A new integrable system on the sphere

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Abstract

We present a new Liouville-integrable natural Hamiltonian system on the (cotangent bundle of the) sphere S^2. The second integral is cubic in the momenta.

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1 Introduction

A Hamiltonian system is called natural if its Hamiltonian is a sum of a positive-definite kinetic energy and a potential. Natural Hamiltonian systems on (cotangent bundles of) closed surfaces admitting integrals polynomial in momenta are interesting for a number of reasons:

1. They are classical: The formulation of the problem is due at least to Darboux [7].
2. If a natural Hamiltonian system admits a real-analytic integral, then it admits an integral polynomial in momenta [18].
3. In particular, all known natural integrable Hamiltonian systems on surfaces have integrals polynomial in momenta.

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4. The existence of the integral polynomial in momenta of degree one or two has a very clear geometric background: the existence of an integral of degree one implies the existence of a one-parametric family of symmetries of the system. The existence of an integral of degree two implies the existence of so-called separating variables.

Natural Hamiltonian systems on closed surfaces that admit a nontrivial integral which is polynomial in momenta of degree one or two are completely understood: there exists a complete description and classification, see, for example, [5]. Up to now, only one family of natural Hamiltonian systems on a closed surface admitting an integral of degree three and admitting no (non-trivial) integral of degree one or two is explicitly known, namely the family of system obtained from the Goryachev-Chaplygin top. We now present a second, distinct, family of natural Hamiltonian systems with a cubic integral.

1.1 The result

Consider the sphere $S^2 \subset \mathbb{R}^3$ of radius 1 with the spherical coordinates $(x, y, z) = (-\sin\theta\cos\phi, -\sin\theta\sin\phi, \cos\theta)$ and the following two functions H (the Hamiltonian) and F (the second integral) on the cotangent bundle of the sphere without poles $z = \pm 1$. Let $A, c, s \in \mathbb{R}$ be parameters with $s > 1$ and define

$$W(z) = z + s, \quad P(z) = 3z^2 + 4sz + 1, \quad Q(z) = 3z^2 + 2sz - 1$$

and

$$G(z) = \frac{P(z)}{(2W(z))^2}.$$

Then, the function H is given by $H := K + V$, where

$$K := \frac{1}{2}\left(\frac{1}{\sin^2\theta} + G(\cos\theta)\right)p_\phi^2 + \frac{1}{2}p_\theta^2$$
$$V := A\frac{\sin\theta}{\sqrt{W(\cos\theta)}}\cos\phi + \frac{c}{W(\cos\theta)},$$

and F is defined as

$$F := 2Hp_\phi - p_\phi^3 + A\cos(\phi)\frac{Q(\cos\theta)}{\sqrt{W(\cos\theta)}\sin\theta}p_\phi + 2A\sin\phi\sqrt{W(\cos\theta)}p_\theta\,.$$

Proposition: *The functions H and F can be analytically continued to the cotangent bundle of the whole sphere. The continuation is also polynomial in momenta (of degree 2 for H and of degree 3 for F).*

We will denote the continuations of H, K, F by the same letters H, K, F; in particular in the theorem below we mean the continued functions defined on the cotangent bundle of the whole sphere.

Theorem: *The following statements hold:*

1. *The functions H and F commute with respect to the standard Poisson bracket on T^*S^2 and are functionally independent*

2. *The kinetic energy K is positive definite*

3. *If $A \neq 0$, the Hamiltonian H does not admit a (smooth) nontrivial integral which is polynomial in velocities of degree less than three and which is linearly independent of H.*

In other words, we found a new 1) Liouville integrable, 2) natural Hamiltonian system on the sphere with 3) an integral cubic in momenta.

