Complete Einstein metrics are geodesically rigid

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Abstract

We prove that every complete Einstein (Riemannian or pseudo-Riemannian) metric g of nonconstant curvature is geodesically rigid: if any other complete metric $\bar{g}$ has the same (unparametrized) geodesics with g, then the Levi-Civita connections of g and $\bar{g}$ coincide.

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1 Introduction

1.1 Definitions and results

Let (M^n, g) be a connected Riemannian (= g is positive definite) or pseudo-Riemannian manifold of dimension $n \geq 3$. We say that a metric $\bar{g}$ on M^n is *geodesically equivalent* to g, if every geodesic of g is a (reparametrized) geodesic of $\bar{g}$. We say that they are *affine equivalent*, if their Levi-Civita connections coincide. We say that g is *Einstein*, if $R_{ij} = \frac{R}{n} \cdot g_{ij}$, where R_{ij} is the Ricci tensor of the metric g, and $R := R_{ij}g^{ij}$ is the scalar curvature. Our main result is

Theorem 1. *Let g and $\bar{g}$ be complete geodesically equivalent metrics on a connected manifold M^n, $n \geq 3$. If g is Einstein, then at least one of the following possibilities holds:*

- *they are affine equivalent, or*
- *for certain constants c, $\bar{c} \in \mathbb{R} \setminus \{0\}$ the metrics $c \cdot g$ and $\bar{c} \cdot \bar{g}$ are Riemannian metrics of curvature 1 (and, in particular, the manifolds $(M^n, c \cdot g)$ and $(M^n, \bar{c} \cdot \bar{g})$ are finite quotients of the standard sphere with the standard metric).*

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For dimension ≥ 5, the assumption that the metrics are complete is important: if one of them is not complete, one can construct counterexamples (essentially due to [16, 42]). For dimensions 3 and 4, (a natural modification of) Theorem 1 is true also locally:

Theorem 2. *Let g and $\bar{g}$ be geodesically equivalent metrics on a connected 3- or 4-dimensional manifold M. If g is Einstein, then at least one of the following possibilities holds:*

- *the metrics are affine equivalent, or*
- *the metrics g and $\bar{g}$ have constant curvature.*

Remark 1. In dimensions 3 and 4, Einstein metrics admitting nontrivial affine equivalent one are completely understood [47, 48]

Theorem 2 was announced in [25, 44], with the extended sketch of the proof. The proof from [25, 44] is very complicated: they prolonged (= covariantly differentiated) the basic equations (8) 6 times, and used the condition that the metric is Einstein at every stage of the prolongation. A partial case of Theorem 2 is also proved in [20].

Our proof of Theorem 2 is a relatively easy Linear Algebra (inspired by [9, 14]) combined with a certain statement which is a relatively easy generalization of a certain result of Levi-Civita.

Remark 2. Theorem 1 is also true in dimension 2 provided the scalar curvature of g is constant. Without this additional assumption Theorem 1 is evidently wrong, since every 2-dimensional metric satisfies $R_{ij} = \frac{R}{2} \cdot g_{ij}$.

1.2 History and motivation

The first examples of geodesically equivalent metrics are due to Lagrange [24]. He observed that the radial projection $f(x, y, z) = \left(-\frac{x}{z}, -\frac{y}{z}, -1\right)$ takes geodesics of the half-sphere $S^2 := \{(x, y, z) \in \mathbb{R}^3 : \ \ x^2 + y^2 + z^2 = 1, \ z < 0\}$ to the geodesics of the plane $E^2 := \{(x, y, z) \in \mathbb{R}^3 : \ \ z = -1\}$, see the left-hand side of Figure 1, since the geodesics of both metrics are intersection of the 2-plane containing the point $(0, 0, 0)$ with the surface. Later, Beltrami [5] generalized the example for the metrics of constant negative curvature, and for the pseudo-Riemannian metrics of constant curvature. In the example of Lagrange, he replaced the half sphere by the half of one of the hyperboloids $H^2_{\pm} := \{(x, y, z) \in \mathbb{R}^3 : \ \ x^2 + y^2 - z^2 = \pm 1\}$, with the restriction of the Lorentz metrics $dx^2 + dy^2 - dz^2$ to it. Then, the geodesics of the metric are also intersections of the 2-planes containing the point $(0, 0, 0)$ with the surface,

