,4)). If all the symmetries coincide, a linear combination of the integrals is as in model 2 or as in model 1, so we reduced case (3,3) to cases (1,3), (2,3).

Case (4,4). In this case we have a two-parameter group of isometries, which is possible only if the metric has constant curvature. Theorem 7 is proved if both metrics are as in model 4.

We considered all possible cases. In every case we have proved that either the metric is as in Examples 1, 2 or has constant curvature. Theorem 7 is proved. □

4. Proof of Theorem 1

Let (M^2, g) be a connected complete oriented Riemannian surface. Suppose the Lie group $(\mathbb{R}, +)$ acts on M^2 by projective transformations. Let the element t of $(\mathbb{R}, +)$ act by the diffeomorphism F_t. Suppose there exists t_0 such that F_{t_0} is not a homothety. Then the metric $F_{t_0}^*g$ is projectively equivalent to g and is non-proportional to g, and the space $\mathcal{I}(M^2, g)$ is at least two-dimensional. We will first prove Theorem 1 under the assumption that the dimension of $\mathcal{I}(M^2, g)$ is precisely two.

Consider the integral from Theorem 2 (constructed for g and $\bar{g} := F_{t_0}^*g$). It is linear independent of the energy integral $E(\xi) := g(\xi, \xi)$. Therefore, the energy integral and the integral I are basis vectors of $\mathcal{I}(M^2, g)$.

Take a point P such that the metrics g and $\bar{g}$ are non-proportional. By Theorem 3, there exist coordinates (x, y) in a neighbourhood of P such that g and $\bar{g}$ have the form (2), (3), respectively.

Consider the vector $v = (v_1, v_2) := \left(\frac{d}{dt}F_t\right)_{|t=0}$. It is a projective vector field for g and for g_{t_0}. By Lemma 1 and Corollary 1, the Lie-derivatives $\mathcal{L}_v g$ and $\mathcal{L}_v \bar{g}$ are diagonal in the coordinates (x, y). We have

$$(\mathcal{L}_v g)_{ij} = -\sum_{k=1}^{2}\left(\frac{\partial g_{ij}}{\partial x_k}v_k + g_{ik}\frac{\partial v_k}{\partial x_j} + g_{jk}\frac{\partial v_k}{\partial x_i}\right).$$

Then

$$-\mathcal{L}_v g = \begin{pmatrix} X'v_1 - Y'v_2 + 2(X-Y)\frac{\partial v_1}{\partial x} & (X-Y)(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}) \\ (X-Y)(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}) & X'v_1 - Y'v_2 + 2(X-Y)\frac{\partial v_2}{\partial y} \end{pmatrix},$$

and

$$-\mathcal{L}_v \bar{g} = \begin{pmatrix} \text{something} & \frac{X-Y}{XY}(\frac{1}{X}\frac{\partial v_1}{\partial y} + \frac{1}{Y}\frac{\partial v_2}{\partial x}) \\ \frac{X-Y}{XY}(\frac{1}{X}\frac{\partial v_1}{\partial y} + \frac{1}{Y}\frac{\partial v_2}{\partial x}) & \text{something} \end{pmatrix}.$$

Since the Lie-derivatives $\mathcal{L}_v g$, $\mathcal{L}_v \bar{g}$ are diagonal, the elements that are not of the diagonal must be zero so that $\frac{X-Y}{XY}\big(\frac{1}{X}\frac{\partial v_1}{\partial y} + \frac{1}{Y}\frac{\partial v_2}{\partial x}\big) = 0$ and $(X-Y)\big(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\big) = 0$. Hence,

$$\frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial x} = 0. \tag{13}$$

By assumptions, the integral from Corollary 1 has the form (5); this gives us the following system of differential equations:

$$\begin{aligned} -\frac{1}{3}X'v_1 + \frac{1}{3}Y'v_2 + \frac{2}{3}(X-Y)\frac{\partial v_1}{\partial x} - \frac{4}{3}(X-Y)\frac{\partial v_2}{\partial y} &= (X-Y)(\beta - \alpha Y),\\ -\frac{1}{3}X'v_1 + \frac{1}{3}Y'v_2 + \frac{2}{3}(X-Y)\frac{\partial v_2}{\partial y} - \frac{4}{3}(X-Y)\frac{\partial v_1}{\partial x} &= (X-Y)(\beta - \alpha X). \end{aligned} \tag{14}$$

We will prove that if α is not zero, then the metric has constant curvature.

