METRIC CONNECTIONS IN PROJECTIVE DIFFERENTIAL GEOMETRY

MICHAEL EASTWOOD AND VLADIMIR MATVEEV

In memory of Thomas Branson

Abstract. We search for Riemannian metrics whose Levi-Civita connection belongs to a given projective class. Following Sinjukov and Mikeš, we show that such metrics correspond precisely to suitably positive solutions of a certain projectively invariant finite-type linear system of partial differential equations. Prolonging this system, we may reformulate these equations as defining covariant constant sections of a certain vector bundle with connection. This vector bundle and its connection are derived from the Cartan connection of the underlying projective structure.

1. Introduction

We shall always work on a smooth oriented manifold M of dimension n. Suppose that ∇ is a torsion-free connection on the tangent bundle of M. We may ask whether there is a Riemannian metric on M whose geodesics coincide with the geodesics of ∇ as unparameterised curves. We shall show that there is a linear system of partial differential equations that precisely controls this question.

To state our results, we shall need some terminology, notation, and preliminary observations. Two torsion-free connections ∇ and $\hat{\nabla}$ are said to be projectively equivalent if they have the same geodesics as unparameterised curves. A projective structure on M is a projective equivalence class of connections. In these terms, we are given a projective structure on M and we ask whether it may be represented by a metric connection. Questions such as this have been addressed by many authors. Starting with a metric connection, Sinjukov [9] considered the existence of other metrics with the same geodesics. He found a system of equations that controls this question and Mikeš [7] observed that essentially the same system pertains when starting with an arbitrary projective structure.

We shall use Penrose's abstract index notation [8] in which indices act as markers to specify the type of a tensor. Thus, ω_a denotes a 1-form whilst X^a denotes a vector field. Repeated indices denote the canonical pairing between vectors and co-vectors. Thus, we shall write $X^a\omega_a$ instead of $X \lrcorner\, \omega$. The tautological 1-form with values in the tangent bundle is denoted by the Kronecker delta $\delta_a{}^b$.

2000 *Mathematics Subject Classification.* Primary 53A20; Secondary 58J70.

Key words and phrases. Projective differential geometry, Metric connection, Tractor.

This work was undertaken during the 2006 Summer Program at the Institute for Mathematics and its Applications at the University of Minnesota. The authors would like to thank the IMA for hospitality during this time. The first author is supported by the Australian Research Council.

As is well-known [5], the geometric formulation of projective equivalence may be re-expressed as

$$\hat{\nabla}_a X^b = \nabla_a X^b + \Upsilon_a X^b + \delta_a{}^b \Upsilon_c X^c \tag{1.1}$$

for an arbitrary 1-form Υ_a. We shall also adopt the curvature conventions of [5]. In particular, it is convenient to write

$$(\nabla_b \nabla_a - \nabla_a \nabla_b) X^b = R_{ab} X^b,$$

where R_{ab} is the usual Ricci tensor, as

$$(\nabla_b \nabla_a - \nabla_a \nabla_b) X^b = (n-1) \mathrm{P}_{ab} X^b - \beta_{ab} X^b \quad \text{where } \beta_{ab} = \mathrm{P}_{ba} - \mathrm{P}_{ab}.$$

If a different connection is chosen in the projective class according to (1.1), then

$$\hat{\beta}_{ab} = \beta_{ab} + \nabla_a \Upsilon_b - \nabla_b \Upsilon_a.$$

Therefore, as a 2-form β_{ab} changes by an exact form. On the other hand, the Bianchi identity implies that β_{ab} is closed. Thus, there is a well-defined de Rham cohomology class $[\beta] \in H^2(M, \mathbb{R})$ associated to any projective structure.

Proposition 1.1. *The class $[\beta] \in H^2(M, \mathbb{R})$ is an obstruction to the existence of a metric connection in the given projective class.*

Proof. The Ricci tensor is symmetric for a metric connection. □

In searching for a metric connection in a given projective class, we may as well suppose that the obstruction $[\beta]$ vanishes. For the remainder of this article we suppose that this is the case and we shall consider only representative connections with symmetric Ricci tensor. In other words, all connections from now on enjoy

$$(\nabla_b \nabla_a - \nabla_a \nabla_b) X^b = (n-1) \mathrm{P}_{ab} X^b \quad \text{where } \mathrm{P}_{ab} = \mathrm{P}_{ba}. \tag{1.2}$$

A convenient alternative characterisation of such connections as follows.