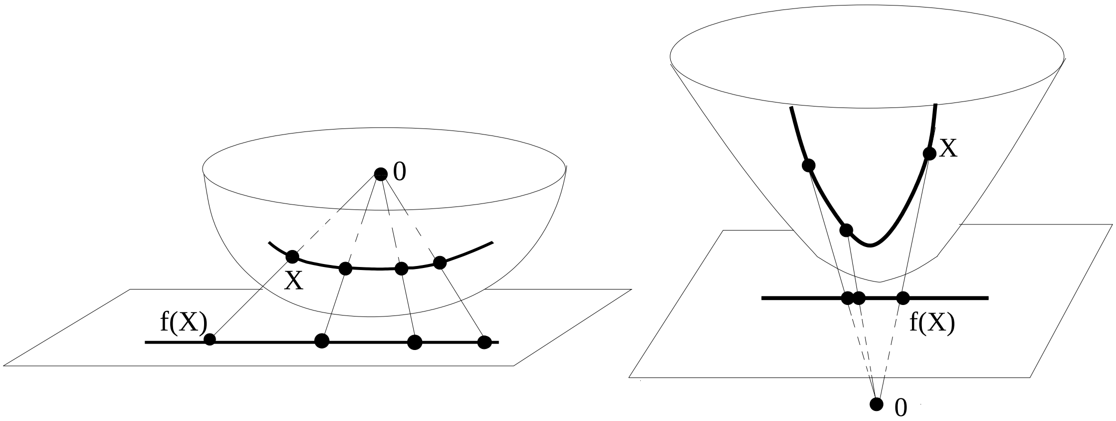

Figure 1: Surfaces of constant curvature are (locally) geodesically equivalent

and, therefore, the stereographic projection sends it to the straight lines of the appropriate plane, see the right-hand side of Figure 1 with the (half of the) hyperboloid H^2_-.

Though the examples of the Lagrange and Beltrami are two-dimensional, one can easily generalize them for every dimension.

One of the possibilities in Theorem 1 is geodesically equivalent metrics of constant positive Riemannian curvature on closed manifold. Examples of such metrics are also due to Beltrami [4], we describe their natural multi-dimensional generalization. Consider the sphere

$$S^n \stackrel{\text{def}}{=} \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} : \; x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1\}$$

with the metric g which is the restriction of the Euclidean metric to the sphere. Next, consider the mapping $a : S^n \to S^n$ given by $a : v \mapsto \frac{A(v)}{\|A(v)\|}$, where A is an arbitrary non-degenerate linear transformation of $\mathbb{R}^{n+1}$.

The mapping is clearly a diffeomorphism taking geodesics to geodesics. Indeed, the geodesics of g are great circles (the intersections of 2-planes that go through the origin with the sphere). Since A is linear, it takes planes to planes. Since the normalization $w \mapsto \frac{w}{\|w\|}$ takes punctured planes to their intersections with the sphere, the mapping a takes great circles to great circles. Thus, the pullback a^*g is geodesically equivalent to g. Evidently, if A is not proportional to an orthogonal transformation, a^*g is not affine equivalent to g.

The success of general relativity suggested (see for example the popular paper [57]) to look for geodesically equivalent Einstein metrics. In particular, the classical textbook [15] has a chapter on geodesic equivalence. In our paper, in the proof of Corollary 1, we will use the following classical result of Weyl [56]: he proved that two conformally and geodesically equivalent metric are proportional with a constant coefficient of the proportionality.