1.2 Why the system is new

The previously known family comes from the Goryachev-Chaplygin top using symplectic reduction, see [3] for the details. It is a natural Hamiltonian system on the sphere. In the standard spherical coordinates its Hamiltonian H_1 is $K_1 + V_1$, where the kinetic energy K_1 and the potential V_1 are

$$\begin{aligned} K_1 &:= \frac{1}{2}\left(\frac{\cos^2\theta}{\sin^2\theta} + 4\right) p_\phi^2 + \frac{1}{2} p_\theta^2 \\ V_1 &:= A_1 \sin\theta \sin\phi\,, \end{aligned}$$

see, for example, [9]. The integral F_1 for the Hamiltonian H_1 is given by

$$F_1 := H_1 p_\phi - 2p_\phi^3 - \frac{A_1}{2} p_\theta \cos(\phi)\cos(\theta) + \frac{A_1}{2}\frac{\sin\phi\left(3\,\left(\cos\theta\right)^2 - 2\right)}{\sin\theta} p_\phi.$$

It is easy to see that the system we found is essentially different from the Goryachev-Chaplygin system, i.e. that there exists no diffemorphism $D: S^2 \to S^2$ such that $D_* K = \alpha K_1$ and $D_* V = \beta V_1$ for certain constants α, β.

Consider the metrics g, g_1 corresponding to the kinetic energies K, K_1, respectively. They are given by the formulae:

$$ds^2 = d\theta^2 + \frac{d\phi^2}{1/\sin^2\theta + G(\cos\theta)} \tag{1}$$

$$ds_1^2 = d\theta^2 + \frac{d\phi^2}{4 + \cot^2\theta} = d\theta^2 + \frac{d\phi^2}{1/\sin^2\theta + 3}. \tag{2}$$

By direct calculation, it is easy to see that level lines of the curvature of both metric are the lines $\{\theta = \text{const}\}$. More precisely, since the formulae above do not depend on ϕ, the curvature is constant along every such line. It is easy to see that the curvature is not constant.

Then, the diffeomorphism D, if it exists, must take the lines $\{\theta = \text{const}\}$ of the first system to the lines $\{\theta = \text{const}_1\}$ of the second system. Since the lines $\{\theta = \text{const}\}$ are orthogonal to the lines $\{\phi = \text{const}\}$ in the first and in the second metric, the diffeomorphism D must also take the lines $\{\phi = \text{const}\}$ of the first system to the lines $\{\phi = \text{const}_1\}$ of the second system. Using that the coordinate ϕ is periodic with the period 2π, we obtain that the diffeomorphism D must be given by formulae

$$\theta_{new} = \theta_{new}(\theta), \quad \phi_{new} = \pm\phi + \phi_0 .$$

Then, the pull-back of the metric g_1 has the form

$$\left(\frac{\partial\theta_{new}}{\partial\theta}\right)^2 d\theta^2 + \frac{d\phi^2}{4 + \cot^2\theta_{new}}.$$

Then, $\frac{\partial\theta_{new}}{\partial\theta} = \pm 1$, and the diffeomorphism D must be given by the formula

$$\theta_{new} = \theta \qquad \text{or} \quad \theta_{new} = \pi - \theta, \qquad \phi_{new} = \pm\phi + \phi_0 .$$

Clearly, the pull-back of the metric g_1 with the help of such diffeomorphism is not proportional to g. Otherwise G would need to become a constant. But then P would have a double root at $z = -s$, which is impossible. Thus, our system is essentially different from the reduced Goryachev-Chaplygin top.

Selivanova [17] proved the existence of an additional family of natural Hamiltonian systems admitting integrals of degree 3 in momenta. This family is not explicit. Instead, a nonlinear differential equation is derived and it is proved that certain solutions of this equation allow to construct a geodesic flow admitting an integral of degree three and admitting no integral of smaller degree.