Proposition 1.2. *A torsion-free affine connection has symmetric Ricci tensor if and only if it induces the flat connection on the bundle of n-forms.*

Proof. If $\epsilon^{pqr\cdots s}$ has n indices and is totally skew then

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \epsilon^{pqr\cdots s} = \kappa_{ab} \epsilon^{pqr\cdots s}$$

for some 2-form κ_{ab}. But, by the Bianchi symmetry,

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \epsilon^{abr\cdots s} = -2 R_{ab} \epsilon^{abr\cdots s},$$

which vanishes if and only if R_{ab} is symmetric. □

Having restricted our attention to affine connections that are flat on the bundle of n-forms, we may as well further restrict to connections ∇_a for which there is a volume form $\epsilon_{bc\cdots d}$ (unique up to scale) with $\nabla_a \epsilon_{bc\cdots d} = 0$. We shall refer to such connections as special. The freedom in special connections within a given projective class is given by (1.1) where $\Upsilon_a = \nabla_a f$ for an arbitrary smooth function f. Following [5], the full curvature of a special connection may be conveniently decomposed:–

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) X^c = W_{ab}{}^c{}_d X^d + \delta_a{}^c \mathrm{P}_{bd} X^d - \delta_b{}^c \mathrm{P}_{ad} X^d, \tag{1.3}$$

where $W_{ab}{}^c{}_d$ is totally trace-free and P_{ab} is symmetric. The tensor $W_{ab}{}^c{}_d$ is known as the Weyl curvature and is projectively invariant.

2. A linear system of equations

In this section we present, as Proposition 2.1, an alternative characterisation of the Levi-Civita connection. The advantage of this characterisation is that it leads, almost immediately, to a system of linear equations that controls the metric connections within a given projective class. The precise results are Theorems 2.2 and 2.3.

Proposition 2.1. *Suppose g^{ab} is a metric on M with volume form $\epsilon_{bc\cdots d}$. Then a torsion-free connection ∇_a is the metric connection for g^{ab} if and only if*

- $\nabla_a g^{bc} = \delta_a{}^b \mu^c + \delta_a{}^c \mu^b$ *for some vector field μ^a*
- $\nabla_a \epsilon_{bc\cdots d} = 0.$

Proof. Write D_a for the metric connection of g^{ab}. Then

$$\nabla_a \omega_b = D_a \omega_b - \Gamma_{ab}{}^c \omega_c \tag{2.1}$$

for some tensor $\Gamma_{ab}{}^c = \Gamma_{ba}{}^c$. We compute

$$\epsilon^{bc\cdots d} \nabla_a \epsilon_{bc\cdots d} = -n \epsilon^{bc\cdots d} \Gamma_{ab}{}^e \epsilon_{ec\cdots d} = -n!\, \Gamma_{ab}{}^b$$

and so $\Gamma_{ab}{}^b = 0$. Similarly,

$$\nabla_a g^{bc} = \Gamma_{ad}{}^b g^{dc} + \Gamma_{ad}{}^c g^{bd}$$

and so

$$\Gamma_{ad}{}^b g^{dc} + \Gamma_{ad}{}^c g^{bd} = \delta_a{}^b \mu^c + \delta_a{}^c \mu^b. \tag{2.2}$$

Let g_{ab} denote the inverse of g^{ab} and contract (2.2) with g_{bc} to conclude that

$$2\Gamma_{ab}{}^b = 2\mu_a \quad \text{where } \mu_a = g_{ab}\mu^b$$

and hence that $\mu^a = 0$. If we let $\Gamma_{abc} = \Gamma_{ab}{}^d g_{cd}$, then (2.2) now reads

$$\Gamma_{acb} + \Gamma_{abc} = 0.$$

Together with $\Gamma_{abc} = \Gamma_{bac}$, this implies that $\Gamma_{abc} = 0$. From (2.1) we see that $\nabla_a = D_a$, which is what we wanted to show. □

Theorem 2.2. *Suppose ∇_a is a special torsion-free connection and there is a metric tensor σ^{ab} such that*

$$\nabla_a \sigma^{bc} = \delta_a{}^b \mu^c + \delta_a{}^c \mu^b \quad \textit{for some vector field } \mu^a. \tag{2.3}$$

Then ∇_a is projectively equivalent to a metric connection.