Later, geodesic equivalence of Einstein metrics was studied by many geometers and physicists (a simple search in mathscinet gives about 50 papers and few books). In particular, Petrov [46] proved that 4-dimensional Ricci-flat metrics of Lorenz signature can not be geodesically equivalent, unless they are affine equivalent. It is one of the results he obtained the Lenin prize (the most important scientific award of the Soviet Union) in 1972 for. He also explicitly asked [47, Problem 5 on page 355] whether the result remains true in other dimensions.

As we will prove in Lemma 3, the assumption that the second metric is Einstein is not important, since it is automatically fulfilled. By Theorem 2, the result of Petrov remains true for 4-dimensional metrics of other signatures. As we already mentioned in Section 1.1, the counterexamples independently constructed by Mikes [42] and Formella [16] show, that the result of Petrov fails in higher dimensions (so one indeed needs certain additional assumptions, for example the assumption that the metrics are complete as in Theorem 1, which is a standard assumption in problems motivated by physics.)

Recent references include Barnes [3], Hall and Lonie [17, 21], Hall [18, 19]. They in particular studied the existence of projective transformations of Ricci-flat, Einstein, and FRW metrics, which is a stronger condition than the existence of geodesically equivalent metrics. Indeed, projective transformation of g allows to construct $\bar{g}$ geodesically equivalent to g. Moreover, if g is Einstein, then $\bar{g}$ is automatically Einstein as well, which essentially simplifies all formulas.

One can find more historical details in the surveys [2, 7, 13, 43], and in the introductions to the papers [33, 34, 38, 39, 41, 40].

Acknowledgments. The results were obtained because Gary Gibbons asked the second author to check whether certain explicitly given Einstein metrics admit geodesic equivalence (these metrics admit integrals quadratic in velocities, and geodesic equivalence could lay behind the existence of such integrals, see [22, 23, 28, 29, 30, 31, 32, 35, 36, 37, 49]).

There exists a algorithmic method to understand whether an explicitly given metric admits an nontrivial geodesic equivalence (assuming we can explicitly differentiate components of the metrics, and perform algebraic operations). Unfortunately, the method is highly computational, and applying it to the metrics suggested by Gibbons, which are given by quite complicated formulas, resulted so huge output, that we could not convince even ourself that everything is correct. Therefore, we started to look for a theory that could simplify the calculations, and solved the problem in the whole generality.

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2 Proof of Theorem 1

2.1 Schema of the proof

In Section 2.2 we list standard facts from theory of geodesically equivalent metrics, and introduce notation we will use through the paper. Most of these facts can be found in the book of Sinjukov [52], but unfortunately they are spread over the text, and it in not clear under which assumption they are true (Sinjukov always assumes real-analicity, but actually needs smoothness). All the facts could be obtained by relatively simple tensor calculations, we will indicate how.

The main result of Section 2.3 are Corollaries 3, 4. In Section 2.4 we explain that the ODE along geodesics given by Corollary 4 (that controls the reparametrization that makes g-geodesics from $\bar g$-geodesics) can not have solutions such that they satisfy the condition that both metrics are complete provided that the Einstein metric g is pseudo-Riemannian, or Riemannian of nonpositive scalar curvature.

Corollary 3 will be used in Section 2.5: we will see that combining Corollary 3 with an nontrivial result of Tanno [54] immediately gives Theorem 1 under additional assumption that the metric is Riemannian of positive scalar curvature.

2.2 Standard formulas we will use

We work in tensor notations with the background metric g. That means, we sum with respect to repeating indexes, use g for raising and lowing indexes (unless we explicitly mention), and use the Levi-Civita connection of g for covariant differentiation.