The first equation minus the second gives us

$$2(X-Y)\left(\frac{\partial v_1}{\partial x} - \frac{\partial v_2}{\partial y}\right) = (X-Y)(\alpha X - \alpha Y).$$

Using (13), we obtain

$$\frac{\partial v_1}{\partial x} = \frac{\alpha}{2}(X+\gamma), \quad \frac{\partial v_2}{\partial y} = \frac{\alpha}{2}(Y+\gamma), \tag{15}$$

where γ is a constant. Substitution into (14) gives

$$X'v_1 - Y'v_2 = \alpha(X^2 - Y^2) + (X-Y)(\alpha\gamma - 3\beta).$$

Using (13), we obtain the following equations for the constant $b := \alpha\gamma - 3\beta$ and for a constant c:

$$X'v_1 = \alpha X^2 + bX + c, \tag{16}$$

$$Y'v_2 = \alpha Y^2 + bY + c. \tag{17}$$

Let $O(P)$ be the orbit of $(\mathbb{R}, +)$ that goes through P. By Corollary 2, the orbit $O(P)$ has no points such that g_{t_0} and g are proportional. Then the equations (16), (17) hold at every point of the orbit. Since the functions X, Y and the vector field v are globally defined, the constants α, b, c are universal along the whole orbit.

Let t be the standard coordinate on $\mathbb{R}$. Since X (Y, respectively) depends on x (y, respectively) only, we have $X'v_1 = \frac{d}{dt}X := \dot{X}$, $Y'v_2 = \frac{d}{dt}Y := \dot{Y}$. Hence

$$\dot{X} = \alpha X^2 + bX + c,$$
$$\dot{Y} = \alpha Y^2 + bY + c.$$

These differential equations can be solved for $\alpha \neq 0$: the constant solutions are $-\frac{b}{2\alpha} \pm \frac{\sqrt{b^2/4 - \alpha c}}{\alpha}$, and the nonconstant solutions are as follows:

(1) For $D := b^2/4 - \alpha c < 0$, every nonconstant solution is the function
$-\frac{b}{2\alpha} + \frac{\sqrt{-D}}{\alpha}\tan(\sqrt{-D}(t+d_1))$.

(2) For $D := b^2/4 - \alpha c > 0$, every nonconstant solution is one of the functions
(a) $-\frac{b}{2\alpha} - \frac{\sqrt{D}}{\alpha}\tanh(\sqrt{D}(t+d_2))$,
(b) $-\frac{b}{2\alpha} - \frac{\sqrt{D}}{\alpha}\coth(\sqrt{D}(t+d_3))$.

(3) For $D := b^2/4 - \alpha c = 0$, every nonconstant solution is the function
$-\frac{b}{2\alpha} - \frac{1}{\alpha(t+d_4)}$.

The solutions (1), (2), (3) explode in finite time. This gives us a contradiction: the metrics g and $\bar g := g_{t_0}$ are smooth and therefore the eigenvalues of $\bar g^{i\alpha} g_{\alpha j}$ are finite.

If the functions $X(t)$ and $Y(t)$ have the form (2), then there exist points $Q_1, Q_2 \in O(P)$, such that $X(Q_1) < Y(Q_2)$. This gives a contradiction with Corollary 3.

Thus, either both solutions are constant, or one solution is constant $-\frac{b}{2\alpha} \pm \frac{\sqrt{D}}{\alpha}$ and the other has the form (2). By Lemma 2, the points where the metrics are proportional are isolated. Since the functions X and Y must be as listed above, if two sufficiently small neighbourhoods intersect and if in one neighbourhood one of the functions X, Y is constant, the same function must be constant in the second neighbourhood as well (because it is given by (2) which can not be constant on the intersection of the neighbourhoods). Thus, either near every point one function has the form (2) and the other is a constant, or at every point both functions are constant. In the second case, the metric is flat.