At present we do not know whether our system overlaps with the systems considered by Selivanova. Our guess is that this is not the case. Here are some reasons supporting this claim:

1. It is not clear whether the system of Selivanova is real-analytic; she claims the smoothness only.

2. Selivanova's family contains (the reduction of) Goryachev-Chaplygin system; our family does not.

3. The ansatz of Selivanova is more restrictive than the form of our system.

2 Proof of the proposition

2.1 Analytic proof

Clearly, the functions V, K, F depend real-analytically on (ϕ, θ) and on the corresponding momenta p_ϕ, p_θ, so the only possible problem could appear near the points $z = \pm 1$. We consider (x, y) as a local coordinate system near these points of the sphere. The coordinates (x, y) are related to coordinates (ϕ, θ) by the following real-analytic change of variables:

$$\begin{aligned} \sin\theta &= \sqrt{x^2+y^2}, & p_\theta &= (xp_x + yp_y)\sqrt{(x^2+y^2)^{-1} - 1} \\ \tan\phi &= y/x, & p_\phi &= xp_y - yp_x\,. \end{aligned}$$

In the new local coordinates the kinetic energy and potential are

$$K = \frac{1}{2}\left(p_x^2 + p_y^2 - (xp_x + yp_y)^2 + (xp_y - yp_x)^2 G(z)\right),$$

$$V(x, y) = -A\frac{x}{\sqrt{W(z)}} + \frac{c}{W(z)}, \quad \text{where} \quad z = \pm\sqrt{1 - x^2 - y^2}\,.$$

The sign of z depends on whether a neighbourhood of the north or south pole is considered. Clearly, K and V are real-analytic. Thus, the Hamiltonian H is a real-analytic function on the whole T^*S^2. Since we will need it for the proof of the second statement of the Theorem, we remark that the kinetic energy is positive definite near the poles, where x and y are small.

Now let us show that the second integral also is a real-analytic function on the whole T^*S^2. The first two terms of F are real-analytic, since the momentum p_ϕ is $xp_y - yp_x$ in the new coordinate system, and therefore is real-analytic, and we already have proven that H is real-analytic. The sum of the third and the fourth terms in the new coordinate system is

$$A\left(\frac{-xy}{\sqrt{\sqrt{1-x^2-y^2}+s}}p_x + \frac{2 - 3x^2 - 2y^2 + 2s\sqrt{1-x^2-y^2}}{\sqrt{\sqrt{1-x^2-y^2}+s}}p_y\right),$$

and this is clearly real-analytic.

2.2 Geometric proof

The second integral F is cubic in the momenta, like the Goryachev-Chaplygin top. This top does not appear as a limiting case in our family, but is serves as a motivation for the following coordinate transformation, which uses global coordinates (x, y, z) on a sphere embedded in $\mathbb{R}^3$. The angular momentum

(L_x, L_y, L_z) where $L_z = xp_y - yp_x = p_\phi$ etc. with cyclic permutations, satisfies the usual Poisson structure on the sphere with non-vanishing brackets

$$\{L_x, L_y\} = L_z, \quad \{x, L_y\} = z, \; \{z, L_y\} = -x$$

and cyclic permutations thereof. This bracket has Casimirs $x^2 + y^2 + z^2 = 1$ and $xL_x + yL_y + zL_z = 0$. The global Hamiltonian is $H = K + V$, where

$$K = \frac{1}{2}\left(L_x^2 + L_y^2 + (1 + G(z))L_z^2\right)$$

$$V(x, y, z) = -A\frac{x}{\sqrt{W(z)}} + \frac{c}{W(z)},$$

and is clearly real-analytic. The integral reads

$$\begin{aligned} F &= 2HL_z - L_z^3 - A\frac{Q}{\sqrt{W}}\frac{xL_z}{1 - z^2} - 2A\sqrt{W}\frac{y(xL_y - yL_x)}{1 - z^2} \\ &= 2HL_z - L_z^3 + \frac{A}{\sqrt{W}}(xL_z + 2WL_x)\,. \end{aligned}$$

In the second line the apparent singularity in F is removed by using both Casimirs $x^2 + y^2 + z^2 = 1$ and $xL_x + yL_y + zL_z = 0$ and in addition the identity $Q = 2zW + z^2 - 1$. The integral is cubic in the momenta, and clearly is real-analytic.