Proof. Consider the projectively equivalent connection

$$\hat{\nabla}_a X^b = \nabla_a X^b + \Upsilon_a X^b + \delta_a{}^b \Upsilon_c X^c \quad \text{where } \Upsilon_a = \nabla_a f$$

for some function f. If we let $\hat{\sigma}^{ab} \equiv e^{-2f}\sigma^{ab}$, then

$$\begin{aligned} \hat{\nabla}_a \hat{\sigma}^{bc} &= e^{-2f}\left(-2\Upsilon_a \sigma^{bc} + \nabla_a \sigma^{bc} + 2\Upsilon_a \sigma^{bc} + \delta_a{}^b \Upsilon_d \sigma^{dc} + \delta_a{}^c \Upsilon_d \sigma^{bd}\right) \\ &= e^{-2f}\left(\delta_a{}^b \mu^c + \delta_a{}^c \mu^b + \delta_a{}^b \Upsilon_d \sigma^{dc} + \delta_a{}^c \Upsilon_d \sigma^{bd}\right) \end{aligned}$$

and so

$$\hat\nabla_a\hat\sigma^{bc}=\delta_a{}^b\hat\mu^c+\delta_a{}^c\hat\mu^b \quad \text{where } \hat\mu^a=e^{-2f}\left(\mu^a+\Upsilon_b\sigma^{ab}\right). \tag{2.4}$$

Similarly, if we choose a volume form $\epsilon_{bc\cdots d}$ killed by ∇_a and let $\hat\epsilon_{bc\cdots d}\equiv e^{(n+1)f}\epsilon_{bc\cdots d}$, then

$$\hat\nabla_a\hat\epsilon_{bc\cdots d}=e^{(n+1)f}\left(\nabla_a\epsilon_{bc\cdots d}+\Upsilon_{[a}\epsilon_{bc\cdots d]}\right)=e^{(n+1)f}\nabla_a\epsilon_{bc\cdots d}=0. \tag{2.5}$$

Define

$$\det(\sigma)\equiv\epsilon_{a\cdots b}\epsilon_{c\cdots d}\sigma^{ac}\cdots\sigma^{bd}$$

and compute

$$\begin{aligned}\widehat{\det}(\hat\sigma) &= \hat\epsilon_{a\cdots b}\hat\epsilon_{c\cdots d}\hat\sigma^{ac}\cdots\hat\sigma^{bd}\\ &= e^{2(n+1)f}e^{-2nf}\epsilon_{a\cdots b}\epsilon_{c\cdots d}\sigma^{ac}\cdots\sigma^{bd} \;=\; e^{2f}\det(\sigma).\end{aligned}$$

Therefore, if we take

$$f=-\tfrac{1}{2}\log\det(\sigma),$$

then we have arranged that $\widehat{\det}(\hat\sigma)=1$. This is precisely the condition that $\hat\epsilon_{bc\cdots d}$ be the volume form for the metric $\hat\sigma^{ab}$. With (2.4) and (2.5) we are now in a position to use Proposition 2.1 to conclude that $\hat\nabla_a$ is the metric connection for $\hat\sigma^{ab}$. We have shown that our original connection ∇_a is projectively equivalent to the Levi-Civita connection for the metric $g^{ab}\equiv\det(\sigma)\,\sigma^{ab}$. □

Evidently, the equations (2.3) precisely control the metric connections within a given special projective class. Precisely, if g_{ab} is a Riemannian metric with associated Levi-Civita connection ∇_a, then

$$\hat\nabla_a\hat g^{bc}=\delta_a{}^b\hat\mu^c+\delta_a{}^c\hat\mu^b,$$

where $\hat\nabla_a$ is projectively equivalent to ∇_a according to (1.1) with $\Upsilon_a=\nabla_a f$ and where $\hat g^{bc}=e^{-2f}g^{bc}$. In other words, we have shown (cf. [7, 9]):–

Theorem 2.3. *There is a one-to-one correspondence between solutions of* (2.3) *for positive definite σ^{bc} and metric connections that are projectively equivalent to ∇_a.*

3. Prolongation

Let us consider the system of equations (2.3) in more detail. It is a linear system for any symmetric contravariant 2-tensor σ^{bc}. Specifically, we may write (2.3) as

$$\text{the trace-free part of } (\nabla_a\sigma^{bc})=0 \tag{3.1}$$

or, more explicitly, as

$$\nabla_a\sigma^{bc}-\tfrac{1}{n+1}\delta_a{}^b\nabla_d\sigma^{cd}-\tfrac{1}{n+1}\delta_a{}^c\nabla_d\sigma^{bd}=0.$$