Let us consider the first case. Without loss of generality, we can assume that Y equals the constant $-\frac{b}{2\alpha} - \frac{\sqrt{D}}{\alpha}$, and that the behaviour of X on every orbit is given by $X(t) = -\frac{b}{2\alpha} - \frac{\sqrt{D}}{\alpha}\tanh(\sqrt{D}\,t)$ (the constant d_2 from (2) (a) depends on the choice of the initial point of the orbit and can be made zero). Let us show that the constant γ from (15) is equal to $-Y$.

Note that the constant γ is only locally defined. For example, if we change y to $-y$, the constant γ must be changed to $-\gamma - 2Y$. But this is essentially the only freedom we have, since in the coordinate system x, y the vector $(0, 1)$ is an eigenvector of $G = \bar g^{-1} g$ corresponding to the smaller eigenvalue, and since its length is $\sqrt{X - Y}$. Clearly, the property of γ to be equal to $-Y$ does not depend on whether we take γ or $-\gamma - 2Y$. Moreover, if two neighbourhoods intersect, and in one neighbourhood $\gamma = -Y$, then in the second neighbourhood $\gamma = -Y$ as well. Then the equation $\gamma = -Y$ is globally defined on the whole manifold.

We assume $\gamma \neq -Y$ and find a contradiction. We will first show that the metric is as in model 1b. Then we will solve the equation (15), find the metric g and show that it is not complete or not smooth.

Take a regular value r of the function X (considered as a function on M^2). Consider a connected component W of the set $\{P \in M^2 \mid X(P) = r\}$. It is a 1-dimensional submanifold of M^2. At every point $P \in W$, denote by $\mathrm{Proj}(v)$ the orthogonal projection of the vector v to the tangent space to W. Consider the function

$$f\colon W \to \mathbb{R}, \quad f(P) := g(\mathrm{Proj}(v), \mathrm{Proj}(v)).$$

In the coordinates x, y, the submanifold W coincides with a line $\{x = \text{const}\}$. Hence, $\mathrm{Proj}(v) = (0, v_2)$ and the function f is $(r - Y)v_2^2$. By the second part of (15), since by assumption $\gamma \neq -Y$, locally the function v_2 is monotone as a function of y. Then the function f is monotone near every point such that $f \neq 0$. Then the set W can not be compact. Thus the manifold M^2 is not compact. Therefore, without loss of generality we can assume that the manifold is homeomorphic to the disk. Since the function Y is constant, the linear combination $I - YE$ is an integral. Clearly, it is a square of a function linear in velocities, which should be an integral as well. Then the metric is as in model 1b or as in model 4. Since W is not compact, the metric is as in model 4. Thus, we may think that M^2 is $\mathbb{R}^2$ and that coordinates x, y are standard coordinates of $\mathbb{R}^2$. In particular, there exists no point such that $X = Y$.

Let us show that the metric is not complete. By (15), v_2 is a linear function of y. Then there exists a point such that $v_2 = 0$. Without loss of generality we can assume $v_2(0, 0) = 0$.

Consider the coordinate line $\{y = 0\}$. At every point of this line we have $v_2 = 0$. Suppose first $v_1 = 0$ at no point of the line. Then the vector field v never vanishes on the line $\{y = 0\}$ and is tangent to it, so that the line $\{y = 0\}$ is an orbit. In particular, the parameter t of the group $(\mathbb{R}, +)$ parametrises the line. Let us show that its length in one direction is finite.

Clearly, the length of line between points $t_0 < t_1$ is given to

$$\int_{t_0}^{t_1} \sqrt{g(v(t), v(t))}dt = \int_{t_0}^{t_1} |v_1(t)|\sqrt{X(t) - Y}dt. \tag{18}$$

In order to calculate $|v_1(t)|$, we will use the first equation of (15). By multiplying it by v_1, and using that $\frac{\partial v_1}{\partial x} v_1 = \frac{dv_1}{dt}$, we obtain

$$\frac{dv_1}{dt} = \frac{\alpha}{2}\,(X(t) + \gamma)\,v_1.$$

This is an ordinary differential equation for the unknown function $v_1(t)$. It can be solved, the solution is (C is a positive constant)

$$|v_1(t)| = C\sqrt{\frac{\exp(\sqrt{D}t)}{\exp(\sqrt{D}t) + \exp(-\sqrt{D}t)}}\exp\left(\frac{\alpha}{2}(Y + \gamma)t\right). \tag{19}$$