Like the Goryachev-Chaplygin top the system is only integrable on the level set of the Casimir $xL_x + yL_y + zL_z = 0$. Writing the Hamiltonian in this form suggests an interpretation of a spinning top with zero angular momentum (the vanishing Casimir), an orientation dependent moment of inertia, and a non-standard potential. Note, however, that $1 + G$ is not positive for all $z \in [-1, 1]$ for certain values of s, which means that a moment of inertia may pass through infinity for such value of s. Nevertheless, the kinetic energy in H is positive definite for all s, see Section 3.2. This is possible because even when $1 + G < 0$ because $xL_x + yL_y + zL_z = 0$.

Introducing $\xi = x + iy$ and $\eta = L_x + iL_y$ and writing $V(x, y, z) = xU(z)$ the equations of motion are

$$-i\dot{\xi} = -L_z(1 + G)\xi + z\eta \tag{3}$$

$$-i\dot{\eta} = -\frac{1}{2}G'L_z^2\xi - L_zG\eta + zU - \frac{1}{2}U'\left(\xi^2 + \xi\bar{\xi}\right). \tag{4}$$

Here $z^2 = 1 - \xi\bar{\xi}$ and $-2zL_z = \xi\bar{\eta} + \eta\bar{\xi}$, or the equations are completed with $\dot{z} = xL_y - yL_x$ and $\dot{L}_z = -yU$.

3 Proof of the Theorem

3.1 Liouville integrable

It is tedious but easy to check that the functions H and F commute, since they are given by explicit formulae. The calculation goes as follows. The canonical bracket is a quadratic polynomial in the momenta. The term independent of the momenta vanishes because $Q = 2zW - 1 + z^2$. The coefficient of $p_\theta p_\phi$ vanishes because $P = 4zW + 1 - z^2$. The nontrivial fact is that the coefficients of p_ϕ^2 which contains P, Q and derivatives thereof vanishes.

H and F are independent because H has terms quadratic in momenta (and not a square of a linear function of momenta) and F has terms that are cubic in momenta (and not a square of a linear function of momenta). So H and F have vanishing Poisson bracket and are functionally independent, hence the system is Liouville integrable.

3.2 Natural

As we already have remarked in Section 2.1, the kinetic energy K is positive near the poles of the sphere. Thus, it is sufficient to show that the metric (1) is positive definite at regular points of the spherical coordinate systems, i.e. we have to show that $C := 1/\sin^2\theta + G(\cos\theta)$ is positive for $0 < \theta < \pi$. Substituting z instead of $\cos\theta$, we obtain

$$C(z) = \frac{1}{4}\frac{6z^2 + 12sz + 4s^2 - 3z^4 - 4sz^3 - 1}{(1-z^2)(z+s)^2}.$$

Since the denominator $(1-z^2)\,(z+s)^2$ is always positive for $-1 < z = \cos(\theta) < 1$, we need to prove that the numerator

$$R(z) := \frac{3}{2}z^2 + 3sz + s^2 - \frac{3}{4}z^4 - sz^3 - \frac{1}{4}$$

is positive. Its value at $z = -1$ is $(s-1)^2$ and hence positive; its derivative is

$$\frac{\partial R(z)}{\partial z} = 3(1-z^2)(z+s)\,.$$

Since the derivative is also positive for $-1 < z = \cos(\theta) < 1$, $R(z)$ is positive for $-1 < z = \cos(\theta) < 1$. Hence the metric (1) is positive definite.

3.3 Cubic

Let us prove that, for $A \neq 0$, our system does not admit an integral linear in momenta, and that every integral quadratic in momenta is proportional to the Hamiltonian.