According to [2], this equation is of finite-type and may be prolonged to a closed system as follows. According to (1.2) and (1.3) we have

$$(\nabla_a\nabla_b-\nabla_b\nabla_a)\sigma^{bc}=W_{ab}{}^c{}_d\sigma^{bd}+\delta_a{}^c\mathrm{P}_{bd}\sigma^{bd}-n\mathrm{P}_{ad}\sigma^{cd}.$$

On the other hand, from (2.3) we have

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\sigma^{bc} = (n+1)\nabla_a\mu^c - \nabla_b(\delta_a{}^b\mu^c + \delta_a{}^c\mu^b) = n\nabla_a\mu^c - \delta_a{}^c\nabla_b\mu^b.$$

We conclude that

$$n\nabla_a\mu^c = \delta_a{}^c\left(\nabla_b\mu^b + \mathrm{P}_{bd}\sigma^{bd}\right) - n\mathrm{P}_{ad}\sigma^{cd} + W_{ab}{}^c{}_d\sigma^{bd}$$

or, equivalently, that

$$\nabla_a\mu^c = \delta_a{}^c\rho - \mathrm{P}_{ad}\sigma^{cd} + \tfrac{1}{n}W_{ab}{}^c{}_d\sigma^{bd}, \tag{3.2}$$

for some function ρ. To complete the prolongation, we use (1.2) to write

$$(\nabla_c\nabla_a - \nabla_a\nabla_c)\mu^c = (n-1)\mathrm{P}_{ac}\mu^c$$

whereas from (3.2) we also have

$$(\nabla_c\nabla_a - \nabla_a\nabla_c)\mu^c = \nabla_c\left(\delta_a{}^c\rho - \mathrm{P}_{ad}\sigma^{cd} + \tfrac{1}{n}W_{ab}{}^c{}_d\sigma^{bd}\right) - \nabla_a\left(n\rho - \mathrm{P}_{cd}\sigma^{cd}\right).$$

Therefore,

$$(n-1)\mathrm{P}_{ac}\mu^c = \nabla_c\left(-\mathrm{P}_{ad}\sigma^{cd} + \tfrac{1}{n}W_{ab}{}^c{}_d\sigma^{bd}\right) - (n-1)\nabla_a\rho + \nabla_a(\mathrm{P}_{cd}\sigma^{cd}). \tag{3.3}$$

The terms involving Weyl curvature

$$\nabla_c(W_{ab}{}^c{}_d\sigma^{bd}) = (\nabla_c W_{ab}{}^c{}_d)\sigma^{bd} + W_{ab}{}^c{}_d\nabla_c\sigma^{bd}$$

may be dealt with by (2.3) and a Bianchi identity

$$\nabla_c W_{ab}{}^c{}_d = (n-2)(\nabla_a\mathrm{P}_{bd} - \nabla_b\mathrm{P}_{ad}).$$

We see that

$$\nabla_c(W_{ab}{}^c{}_d\sigma^{bd}) = (n-2)(\nabla_a\mathrm{P}_{bd} - \nabla_b\mathrm{P}_{ad})\sigma^{bd}$$

and (3.3) becomes

$$(n-1)\mathrm{P}_{ac}\mu^c = \tfrac{n-2}{n}(\nabla_a\mathrm{P}_{bd} - \nabla_b\mathrm{P}_{ad})\sigma^{bd} - \nabla_c(\mathrm{P}_{ad}\sigma^{cd}) - (n-1)\nabla_a\rho + \nabla_a(\mathrm{P}_{cd}\sigma^{cd})$$

or, equivalently,

$$\mathrm{P}_{ac}\mu^c = \tfrac{2}{n}(\nabla_a\mathrm{P}_{bd} - \nabla_b\mathrm{P}_{ad})\sigma^{bd} - \tfrac{1}{n-1}\mathrm{P}_{ad}\nabla_c\sigma^{cd} - \nabla_a\rho + \tfrac{1}{n-1}\mathrm{P}_{cd}\nabla_a\sigma^{cd}. \tag{3.4}$$