As it was known already to Levi-Civita [26], two connections $\Gamma = \Gamma^i_{jk}$ and $\bar\Gamma = \bar\Gamma^i_{jk}$ have the same unparameterized geodesics, if and only if their difference is a pure trace: there exists a $(0,1)$-tensor ϕ such that

$$\bar\Gamma^i_{jk} = \Gamma^i_{jk} + \delta^i_k\phi_j + \delta^i_j\phi_k. \tag{1}$$

The reparameterization of the geodesics for Γ and $\bar\Gamma$ connected by (1) is done according to the following rule: for a parametrized geodesic $\gamma(\tau)$ of $\bar\Gamma$, the curve $\gamma(\tau(t))$ is a parametrized geodesic of Γ, if and only if the parameter transformation $\tau(t)$ satisfies the following ODE:

$$\phi_\alpha\dot\gamma^\alpha = \frac{1}{2}\frac{d}{dt}\left(\log\left(\left|\frac{d\tau}{dt}\right|\right)\right). \tag{2}$$

(We denote by $\dot\gamma$ the velocity vector of γ with respect to the parameter t, and assume summation with respect to the repeating index α.)

If Γ and $\bar\Gamma$ related by (1) are Levi-Cevita connections of metrics g and $\bar g$, then one can find explicitly (following Levi-Civita [26]) a function ϕ on the manifold such that its differential $\phi_{,i}$ coincides with the covector ϕ_i: indeed, contracting (1) with respect to i and j, we obtain $\bar\Gamma^\alpha_{\alpha i} = \Gamma^\alpha_{\alpha i} + (n+1)\phi_i$. From the other side, for the Levi-Civita connection Γ of a metric g we have $\Gamma^\alpha_{\alpha k} = \frac{1}{2}\frac{\partial \log(|det(g)|)}{\partial x_k}$. Thus,

$$\phi_i = \frac{1}{2(n+1)}\frac{\partial}{\partial x_i}\log\left(\left|\frac{\det(\bar g)}{\det(g)}\right|\right) = \phi_{,i} \tag{3}$$

for the function $\phi : M \to \mathbb{R}$ given by

$$\phi := \frac{1}{2(n+1)}\log\left(\left|\frac{\det(\bar g)}{\det(g)}\right|\right). \tag{4}$$

In particular, the derivative of ϕ_i is symmetric, i.e., $\phi_{i,j} = \phi_{j,i}$.

The formula (1) implies that two metrics g and $\bar g$ are geodesically equivalent if and only if for a certain ϕ_i (which is, as we explained above, the differential of ϕ given by (4)) we have

$$\bar g_{ij,k} - 2\bar g_{ij}\phi_k - \bar g_{ik}\phi_j - \bar g_{jk}\phi_i = 0, \tag{5}$$

where "comma" denotes the covariant derivative with respect to the connection Γ. Indeed, the left-hand side of this equation is the covariant derivative with respect to $\bar\Gamma$, and vanishes if and only if $\bar\Gamma$ is the Levi-Civita connection for $\bar g$.

The equations (5) can be linearized by a clever substitution: consider a_{ij} and λ_i given by

$$a_{ij} = e^{2\phi}\bar g^{\alpha\beta}g_{\alpha i}g_{\beta j}, \tag{6}$$

$$\lambda_i = -e^{2\phi}\phi_\alpha\bar g^{\alpha\beta}g_{\beta i}, \tag{7}$$

where $\bar g^{\alpha\beta}$ is the tensor dual to $\bar g_{\alpha\beta}$: $\bar g^{\alpha i}\bar g_{\alpha j} = \delta^i_j$. It is an easy exercise to show that the following linear equations on the symmetric $(0,2)-$tensor a_{ij} and $(0,1)-$tensor λ_i are equivalent to (5).

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}. \tag{8}$$

Remark 3. For dimension 2, the substitution (6,7) was already known to R. Liouville [27] and Dini [12], see [10, Section 2.4] for details and a conceptual explanation. For arbitrary dimension, the substitution (6,7) and the equation (8) are due to Sinjukov [52]. The background geometry is explained in [14].

Note that it is possible to find a function λ such that its differential is precisely the the $(0,1)-$tensor λ_i: indeed, multiplying (8) by g^{ij} and summing with respect to repeating indexes i,j we obtain $(g^{ij}a_{ij})_{,k} = 2\lambda_k$. Thus, λ_i is the differential of the function

$$\lambda := \tfrac{1}{2}g^{\alpha\beta}a_{\alpha\beta}. \tag{9}$$

In particular, the covariant derivative of λ_i is symmetric: $\lambda_{i,j} = \lambda_{j,i}$.