Suppose the function $P_2 + P_1 + P_0$, where every P_i is a homogeneous polynomial in momenta of degree i, is an integral for $H = K + V$. Then, the Poisson bracket $\{K + V, P_2 + P_1 + P_0\}$ must vanish. We have

$$\begin{aligned} 0 &= \{K + V, P_2 + P_1 + P_0\} \\ &= \{K, P_2\} + \{K, P_1\} + \{K, P_0\} + \{V, P_2\} + \{V, P_1\} + \{V, P_0\}\,. \end{aligned} \qquad (5)$$

Since the Poisson bracket of two functions $F_1, F_2 : T^*M^2 \to \mathbb{R}$ is given by

$$\{F_1, F_2\} = \frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial p_x} + \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial p_y} - \frac{\partial F_1}{\partial p_x}\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial p_y}\frac{\partial F_2}{\partial y},$$

where x, y are local coordinates on M^2 and p_x, p_y are the corresponding momenta, every term on the right-hand side of (5) is polynomial and homogeneous in momenta: $\{K, P_2\}$ of degree 3; $\{K, P_1\}$ of degree 2; $\{K, P_0\}+\{V, P_2\}$ of degree 1 and $\{V, P_1\}$ of degree 0. The Poisson bracket $\{V, P_0\}$ evidently vanishes. For (5) to vanish all the other degree 0, 1, 2, and 3 polynomials must also vanish identically.

Let us prove that the linear term P_1 must be zero. Vanishing of $\{K, P_1\}$ implies that P_1 is a linear integral of the geodesic flow of the metric given by K. In the spherical coordinates ϕ, θ on the sphere the metric is given by (1). We have already remarked in Section 1.2 that the level curves of the curvature are the curves $\{\theta = \text{const}\}$.

Suppose the integral P_1 has the form $v_1(\theta, \phi)p_\theta + v_2(\theta, \phi)p_\phi$. Then, by Noether's Theorem, the vector field $v := (v_1, v_2)$ is a Killing vector field, i.e. its flow preserves the metric. Then, the flow of v preserves the curvature, which implies that the component v_1 must be zero.

Now, using that $\{V, P_1\}$ vanishes, we have that $\{V, v_2 p_\phi\}$ is zero, which in the case $v_2 \not\equiv 0$ implies $\frac{\partial V}{\partial \phi} = 0$, which is definitely not the case. Thus, $v_2 \equiv 0$, and $P_1 \equiv 0$.

Now let us show that $P_2 + P_0$ is proportional to the Hamiltonian H. Since $\{K, P_2\} = 0$, P_2 is an integral of the geodesic flow for the metric (1). As we already mentioned, for geodesic flows on closed surfaces the integrals quadratic in velocities are completely understood. In particular, if the curvature of the surface is not constant, then the space of quadratic integrals is at most two-dimensional [14, 12].

The curvature of the metric (1) is not constant, and the following two linearly independent functions are integrals of its geodesic flow: the first is K itself, and the second is p_ϕ^2. Then, for certain constants β, α, we have $P_2 = \alpha p_\phi^2 + \beta K$. Our goal is to show that $P_2 + P_0$ is proportional to H. ¿From $\{K, P_0\} + \{V, P_2\} = 0$ we find $\{K, P_0 - \beta V\} + \alpha\{V, p_\phi^2\} = 0$. Hence,

$$p_\theta \frac{\partial(P_0 - \beta V)}{\partial \theta} + p_\phi \frac{1}{\sin^2\theta + G(\cos\theta)} \frac{\partial(P_0 - \beta V)}{\partial \phi} - 2\alpha p_\phi \frac{\partial V}{\partial \phi} = 0\,.$$

Therefore $\frac{\partial(P_0-\beta V)}{\partial\theta} = 0$, and then $\frac{\partial(P_0-\beta V)}{\partial\phi} = 2\alpha(\sin^2\theta + G(\cos\theta))\frac{\partial V}{\partial\phi}$. By cross differentiation we find

$$2\alpha\frac{\partial\left((\sin^2\theta + G(\cos\theta))\frac{\partial V}{\partial\phi}\right)}{\partial\theta} = 0,$$

which is possible only if $\alpha = 0$ or $A = 0$. If $\alpha = 0$ then $P_0 - \beta V = \text{const}$, and so after absorbing the constant in the Hamiltonian we have $\beta = 1$ and the integral $P_2 + P_1 + P_0$ is proportional to H. This proves the Theorem.

4 Geodesic flows with cubic integrals

A geodesic flow is a natural Hamiltonian system whose potential energy is identically zero.