Again, we substitute from (2.3) to rewrite

$$\mathrm{P}_{cd}\nabla_a\sigma^{cd} - \mathrm{P}_{ad}\nabla_c\sigma^{cd} = \mathrm{P}_{cd}(\delta_a{}^c\mu^d + \delta_a{}^d\mu^c) - (n+1)\mathrm{P}_{ad}\mu^d = -(n-1)\mathrm{P}_{ad}\mu^d$$

and (3.4) becomes

$$\mathrm{P}_{ac}\mu^c = \tfrac{2}{n}(\nabla_a\mathrm{P}_{bd} - \nabla_b\mathrm{P}_{ad})\sigma^{bd} - \mathrm{P}_{ad}\mu^d - \nabla_a\rho,$$

which we may rearrange as

$$\nabla_a\rho = -2\mathrm{P}_{ab}\mu^b + \tfrac{2}{n}(\nabla_a\mathrm{P}_{bd} - \nabla_b\mathrm{P}_{ad})\sigma^{bd}.$$

Together with (2.3) and (3.2), we have a closed system, essentially as in [7, 9]:–

$$\begin{aligned} \nabla_a\sigma^{bc} &= \delta_a{}^b\mu^c + \delta_a{}^c\mu^b \\ \nabla_a\mu^b &= \delta_a{}^b\rho - \mathrm{P}_{ac}\sigma^{bc} + \tfrac{1}{n}W_{ac}{}^b{}_d\sigma^{cd} \\ \nabla_a\rho &= -2\mathrm{P}_{ab}\mu^b + \tfrac{4}{n}Y_{abc}\sigma^{bc} \end{aligned} \tag{3.5}$$

where $Y_{abc} = \frac{1}{2}(\nabla_a \mathrm{P}_{bc} - \nabla_b \mathrm{P}_{ac})$, the Cotton-York tensor. The three tensors σ^{bc}, μ^b, and ρ may be regarded together as a section of the vector bundle

$$\mathcal{T} = \bigodot^2 TM \oplus TM \oplus \mathbb{R}$$

where $\bigodot$ denotes symmetric tensor product and $\mathbb{R}$ denotes the trivial bundle. We have proved:–

Theorem 3.1. *If we endow $\mathcal{T}$ with the connection*

$$\begin{pmatrix} \sigma^{bc} \\ \mu^b \\ \rho \end{pmatrix} \longmapsto \begin{pmatrix} \nabla_a \sigma^{bc} - \delta_a{}^b \mu^c - \delta_a{}^c \mu^b \\ \nabla_a \mu^b - \delta_a{}^b \rho + \mathrm{P}_{ac}\sigma^{bc} - \frac{1}{n} W_{ac}{}^b{}_d \sigma^{cd} \\ \nabla_a \rho + 2\mathrm{P}_{ab}\mu^b - \frac{4}{n} Y_{abc}\sigma^{bc} \end{pmatrix}. \tag{3.6}$$

then there is a one-to-one correspondence between covariant constant sections of $\mathcal{T}$ and solutions σ^{bc} of (2.3).

4. Projective invariance

The equation (2.3) is projectively invariant in the following sense. Following [5], let $\mathcal{E}^{(ab)}(w)$ denote the bundle of symmetric contravariant 2-tensors of projective weight w. Thus, in the presence of a volume form $\epsilon_{bc\cdots d}$, a section $\sigma^{ab} \in \Gamma(M, \mathcal{E}^{(ab)}(w))$ is an ordinary symmetric contravariant 2-tensor but if we change volume form

$$\epsilon_{bc\cdots d} \mapsto \hat{\epsilon}_{bc\cdots d} = e^{(n+1)f} \epsilon_{bc\cdots d} \quad \text{for any smooth function } f,$$

then we are obliged to rescale σ^{ab} according to $\hat{\sigma}^{ab} = e^{wf}\sigma^{ab}$. Equivalently, we are saying that $\mathcal{E}^{(ab)}(w) = \bigodot^2 TM \otimes (\Lambda^n)^{-w/(n+1)}$, where Λ^n is the line-bundle of n-forms on M. The projectively weighted irreducible tensor bundles are fundamental objects on a manifold with projective structure.