Substituting it and the formula for $X(t)$ in (18), we rewrite (18) as

$$\frac{\sqrt[4]{D}C}{\sqrt{\alpha}}\int_{t_0}^{t_1}\frac{\exp\left(\frac{\alpha}{2}(Y+\gamma)t\right)}{\exp(\sqrt{D}t)+\exp(-\sqrt{D}t)}dt. \tag{20}$$

We see that the length of the line is finite at least in one direction, so that the metric is not complete. This contradiction shows that there exists a point of the line such that v_1 does not vanish there.

Now suppose there exist points of the line such that $v_1 = 0$. Then, by (16), $X = -\frac{b}{2\alpha} \pm \frac{\sqrt{D}}{\alpha}$ at these points. Since X can not be equal to $Y = -\frac{b}{2\alpha} - \frac{\sqrt{D}}{\alpha}$, it must be equal to $-\frac{b}{2\alpha} + \frac{\sqrt{D}}{\alpha}$. We will show that in this case the scalar curvature explodes at (at least one of) these points, so the metric has a singularity.

Indeed, without loss of generality we can suppose that at the point $(0, 0)$ the value of X is not $-\frac{b}{2\alpha} + \frac{\sqrt{D}}{\alpha}$. Consider the orbit of the action passing through this point. It lies on the line $\{y = 0\}$. Since the function X is monotone on this orbit (as the function of t), one end of this orbit (when $t \to +\infty$) goes to infinity, and the other end (when $t \to -\infty$) ends at a point such that $X = -\frac{b}{2\alpha} + \frac{\sqrt{D}}{\alpha}$. We will show that the scalar curvature explodes at this point.

As we have shown above, the component v_1 of the vector field is given by (19), and the length of the segment of the orbit between t_0 and t_1 is given by (20). Since the length of the orbit should be finite in direction $t \to -\infty$,

$$\frac{\alpha}{2}(Y+\gamma) > \sqrt{D}. \tag{21}$$

Since the metric is given by $(X(x) - Y)(dx^2 + dy^2)$, and since $\frac{dx}{dt} = v_1$ so that $dx^2 = v_1^2 dt^2$, the metric is given by

$$\frac{\sqrt{D}}{\alpha}\big(1-\tanh(\sqrt{D}\,t)\big)\left(C^2\frac{\exp(\sqrt{D}t)\exp(\alpha(Y+\gamma)t)}{\exp(\sqrt{D}t)+\exp(-\sqrt{D}t)}\,dt^2+dy^2\right). \tag{22}$$

Then its scalar curvature is

$$-\frac{\alpha^2}{2}(Y+\gamma)\exp(\sqrt{D}t)\exp(-\alpha(Y+\gamma)t)\cosh(\sqrt{D}t)+\frac{\alpha\sqrt{D}}{2}\exp(-\alpha(Y+\gamma)t).$$

By (21), the scalar curvature goes to infinity when $t \to -\infty$. This contradicts the smoothness of the metric. Finally, $\gamma = -Y$.

Substituting $\gamma = -Y$ in (22), we obtain that the metric g is proportional to

$$\big(1-\tanh(\sqrt{D}\,t)\big)\left(\frac{\exp(\sqrt{D}t)}{\exp(\sqrt{D}t)+\exp(-\sqrt{D}t)}\,dt^2+dy^2\right),$$

and its curvature is constant.

Finally, $\alpha = 0$ and the Lie-derivative $\mathcal{L}_v g$ is proportional to g. Then all metrics $F_t^* g$ are proportional to g. By Corollary 4, the group $\mathbb{R}$ acts by homotheties. Then, by [15], either the group acts by isometries or the metric is flat. Theorem 1 is proved under the assumption that the dimension of $\mathcal{I}(M^2, g)$ is precisely two.