According to Maupertuis's principle, an integrable natural Hamiltonian system immediately gives a family of integrable geodesic flow, see [3] for details. If the integral of the system is polynomial in momenta, the integrals of the geodesic flows are also polynomial of the same degree.

In our case, the Hamiltonian of the geodesic flow is given by

$$H_{geod} = \frac{p_\theta^2 + p_\phi^2\big(1/\sin^2\theta + G(\cos\theta)\big)}{2(h-V)}.$$

Here h is a constant such that h is larger than the maximum of the potential $V(x)$ on the sphere S^2. The integral of the third degree is

$$\begin{aligned} F_{geod} =& 2Hp_\phi - 2Vp_\phi + p_\phi^3 + H_{geod}\Big(2Vp_\phi \\ &+ A\cos(\phi)\frac{Q(\cos\theta)}{\sqrt{W(\cos\theta)}\sin\theta}p_\phi + 2A\sin\phi\sqrt{W(\cos\theta)}p_\theta\Big). \end{aligned} \tag{6}$$

The metric corresponding to H_{geod} is

$$ds^2_{geod} = (h-V)\left(d\theta^2 + \frac{d\phi^2}{1/\sin^2\theta + G(\cos\theta)}\right). \tag{7}$$

The investigation of geodesic flows that admit an integral polynomial in momenta is a very classical subject. If the degree of integrals is one or two, they were well-understood already in 19th century, see for example [7]. There are a lot of local examples of geodesic flows admitting integral polynomial in velocities of degree 3, see the survey [11] for details, and only very few global, i.e. on closed or complete surfaces.

Let us recall the known results about geodesic flows on closed surfaces admitting an integral polynomial in momenta. For a more detailed review see [5].

First of all, in view of results of Kolokoltsov [15], a geodesic flow of a surfaces of genus greater than two cannot admit a nontrivial integral polynomial in momenta. Then, an orientable surfaces admitting such geodesic flows must be the sphere or the torus.

We collect the main results about existence and classification of metrics on the torus and on the sphere whose geodesic flow admits a nontrivial integral polynomial in momenta in the following table:

	Sphere S^2	Torus T^2
Degree 1	All is known	All is known
Degree 2	All is known	All is known
Degree 3	Series of examples	Partial negative results
Degree 4	Series of examples	Partial negative results
Degree ≥ 5	Nothing is known	Nothing is known

The following notation is used in the table. "Degree" means the smallest degree of an integral polynomial in velocities. "All is known" means that there exists an effective description and classification (can be found in [4]).

There exists only one explicit "Series of examples" for degree three. It comes from the Goryachev-Chaplygin case of rigid body motion (by applying symplectic reduction and Maupertuis's principle), see [3]. The most valuable "partial negative results" are due to Byalyĭ [6] and Denisova and Kozlov [8]. They proved that if a natural Hamiltonian system on the torus whose kinetic energy is given by a flat metric admits an integral polynomial in momenta of degree three, it admits an integral linear in momenta.

Thus, our system gives one more series of examples of integrable geodesic flows on the sphere whose integral is polynomial in momenta of degree 3. Below, in Sections 4.1, 4.2, we show that these geodesic flows do not admit an integral which is polynomial in momenta of degree less than 3, and that these examples are different from examples coming from Goryachev-Chaplygin. Note, that this does not follow directly from the Theorem.

Kiyohara [13] proved the existence of one more series of examples of integrable geodesic flows on the sphere whose integral is polynomial in momenta of degree 3. Although his examples are not explicit (similar to Selivanova [17] he writes a nonlinear differential equation and claims that its solutions give integrable geodesic flows), it is easy to see that his examples are different from ours, since they have regions of constant curvature and therefore can not be real-analytic.

4.1 Cubic

We need to prove that the Hamiltonian H_{geod} has no integral which is polynomial in velocities of degree 1,2. We will prove it assuming that the energy level h is sufficiently large. Our proof for arbitrary h is based on the technique developed in [5] and is very long. It will be published elsewhere.