Integrability conditions for the equation (8) (we substitute the derivatives of a_{ij} given by (8) in the formula $a_{ij,lk} - a_{ij,kl} = a_{i\alpha} R^\alpha_{jkl} + a_{\alpha j} R^\alpha_{ikl}$, which is true for every $(0,2)-$tensor a_{ij}) were first obtained by Solodovnikov [53] and are

$$a_{i\alpha} R^\alpha_{jkl} + a_{\alpha j} R^\alpha_{ikl} = \lambda_{l,i} g_{jk} + \lambda_{l,j} g_{ik} - \lambda_{k,i} g_{jl} - \lambda_{k,j} g_{il}. \tag{10}$$

For further use let us recall the fact which can also be obtained by simple calculations: the Ricci-tensors of connections related by (1) are connected by the formula

$$\bar{R}_{ij} = R_{ij} - (n-1)(\phi_{i,j} - \phi_i \phi_j), \tag{11}$$

where R_{ij} is the Ricci-tensor of Γ and $\bar{R}_{ij}$ is the Ricci-tensor of $\bar{\Gamma}$.

2.3 Local results

Within the whole paper we work on a smooth manifold of dimension $n \geq 3$.

Lemma 1 (Folklore)**.** *Let a_{ij} be a solution of (8) for the metric g. Then, it commutes with the Ricci-tensor:*

$$a^\alpha_i R_{\alpha j} = a^\alpha_j R_{i\alpha}. \tag{12}$$

Proof. Consider the equations (10). We "cycling" the equation with respect to i, k, l: we sum it with itself after renaming the indexes according to $(i \mapsto k \mapsto l \mapsto i)$ and with itself after renaming the indexes according to $(i \mapsto l \mapsto k \mapsto i)$. The first term at the left-hand side of the equation disappears because of the Bianchi equality $R^\alpha_{ikl} + R^\alpha_{kli} + R^\alpha_{lik} = 0$, the right-hand side vanishes completely, and we obtain

$$a_{\alpha i} R^\alpha_{jkl} + a_{\alpha k} R^\alpha_{jli} + a_{\alpha l} R^\alpha_{jik} = 0. \tag{13}$$

Multiplying with g^{jk}, using the symmetries of the curvature tensor, and summing over the repeating indexes we obtain $a_{\alpha i} R^\alpha_l - a_{\alpha l} R^\alpha_i = 0$ implying the claim, □

Lemma 2. *Suppose the curvature tensor of the metric g satisfies*

$$R^\alpha_{ijk,\alpha} = 0.$$

Then, for every solution a_{ij} of (8) such that $\lambda_i \neq 0$ at a point $p \in M^n$, in a sufficiently small neighborhood $U(p)$ of p we have

$$\lambda_{k,j} = \overset{1}{c}\, g_{kj} + \overset{2}{c}\, R_{kj} + \overset{3}{c}\, a_{kj} + \overset{4}{c}\, a^\alpha_j R_{\alpha k}, \tag{14}$$

where the coefficients $\overset{1}{c}$, $\overset{2}{c}$, $\overset{3}{c}$, $\overset{4}{c}$ *are given by the formulas*

$$\overset{1}{c}=\frac{-\lambda_\alpha a^\alpha_\beta \xi^\beta R + 2\lambda\lambda_{\alpha,\beta}{}^{\beta}\xi^\alpha + a^\alpha_\beta R^\beta_\alpha - 4\lambda_{\alpha,}{}^{\alpha}}{4n}; \quad \overset{2}{c}=\tfrac{1}{4}\lambda_\alpha a^\alpha_\beta \xi^\beta; \quad \overset{3}{c}=-\tfrac{1}{4}\lambda_{\alpha,\beta}{}^{\beta}\xi^\alpha; \overset{4}{c}=-\tfrac{1}{4}a^\alpha_\beta R^\beta_\alpha,$$

where ξ *is an arbitrary vector field such that* $\lambda_i\xi^i = 1$.