Now suppose the space $\mathcal{I}(M^2, g)$ has dimension greater than two. Then, by Theorem 6, if M^2 is closed, the curvature of g is constant as needed. Suppose M^2 is not closed. Since every nonclosed surface is covered by the disk, we can assume that M^2 is homeomorphic to the disk D^2. By Corollary 5, the complex structure on the disk (generated by g) is as that of $\mathbb{C}$. By Theorem 7, if the curvature of g is not constant, the metric g is (proportional to) the metric from Example 1 or Example 2. Let us show that these metrics admit no projective transformations which are not isometries. Actually, we will show that the metrics have no (non-proportional) projectively equivalent complete Riemannian metric. Since the metrics are not flat, they do not admit homotheties, so every projective transformation must be an isometry.

Let us show this for Example 1. First of all, the space $\mathcal{I}(M^2, g)$ for the metric $(x^2 + y^2 + \gamma)(dx^2 + dy^2)$ on $\mathbb{C}$ has dimension four. Indeed, we can present four linear independent integrals. They are (in the standard coordinates (x, y, p_x, p_y) on $T^*\mathbb{C}$)

$$
\begin{aligned}
H &:= \frac{{p_x}^2 + {p_y}^2}{x^2 + y^2 + \gamma},\\
F_1 &:= \frac{{p_x}^2 y^2 - \left(x^2 + \gamma\right) {p_y}^2}{x^2 + y^2 + \gamma},\\
F_2 &:= (x p_y - y p_x)^2,\\
F_3 &:= xyH - p_x p_y.
\end{aligned}
$$

The integrals are clearly linear independent (for example because the form from Lemma 2 for a nontrivial linear combination of the integrals H and F_1 is $C_1\, dz \otimes dz$ and is never equal to the form from Lemma 2 for a nontrivial linear combination of the integrals F_2 and F_3 which is $\frac{1}{C_2 z^2 + C_3 i} dz \otimes dz$).

Since the curvature of g is not constant, the dimension of $\mathcal{I}(M^2, g)$ can not be greater than four by [12].

Therefore, every integral quadratic in velocities is a linear combination of the integrals above. Clearly, the poles of the form from Lemma 2 for the linear combination of the integrals is symmetric with respect to the point 0. Then essentially (i.e. modulo a rotation of the coordinate system and a scaling) we can assume that the form is either $dz \otimes dz$ or $\frac{1}{z^2} dz \otimes dz$ or $\frac{1}{z^2 - 1} dz \otimes dz$. In the first case, the projectively equivalent metric constructed by Theorem 2 from the integral is

$$\pm\left(\frac{1}{C-y^2}-\frac{1}{x^2+\gamma+C}\right)\left(\frac{dx^2}{x^2+\gamma+C}+\frac{dy^2}{C-y^2}\right),$$

and we see that it is not always positive defined. In the second case, the projectively equivalent metric constructed from the integral has the form

$$\pm\left(\frac{1}{C}-\frac{1}{(r^2+\gamma)r^2+C}\right)\left(\frac{dr^2}{r^2\left(r^2+\gamma\right)r^2+C\right)}+\frac{dy^2}{C}\right)$$

in the polar coordinates. We see that the metric is not positive definite or not complete.

In the third case, the projectively equivalent metric constructed from the integral has the form

$$\pm\left(\frac{1}{X}-\frac{1}{Y}\right)\left(\frac{dx^2}{X}+\frac{dy^2}{Y}\right)$$

in the coordinates $x_{\text{new}}+iy_{\text{new}}:=\arcsin(x+iy)$ where the functions X, Y are given by

$$X:=(\cos(x))^2-(\cos(x))^4+\gamma\ (\cos(x))^2+C_1;$$
$$Y:=(\cosh(y))^4-(\cosh(y))^2+\gamma\ (\cosh(y))^2+C_2.$$

We see that the metric is not positive definite or not complete.

Thus, the metric from Example 1 has no non-proportional projectively equivalent Riemannian metrics which are everywhere defined and complete, and therefore every projective transformations is a isometry.

Now let us show that the metric from Example 2 admits no complete non-proportional projectively equivalent Riemannian metric.