It is easy to see that the integral F_{geod} is not the third power of a function linear in momenta and is not the product of a function linear in momenta and the Hamiltonian. Then, if $F_{additional} \not\equiv 0$ is linear or quadratic in momenta integral for H_{geod}, it must be functionally independent of F_{geod}. Then, the system is resonant. Then, all geodesics are closed. The theory of metrics all whose geodesics are closed is well-developed, see [1]. It is known [10] that, on the two-sphere, geodesics of such metrics have no self-intersections. But if h is very big, certain geodesics of our metric must have self-intersections: indeed, for huge h, the geodesics of our metric are very close to the geodesic of the metric (1), and by direct analysis it is easy to check that certain geodesics of (1) do have self-intersections.

Thus, for very big h, the Hamiltonian H_{geod} admits no integral of degree one or two.

4.2 Why ds^2_{geod} is new

The examples coming from Goryachev-Chaplygin are the metrics

$$ds_2^2 = (h_1 - V_1)\left(d\theta^2 + \frac{d\phi^2}{4+\cot^2(\theta)}\right). \tag{8}$$

Their Hamiltonians are

$$H_2 := \frac{p_\theta^2 + \left(4+\cot^2(\theta)\right) p_\phi^2}{2(h_1 - V_1)},$$

and the integral for the system with the Hamiltonian H_2 is

$$\begin{aligned} F_2 :=& H_1 p_\phi - 2p_\phi^3 - V_1 p_\phi \\ &+ \left(V_1 p_\phi - \frac{A_1}{2} p_\theta \cos(\phi)\cos(\theta) + \frac{A_1}{2}\frac{\sin\phi\left(3\,(\cos\theta)^2 - 2\right)}{\sin\theta} p_\phi\right) H_2 \,. \end{aligned}$$

Our goal is to show that there is no diffeomorphism $D : S^2 \to S^2$ that takes the metric (7) to the metric (8) or a constant multiple thereof. Since the potential energy V_1 depends linearly on the parameter A_1, it is sufficient to show that there exists no diffeomorphism $D : S^2 \to S^2$ that takes the metric (7) to the metric (8).

First let us show that such diffeomorphism D must be given by the formulae $\theta_{new} = \theta_{new}(\theta)$, $\phi_{new} = \pm\phi + \phi_0$, where ϕ_0 is a constant. It is known, that the Goryachev-Chaplygin system is nonresonant on every energy level, see for example [16] where the rotation function is calculated explicitely. Then, the geodesic flow of (8) is nonresonant as well. Then, the diffeomorphism D must take the integral (6) to the integral (8) or a constant multiple thereof.

It is known that a Riemannian metric on a (oriented) surface defines a complex structure on it. That is, in a neighbourhood of every point we can find a coordinate $z = x + iy$ such that the metric looks like $\lambda(x, y)(dx^2 + dy^2)$. In such complex coordinates, an integral of degree three has the form

$$a(z)p^3 + b(z)p^2\bar{p} + \bar{b}(z)p\bar{p}^2 + \bar{a}(z)\bar{p}^3. \tag{9}$$

Here a, b are functions; "bar" means the complex conjugation, p is the complex momentum corresponding to z. The functions a, b are a-priory not assumed to be holomorphic. Note that this form is not restrictive: If (9) is a real-valued function, the coefficients at $\bar{p}^3$ and $p^2\bar{p}$ must be complex-conjugate to the coefficients at p^3 and $\bar{p}^2 p$, respectively.

We will use the following result of Kolokoltsov from [15]; in a weaker form, this result has been already known to Birkhoff [2].

Lemma: *Suppose (9) is an integral for the geodesic flow of g and that it is not the product of the Hamiltonian and an integral linear in velocities. Then,*

$$\frac{1}{a(z)}dz^3 \tag{10}$$

is a meromorphic $(3, 0)$*-form.*

Evidently, the form (10) has no zeros. It could have poles. In fact, because of Abel's Lemma, it always has poles on the sphere.