Proposition 4.1. *The differential operator*

$$\mathcal{E}^{(ab)}(-2) \longrightarrow \text{the trace-free part of } \mathcal{E}_a{}^{(bc)}(-2) \tag{4.1}$$

defined by (3.1) *is projectively invariant.*

Proof. This is already implicit in the proof of Theorem 2.2. Explicitly, however, we just compute from (1.1):–

$$\hat{\nabla}_a \hat{\sigma}^{bc} = \nabla_a \hat{\sigma}^{bc} + 2\Upsilon_a \hat{\sigma}^{bc} + \delta_a{}^b \Upsilon_d \hat{\sigma}^{dc} + \delta_a{}^c \Upsilon_d \hat{\sigma}^{bd},$$

where $\Upsilon_a = \nabla_a f$ whilst

$$\nabla_a \hat{\sigma}^{bc} = \nabla_a (e^{-2f}\sigma^{bc}) = e^{-2f}\left(\nabla_a \sigma^{bc} - 2\Upsilon_a \sigma^{bc}\right) = \widehat{\nabla_a \sigma^{bc}} - 2\Upsilon_a \hat{\sigma}^{bc}.$$

It follows that

$$\hat{\nabla}_a \hat{\sigma}^{bc} = \widehat{\nabla_a \sigma^{bc}} + \text{trace terms},$$

which is what we wanted to show. □

In hindsight, it is not too difficult to believe that (3.1) should control the metric connections within a given projective class. There are very few projectively invariant operators. In fact, there are precisely two finite-type first order invariant linear operators on symmetric 2-tensors. One of them is (4.1) and the other is

$$\mathcal{E}_{(ab)}(4) \to \mathcal{E}_{(abc)}(4) \quad \text{given by} \quad \sigma_{ab} \mapsto \nabla_{(a}\sigma_{bc)}. \tag{4.2}$$

In two dimensions, (4.2) and (4.1) coincide. In higher dimensions, however, being in the kernel of (4.2) for positive definite σ_{ab} corresponds to having a metric g_{ab} and a totally trace-free tensor Γ_{abc} with

$$\Gamma_{abc} = \Gamma_{bac} \quad \text{and} \quad \Gamma_{abc} + \Gamma_{bca} + \Gamma_{cab} = 0$$

such that the connection

$$\omega_b \longmapsto D_a\omega_b - \Gamma_{ab}{}^c\omega_c$$

belongs to the projective class of ∇_a, where D_a is the Levi-Civita connection of g_{ab}. The available tensors Γ_{abc} for a given metric have dimension $n(n+2)(n-2)/3$.

5. Relationship to the Cartan connection

On a manifold with projective structure, it is shown in [5] how to associate vector bundles with connection to any irreducible representation of $\mathrm{SL}(n+1,\mathbb{R})$. These are the tractor bundles following their construction by Thomas [10]. Equivalently, they are induced by the Cartan connection [4] of the projective structure. The relevant tractor bundle in our case is induced by $\bigodot^2\mathbb{R}^{n+1}$ where $\mathbb{R}^{n+1}$ is the defining representation of $\mathrm{SL}(n+1,\mathbb{R})$. It has a composition series

$$\mathcal{E}^{(BC)} = \mathcal{E}^{(bc)}(-2) + \mathcal{E}^{b}(-2) + \mathcal{E}(-2)$$

and in the presence of a connection is simply the direct sum of these bundles. Under projective change of connection according to (1.1), however, we decree that

$$\widehat{\begin{pmatrix} \sigma^{bc} \\ \mu^b \\ \rho \end{pmatrix}} = \begin{pmatrix} \sigma^{bc} \\ \mu^b + \Upsilon_c\sigma^{bc} \\ \rho + 2\Upsilon_b\mu^b + \Upsilon_b\Upsilon_c\sigma^{bc} \end{pmatrix}. \tag{5.1}$$

Following [5], the tractor connection on $\mathcal{E}^{(AB)}$ is given by

$$\nabla_a \begin{pmatrix} \sigma^{bc} \\ \mu^b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a\sigma^{bc} - \delta_a{}^b\mu^c - \delta_a{}^c\mu^b \\ \nabla_a\mu^b - \delta_a{}^b\rho + \mathrm{P}_{ac}\sigma^{bc} \\ \nabla_a\rho + 2\mathrm{P}_{ab}\mu^b \end{pmatrix}.$$

Therefore, we have proved:–

Theorem 5.1. *The solutions of* (2.3) *are in one-to-one correspondence with solutions of the following system:–*

$$\nabla_a \begin{pmatrix} \sigma^{bc} \\ \mu^b \\ \rho \end{pmatrix} - \frac{1}{n}\begin{pmatrix} 0 \\ W_{ac}{}^b{}_d\sigma^{cd} \\ 4Y_{abc}\sigma^{bc} \end{pmatrix} = 0. \tag{5.2}$$

Corollary 5.2. *There is a one-to-one correspondence between solutions of* (5.2) *for positive definite* σ^{bc} *and metric connections that are projectively equivalent to* ∇_a.