Remark 4. The assumptions of the lemma are automatically fulfilled for Einstein spaces. Indeed, the second Bianchi identity for the curvature tensor is

$$R^h_{ijk,l} + R^h_{ikl,j} + R^h_{ilj,k} = 0.$$

Contracting with respect to h and l, we obtain

$$R^\alpha_{ijk,\alpha} + \underbrace{R^\alpha_{ik\alpha,j}}_{-R_{ik,j}} + \underbrace{R^\alpha_{i\alpha j,k}}_{R_{ij,k}} = 0.$$

If the metric is Einstein, then the second and the third components of the equation vanishes, and we obtain $R^\alpha_{ijk,\alpha} = 0$. Moreover, we see that actually the condition $R_{ik,j} - R_{ij,k} = 0$ is a necessary and sufficient condition for $R^\alpha_{ijk,\alpha} = 0$.

Remark 5. The tensor $R_{ik,j} - R_{ij,k}$ is called projective Yano tensor, and plays important role in the theory of geodesically equivalent metrics; in particular, it is projectively invariant in dimension 2 [27, 10], and is an essential part of the so-called tractor approach for the investigation of geodesically equivalent metrics [14].

Proof of Lemma 2. Consider the solution a_{ij} of the equation (8). Let us take the covariant derivative of the equations (10) (the index of differentiation is "m"), and replace the covariant derivative of a by formula (8). We obtain

$$\begin{aligned} &\lambda_\alpha R^\alpha_{jkl} g_{im} + \lambda_i R_{mjkl} + a_{\alpha i} R^\alpha_{jkl,m} + \lambda_\alpha R^\alpha_{ikl} g_{jm} + \lambda_j R_{mikl} + a_{\alpha j} R^\alpha_{ikl,m} \\ &= \lambda_{l,im} g_{jk} + \lambda_{l,jm} g_{ik} - \lambda_{k,im} g_{jl} - \lambda_{k,jm} g_{il}. \end{aligned} \tag{15}$$

We multiply with g^{lm}, sum with respect to repeating indexes l, m, and use $R^\alpha_{ijk,\alpha} = 0$. We obtain:

$$\lambda_\alpha R^\alpha_{ikj} + \lambda_\alpha R^\alpha_{jki} - \lambda_i R_{jk} - \lambda_j R_{ik} = \lambda_{i,\alpha}{}^{\alpha} g_{jk} + \lambda_{j,\alpha}{}^{\alpha} g_{ik} - \lambda_{k,ij} - \lambda_{k,ji}. \tag{16}$$

We now skew-symmetrise the equation (16) with respect to k, j to obtain

$$4\lambda_\alpha R^\alpha_{ikj} = \lambda_{j,\alpha}{}^{\alpha} g_{ik} - \lambda_{k,\alpha}{}^{\alpha} g_{ij} - \lambda_k R_{ij} + \lambda_j R_{ik}. \tag{17}$$

Let us now rename the indexes $i \mapsto k \mapsto j \mapsto \alpha$ in (17), multiply the result by a^α_i, use the symmetries of the curvature tensor and sum over the repeating index α. We obtain

$$
\begin{aligned}
4a_i^\alpha R_{\alpha jk\beta}\lambda^\beta &= 4a_i^\alpha R^\beta_{kj\alpha}\lambda_\beta \\
&= a_i^\alpha \left(\lambda_{\alpha,\beta}{}^{\beta} g_{kj} - \lambda_{k,\alpha}{}^{\alpha} g_{ij} - \lambda_{j,\beta}{}^{\beta} g_{k\alpha} - \lambda_j R_{\alpha k} + \lambda_\alpha R_{jk}\right) \\
&= a_i^\alpha \lambda_{\alpha,\beta}{}^{\beta} g_{kj} - \lambda_{j,\beta}{}^{\beta} a_{ki} - \lambda_j a_i^\alpha R_{\alpha k} + \lambda_\alpha a_i^\alpha R_{kj}.
\end{aligned} \tag{18}
$$