The space $\mathcal{I}(M^2,g)$ for the metric $(x^2+\frac{y^2}{4}+\gamma)(dx^2+dy^2)$ has dimension three. Indeed, we can present three linear independent integrals. They are (in the standard coordinates (x,y,p_x,p_y) on $T^*\mathbb{C}$))

$$H:=\frac{{p_x}^2+{p_y}^2}{x^2+\frac{1}{4}y^2+\gamma},$$
$$F_1:=\frac{{p_x}^2\frac{y^2}{4}-\left(x^2+\gamma\right){p_y}^2}{x^2+\frac{1}{4}y^2+\gamma},$$
$$F_2:=\frac{1}{4}xy^2H+p_y(xp_y-yp_x).$$

If there exists a fourth linear independent integral for H, then, by [12], there exists a nontrivial integral linear in velocities. Metrics admitting integrals linear in velocities described in Theorem 5. It is easy to see that the lever curves of the coefficient f from Theorem 5 are generalised circles (i.e. circles or straight lines). Since linear transformations take generalised circles to generalised circles, and since not all level curves of the function $x^2 + \frac{1}{4}y^2 + \gamma$ are generalised circles, the metric from Example 2 admits no integral which is linear in velocities. Thus the dimension of $\mathcal{I}(M^2, g)$ is precisely three.

It is easy to see that the linear combination $AH + BF_1 + CF_2$ of the integrals is nonnegative on the whole $T^*\mathbb{C}$ if and only if $B = C = 0$. Indeed, the coefficient by p_x^2 is $\frac{4A+By^2+Cy^2x}{x^2+\frac{y^2}{4}+\gamma}$. It is nonnegative at every point of $\mathbb{C}$ if and only if $C = 0$, $A \geq 0$, $B \geq 0$. The coefficient by p_y^2 in the linear combination $AH + BF_1$ is $\frac{A-B(x^2+\gamma)}{x^2+\frac{y^2}{4}+\gamma}$. It is nonnegative at every point of $\mathbb{C}$ if and only if $A \geq 0$, $B \leq 0$. Thus, the linear combination $AH + BF_1 + CF_2$ of the integrals is nonnegative on the whole $T^*\mathbb{C}$ if and only if $B = C = 0$.

Since every metric projectively equivalent to g and not proportional to g gives by Theorem 2 a nonnegative quadratic integral independent of the energy integral, there is no metric projectively equivalent to g and non-proportional to g. Theorem 1 is proved. □

Note added in proof. After the paper was submitted, I proved the multidimensional version of Conjecture 1. The proof can be found in [30].

Theorem. *Let a connected Lie group G act on a complete connected Riemannian manifold (M^n, g) of dimension $n \geq 2$ by projective transformations. Then it acts by affine transformations, or g has constant positive sectional curvature.*

References

[1] H. Akbar-Zadeh, R. Couty, Espaces à tenseur de Ricci parallèle admettant des transformations projectives. *Rend. Mat.* (6) **11** (1978), 85–96. Zbl 0393.53023 MR 0511203

[2] E. Beltrami, Resoluzione del problema: riportari i punti di una superficie sopra un piano in modo che le linee geodetische vengano rappresentante da linee rette. *Ann. Mat.* **1** (1865), 185–204.

[3] G. D. Birkhoff, *Dynamical Systems.* Amer. Math. Soc. Colloq. Publ. 9, Amer. Math. Soc., New York 1927. JFM 53.0732.01 MR 0209095

[4] A. V. Bolsinov, V. S. Matveev, A. T. Fomenko, Two-dimensional Riemannian metrics with an integrable geodesic flow. Local and global geometries. *Sb. Math.* **189** (1998), 1441–1466. Zbl 0917.58031 MR 1691292

[5] R. Couty, Sur les transformations des variétés riemanniennes et kählériennes. *Ann. Inst. Fourier. Grenoble* **9** (1959) 147–248. Zbl 0099.37404 MR 0121754