Notice that the extra terms in (5.2) are projectively invariant as they should be. Specifically, it is observed in [5] that

$$\hat{Y}_{abc} = Y_{abc} + \tfrac{1}{2} W_{ab}{}^{d}{}_{c} \Upsilon_d$$

and so

$$4\hat{Y}_{abc}\sigma^{bc} = 4Y_{abc}\sigma^{bc} + 2\Upsilon_b W_{ac}{}^{b}{}_{d}\sigma^{cd}$$

in accordance with (5.1).

It is clear from Theorem 3.1 that, generically, (2.3) has no solutions. Indeed, this is one reason why the prolonged from is so helpful. More generally, we should compute the curvature of the connection (3.6) and the form (5.2) is useful for this task. A model computation along these lines is given in [5]. In our case, the tractor curvature is given by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\left(\begin{array}{c}\sigma^{cd}\\ \mu^c\\ \rho\end{array}\right) = \left(\begin{array}{c}W_{ab}{}^{c}{}_{e}\sigma^{de} + W_{ab}{}^{d}{}_{e}\sigma^{ce}\\ W_{ab}{}^{c}{}_{d}\mu^d + 2Y_{abd}\sigma^{cd}\\ 4Y_{abc}\mu^c\end{array}\right)$$

and we obtain:–

Proposition 5.3. *The curvature of the connection* (3.6) *is given by*

$$\left(\begin{array}{c}\sigma^{cd}\\ \mu^c\\ \rho\end{array}\right) \mapsto \left(\begin{array}{c}W_{ab}{}^{c}{}_{e}\sigma^{de} + W_{ab}{}^{d}{}_{e}\sigma^{ce}\\ W_{ab}{}^{c}{}_{d}\mu^d + 2Y_{abd}\sigma^{cd}\\ 4Y_{abc}\mu^c\end{array}\right) + \frac{1}{n}\left(\begin{array}{c}\delta_a{}^{c}U_b{}^{d} + \delta_a{}^{d}U_b{}^{c} - \delta_b{}^{c}U_a{}^{d} - \delta_b{}^{d}U_a{}^{c}\\ *\\ *\end{array}\right),$$

where $U_b{}^d = W_{be}{}^{d}{}_{f}\sigma^{ef}$ *and* $*$ *denotes expressions that we shall not need.*

Corollary 5.4. *The curvature of the connection* (3.6) *vanishes if and only if the projective structure is flat.*

Proof. Let us suppose that $n \geq 3$. The uppermost entry of the curvature is given by

$$\sigma^{cd} \longmapsto \text{the trace-free part of } (W_{ab}{}^{c}{}_{e}\sigma^{de} + W_{ab}{}^{d}{}_{e}\sigma^{ce})$$

and it is a matter of elementary representation theory to show that if this expression is zero for a fixed $W_{ab}{}^{c}{}_{d}$ and for all σ^{cd}, then $W_{ab}{}^{c}{}_{d} = 0$. Specifically, the symmetries of $W_{ab}{}^{c}{}_{d}$, namely

$$W_{ab}{}^{c}{}_{d} + W_{ba}{}^{c}{}_{d} = 0 \qquad W_{ab}{}^{c}{}_{d} + W_{bd}{}^{c}{}_{a} + W_{da}{}^{c}{}_{b} = 0 \qquad W_{ab}{}^{a}{}_{d} = 0, \tag{5.3}$$

constitute an irreducible representation of $\mathrm{SL}(n,\mathbb{R})$. Hence, the submodule

$$\left\{W_{ab}{}^{c}{}_{d} \text{ s.t. the trace-free part of } (W_{ab}{}^{c}{}_{e}\sigma^{de} + W_{ab}{}^{d}{}_{e}\sigma^{ce}) = 0,\ \forall\, \sigma^{cd}\right\}$$

must be zero since it is not the whole space. We have shown that if the curvature of the connection (3.6) vanishes, then $W_{ab}{}^{c}{}_{d} = 0$. For $n \geq 3$ this is exactly the condition that the projective structure be flat. For $n = 2$, the Weyl curvature $W_{ab}{}^{c}{}_{d}$ vanishes automatically since the symmetries (5.3) are too severe a constraint. Instead, a similar calculation shows that $Y_{abc} = 0$ and this is the condition that the projective structure be flat. □