The metrics (7,8) and the integrals F_{geod}, F_2 are given explicitly. Direct calculation of the form (10) shows that the form (10) for the metric (7) and for the integral F_{geod} is very similar to the form (10) for the metric (8) and for the integral F_2: they have precisely two poles. The poles are located at $\theta = 0, \pi$. Then, the diffeomorpism D takes the points $\theta = 0, \pi$ to the points $\theta = 0, \pi$.

It is known that a holomorphic diffeomorphism of the Riemann sphere $\bar{\mathbb{C}}$ that preserves the points $z = 0$ and $z = \infty$ is a linear transformation $z \mapsto \alpha z$. A holomorphic diffeomorphism that interchange the points is a transformation $z \mapsto \frac{\alpha}{z}$. In both cases, the diffeomorphism D has the form $\theta_{new} = \theta_{new}(\theta)$, $\phi_{new} = \pm\phi + \phi_0$.

Then, the pull-back of the metric (8) has the form

$$(h_1 - A_1 \sin\theta_{new} \sin(\pm\phi + \phi_0)) \left(\left(\frac{\partial \theta_{new}}{\partial \theta}\right)^2 d\theta^2 + \frac{d\phi^2}{4 + \cot^2(\theta_{new})} \right).$$

If this metric coincides with the metric (7), the coefficients of $d\theta^2$ must coincide. Then,

$$\left(\frac{\partial \theta_{new}}{\partial \theta}\right)^2 (h_1 - A_1 \sin(\theta_{new}) \sin(\pm\phi + \phi_0)) = h - \frac{A \sin\theta \cos\phi}{\sqrt{\cos(\theta) + s}} - \frac{c}{\cos(\theta) + s},$$

which implies (for simplicity, we assume $\mathrm{sign}(A) = \mathrm{sign}(A_1)$, the case $\mathrm{sign}(A) = -\mathrm{sign}(A_1)$ can be triated similarly)

$$A_1 \sin(\theta_{new}) \left(\frac{\partial \theta_{new}}{\partial \theta}\right)^2 = \frac{A \sin\theta}{\sqrt{\cos(\theta) + s}}, \tag{11}$$

$$h_1 \left(\frac{\partial \theta_{new}}{\partial \theta}\right)^2 = \left(h - \frac{c}{\cos(\theta) + s}\right). \tag{12}$$

Similarly, comparing coefficients of $d\phi^2$, we obtain

$$\frac{A_1 \sin(\theta_{new})}{4 + \cot^2\theta_{new}} = \frac{A \sin(\theta)}{\sqrt{\cos(\theta) + s}}, \tag{13}$$

$$\frac{h_1}{4 + \cot^2\theta_{new}} = \frac{h - c/(\cos\theta + s)}{1/\sin^2\theta + G(\cos\theta)}. \tag{14}$$

Comparing Equations (11,13) and Equations (12,14), we obtain

$$\left(\frac{\partial \theta_{new}}{\partial \theta}\right)^2 = 4 + \cot^2\theta_{new} \quad \text{and} \quad \left(\frac{\partial \theta_{new}}{\partial \theta}\right)^2 = \frac{4 + \cot^2\theta_{new}}{1/\sin^2\theta + G(\cos\theta)}.$$

Thus, we obtain a contradiction, since clearly $1/\sin^2\theta + G(\cos\theta)$ is not constant. Finally, there exists no diffeomorphism D that takes the metric (7) to metric (8).

5 What to do next

We constructed a new natural integrable Hamiltonian system on the sphere. It suggests the following ways of further investigation:

1. Construction of bifurcation diagram and description of the topological structure of the system.

2. Finding action-angle variables and explicit formulae for orbits.

3. Quantization of the system.

4. Painlevé analysis.

We constructed this new system using methods based on an observation from [9]. We will describe the methods and how we applied them elsewhere. It seems that it is the only new integrable system with cubic integral on a closed manifold containing elementary functions only that can be found using these methods. But still one can construct many new local examples of metrics with integrable geodesic flows, which we will do in a joint paper with Tabov and Topalov.

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