[6] U. Dini, Sopra un problema che si presenta nella teoria generale delle rappresentazione geografice di una superficie su di un'altra. *Ann. Mat.* Ser. 2 **3** (1869), 269–293; see also: Dini, Ulisse, *Opere*, Vol. I, Algebra. Geometria differenziale. Edizioni Cremonese della Casa Editrice Perrella, Roma 1953. JFM 02.0622.02

[7] L. P. Eisenhart, *Riemannian Geometry.* 2d printing, Princeton University Press, Princeton, NJ, 1949. Zbl 0041.29403 MR 0035081

[8] I. Hasegawa, K. Yamauchi, Infinitesimal projective transformations on tangent bundles with lift connections. *Sci. Math. Japan* **57** (2003), 469–483. Zbl 1050.53026 MR 1975964

[9] M. Igarashi, K. Kiyohara, K. Sugahara, Noncompact Liouville surfaces. *J. Math. Soc. Japan* **45** (1993), 459–479. Zbl 0798.53007 MR 1219879

[10] K. Kiyohara, Compact Liouville surfaces. *J. Math. Soc. Japan* **43** (1991), 555–591. Zbl 0751.53015 MR 1111603

[11] K. Kiyohara, Two Classes of Riemannian Manifolds Whose Geodesic Flows Are Integrable. *Mem. Amer. Math. Soc.* **130** (1997), no. 619. Zbl 0904.53007 MR 1396959

[12] G. Koenigs, *Sur les géodésiques à intégrales quadratiques.* A note appearing in "Leçons sur la théorie générale des surfaces", Vol. 3, Chelsea Publishing 1896. JFM 25.1196.03

[13] V. N. Kolokol'tzov, Geodesic flows on two-dimensional manifolds with an additional first integral that is polynomial with respect to velocities. *Math. USSR-Izv.* **21** (1983), 291–306. Zbl 0548.58028 MR 0675528

[14] V. N. Kolokol'tzov, Polynomial integrals of geodesic flows on compact surfaces. Dissertation, Moscow State University 1984.

[15] A. Lichnerowicz, *Geometry of groups of transformations.* Translated from the French and edited by Michael Cole, Noordhoff International Publishing, Leyden 1977. Zbl 0348.53001 MR 0438364

[16] S. Lie, Untersuchungen über geodätische Kurven. Math. Ann. **20** (1882); in *Sophus Lie Gesammelte Abhandlungen*, Band 2, erster Teil, Teubner, Leipzig 1935, 267–374. Zbl 0012.41201

[17] V. S. Matveev, P. J. Topalov, Trajectory equivalence and corresponding integrals. *Regul. Chaotic Dyn.* **3** (1998), 30–45. Zbl 0928.37003 MR 1693470

[18] V. S. Matveev, P. J. Topalov, Geodesic equivalence of metrics on surfaces, and their integrability. *Dokl. Akad. Nauk* **367** (1999), 736–738; English translation *Dokl. Math.* **60** (1999), 112–114. Zbl 1041.37502 MR 1719653

[19] V. S. Matveev and P. J. Topalov, Metric with ergodic geodesic flow is completely determined by unparameterized geodesics. *Electron. Res. Announc. Amer. Math. Soc.* **6** (2000), 98–104. Zbl 0979.53032 MR 1796527

[20] V. S. Matveev and P. J. Topalov, Integrability in theory of geodesically equivalent metrics. *J. Phys. A.* **34** (2001), 2415–2433. Zbl 0983.53024 MR 1831306

[21] V. S. Matveev, P. J. Topalov, Quantum integrability for the Beltrami-Laplace operator as geodesic equivalence. *Math. Z.* **238** (2001), 833–866. Zbl 0998.53025 MR 1872577

[22] V. S. Matveev, Geschlossene hyperbolische 3-Mannigfaltigkeiten sind geodätisch starr. *Manuscripta Math.* **105** (2001), 343–352. Zbl 01690798 MR 1856615

[23] V. S. Matveev, Low-dimensional manifolds admitting metrics with the same geodesics. In *Differential geometry and integrable systems* (Tokyo, 2000), Contemp. Math. 308, Amer. Math. Soc., Providence, RI, 2002, 229–243. Zbl 01857924 MR 1955639