Following Mikeš [7], the dimension of the space of solutions of (2.3) is called the degree of mobility of the projective structure. Theorem 3.1 implies that the degree of mobility is bounded by $(n+1)(n+2)/2$ and Corollary 5.4 implies that this bound is achieved only for the flat projective structure. Of course, the flat projective structure may as well be represented by the flat connection $\nabla_a = \partial/\partial x^a$ on $\mathbb{R}^n$, which is the Levi-Civita connection for the standard Euclidean metric. In this case, we may use (3.5) find the general solution of (2.3):–

$$\sigma^{ab} = s^{ab} + x^a m^b + x^b m^a + x^a x^b r. \tag{5.4}$$

This form is positive definite near the origin if and only if s^{ab} is positive definite. We conclude that the general projectively flat metric near the origin in $\mathbb{R}^n$ is

$$g^{ab} = \det(\sigma)\,\sigma^{ab},$$

where σ^{ab} is as in (5.4) for some positive definite quadratic form s^{ab}. In fact, these metrics are constant curvature. Rather than prove this by calculation, there is an alternative as follows. As already observed, the Weyl curvature $W_{ab}{}^c{}_d$ corresponds to an irreducible representation of $\mathrm{SL}(n,\mathbb{R})$ characterised by (5.3). In the presence of a metric g_{ab}, however, we should decompose $W_{ab}{}^c{}_d$ further under $\mathrm{SO}(n)$.

Proposition 5.5. *In the presence of a metric g_{ab}*

$$W_{ab}{}^c{}_d = C_{ab}{}^c{}_d + \frac{1}{(n-1)(n-2)}\left(\delta_a{}^c\Phi_{bd} - \delta_b{}^c\Phi_{ad}\right) + \frac{1}{n-2}\left(\Phi_a{}^c g_{bd} - \Phi_b{}^c g_{ad}\right) \tag{5.5}$$

where $C_{ab}{}^c{}_d$ is the Weyl part of the Riemann curvature tensor and Φ_{ab} is the trace-free part of the Ricci tensor.

Proof. According to (1.3),

$$R_{ab}{}^c{}_d = W_{ab}{}^c{}_d + \delta_a{}^c\mathrm{P}_{bd} - \delta_b{}^c\mathrm{P}_{ad} \tag{5.6}$$

but the Riemann curvature decomposes according to

$$R_{abcd} = C_{abcd} + g_{ac}Q_{bd} - g_{bc}Q_{ad} + Q_{ac}g_{bd} - Q_{bc}g_{ad}, \tag{5.7}$$

where Q_{ab} is the Schouten tensor

$$Q_{ab} = \frac{1}{n-2}\Phi_{ab} + \frac{1}{2n(n-1)}Rg_{ab}.$$

Comparing (5.6) and (5.7) leads, after a short computation, to (5.5). □

Corollary 5.6. *A projectively flat metric is constant curvature.*

Proof. If $n \geq 3$ and the projective Weyl tensor vanishes then the only remaining part of the Riemann curvature tensor is the scalar curvature. As usual, a separate proof based on Y_{abc} is needed for the case $n = 2$. □

This corollary is usually stated as follows. If a local diffeomorphism between two Riemannian manifolds preserves geodesics and one of them is constant curvature, then so is the other. This is a classical result due to Beltrami [1].

6. Concluding remarks

Results such as Theorem 2.3 and Theorem 5.1 are quite common in projective, conformal, and other parabolic geometries. It is shown in [5], for example, that the Killing equation in Riemannian geometry is projectively invariant and its solutions are in one-to-one correspondence with covariant constant sections of the tractor bundle $\mathcal{E}_{[AB]}$ equipped with a connection that is derived from (but not quite equal to) the tractor connection. The situation is completely parallel for conformal Killing vectors in conformal geometry and, more generally, for the infinitesimal automorphisms of parabolic geometries [3]. It is well-known that having an Einstein metric in a given conformal class is equivalent to having a suitably positive covariant constant section of the standard tractor bundle $\mathcal{E}^A$ equipped with its usual tractor connection. Gover and Nurowski [6] use this observation systematically to find obstructions to the existence of an Einstein metric within a given conformal class. We anticipate a similar use for Theorem 5.1 in establishing obstructions to the existence of a metric connection within a given projective class.

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Department of Mathematics, University of Adelaide, SA 5005, Australia
E-mail address: meastwoo@member.ams.org

Mathematisches Institut, Fakultät für Mathematik und Informatik, Friedrich-Schiller-Universität Jena, 07737 Jena, Germany
E-mail address: matveev@minet.uni-jena.de