[24] V. S. Matveev, *Three-manifolds admitting metrics with the same geodesics. Math. Res. Lett.* **9** (2002), 267–276. Zbl 1013.53026 MR 1909644

[25] V. S. Matveev, Three-dimensional manifolds having metrics with the same geodesics. *Topology* **42** (2003), 1371–1395. Zbl 1035.53117 MR 1981360

[26] V. S. Matveev, Hyperbolic manifolds are geodesically rigid. *Invent. Math.* **151** (2003), 579–609. Zbl 1039.53046 MR 1961339

[27] V. S. Matveev, Die Vermutung von Obata für Dimension zwei. *Arch. Math.* **82** (2004) 273–281. Zbl 02119264 MR 2053631

[28] V. S. Matveev, Solodovnikov's theorem in dimension two. *Dokl. Math.* **69** (2004), 338–341. MR 2115905

[29] V. S. Matveev, Projectively equivalent metrics on the torus. *Differential Geom. Appl.* **20** (2004), 251–265. Zbl 1051.37030 MR 2053913

[30] V. S. Matveev, Projective Lichnerowicz–Obata conjecture. ArXiv.org/math.DG/0407337.

[31] T. Nagano, T. Ochiai, On compact Riemannian manifolds admitting essential projective transformations. *J. Fac. Sci. Univ. Tokyo Sect. IA, Math.* **33** (1986) 233–246. Zbl 0645.53022 MR 0866391

[32] J. A. Schouten, Erlanger Programm und Übertragungslehre. Neue Gesichtspunkte zur Grundlegung der Geometrie. *Rend. Circ. Matem. Palermo* **L** (1926), 142–169. JFM 52.0721.03

[33] A. S. Solodovnikov, Projective transformations of Riemannian spaces. *Uspehi Mat. Nauk* (N.S.) **11** (4) (1956), 45–116. Zbl 0071.15202 MR 0084826

[34] A. S. Solodovnikov, Uniqueness of a maximal K-expansion. *Uspehi Mat. Nauk* **13** (6) (1958), 173–179. Zbl 0087.36905 MR 0103498

[35] A. S. Solodovnikov, The group of projective transformations in a complete analytic Riemannian space. *Dokl. Akad. Nauk SSSR* **186** (1969), 1262–1265. Zbl 0188.54003 MR 0248685

[36] P. J. Topalov and V. S. Matveev, Geodesic equivalence and integrability. MPIM1998-74, *Preprint series of Max-Planck-Institut für Mathematik*, Bonn 1998.

[37] P. J. Topalov, Vladimir S. Matveev, Geodesic equivalence via integrability. *Geom. Dedicata* **96** (2003), 91–115. Zbl 1017.37029 MR 1956835

[38] P. J. Topalov, Commutative conservation laws for geodesic flows of metrics admitting projective symmetry. *Math. Res. Lett.* **9** (2002) 65–72. Zbl 1005.53065 MR 1892314

[39] H. Weyl, *Zur Infinitisimalgeometrie: Einordnung der projektiven und der konformen Auffasung.* Nachrichten von der K. Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse; "Selecta Hermann Weyl", Birkhäuser Verlag, Basel, Stuttgart 1956. JFM 48.0844.04 MR

[40] H. Weyl, On the foundations of infinitesimal geometry. *Bull. Amer. Math. Soc.* **35** (1929), 716–725. JFM 55.1027.01

[41] K. Yamauchi, On infinitesimal projective transformations satisfying the certain conditions. *Hokkaido Math. J.* **7** (1978), 74–77. Zbl 0383.53016 MR 0482584

[42] K. Yamauchi, On Riemannian manifolds admitting infinitesimal projective transformations. *Hokkaido Math. J.* **16** (1987), 115–125. Zbl 0629.53040 MR 0894156

[43] K. Yamauchi, A study of infinitesimal projective transformations of Riemannian manifolds. *Sci. Rep. Kagoshima Univ.* **37** (1988), 25–54. Zbl 0676.53043 MR 0981171

Received March 8, 2004

Vladimir S. Matveev, Mathematisches Institut der Universität Freiburg, Eckerstraße 1, 79104 Freiburg, Germany

E-mail: matveev@email.mathematik.uni-freiburg.de