Now we multiply the equation (10) by λ^l and sum over the repeating index l. We see that the first component of the result is precisely the left-hand side of (18); we replace it by the right-hand side of (18). We obtain

$$
\begin{aligned}
0 \;=\; & \left(a_i^\alpha \lambda_{\alpha,\beta}{}^{\beta} - 4\lambda^\alpha \lambda_{\alpha,i}\right) g_{kj} - \lambda_{j,\beta}{}^{\beta} a_{ki} + \lambda_j\left(-a_i^\alpha R_{\alpha k} + 4\lambda_{k,i}\right) + \lambda_\alpha a_i^\alpha R_{kj} \\
+ & \left(a_j^\alpha \lambda_{\alpha,\beta}{}^{\beta} - 4\lambda^\alpha \lambda_{\alpha,j}\right) g_{ki} - \lambda_{i,\beta}{}^{\beta} a_{kj} + \lambda_i\left(-a_j^\alpha R_{\alpha k} + 4\lambda_{k,j}\right) + \lambda_\alpha a_j^\alpha R_{ki}
\end{aligned} \tag{19}
$$

We now skew-symmetrise (18) with respect to k, j, rename $k \longleftrightarrow i$, and add the result to (19). After dividing by 2 for cosmetic reasons, and using that by Lemma 1 the tensor $a_i^\alpha R_{\alpha k}$ is symmetric with respect to i, k, we obtain

$$
\left(a_i^\alpha \lambda_{\alpha,\beta}{}^{\beta} - 4\lambda^\alpha \lambda_{\alpha,i}\right) g_{kj} + \lambda_\alpha a_i^\alpha R_{kj} - \lambda_{i,\beta}{}^{\beta} a_{kj} + \lambda_i\left(-a_j^\alpha R_{\alpha k} + 4\lambda_{k,j}\right) = 0. \tag{20}
$$

We multiply (20) by g^{kj} and sum over the repeating indexes k, j. We obtain (after dividing by n)

$$
\left(a_i^\alpha \lambda_{\alpha,\beta}{}^{\beta} - 4\lambda^\alpha \lambda_{\alpha,i}\right) = -\tfrac{R}{n}\lambda_\alpha a_i^\alpha + \frac{2\lambda}{n}\lambda_{i,\beta}{}^{\beta} - \lambda_i \frac{\left(-a_\beta^\alpha R_\alpha^\beta + 4\lambda_{\alpha,}{}^{\alpha}\right)}{n} = 0, \tag{21}
$$

where $R := R_{\alpha\beta} g^{\alpha\beta}$ is the scalar curvature of g. Substituting the expression for $\left(a_i^\alpha \lambda_{\alpha,\beta}{}^{\beta} - 4\lambda^\alpha \lambda_{\alpha,i}\right)$ from (21) in (20), we obtain

$$
\begin{aligned}
0 &= \lambda_\alpha a_i^\alpha \left(R_{kj} - \tfrac{R}{n} g_{kj}\right) + \lambda_{i,\beta}{}^{\beta}\left(\tfrac{2\lambda}{n} g_{kj} - a_{kj}\right) \\
&- \lambda_i \left(\tfrac{-a_\beta^\alpha R_\alpha^\beta + 4\lambda_{\alpha,}{}^{\alpha}}{n} g_{kj} + a_j^\alpha R_{\alpha k} - 4\lambda_{k,j}\right)
\end{aligned} \tag{22}
$$

Since $\lambda_i \neq 0$ at a point p, then $\lambda_i \xi^i = 1$ for a certain vector field ξ in a sufficiently small neighborhood $U(p)$. Contracting the equation (22) with this ξ^i, we obtain

$$
\begin{aligned}
0 &= \lambda_\alpha a_i^\alpha \xi^i \left(R_{kj} - \tfrac{R}{n} g_{kj}\right) + \xi^i \lambda_{i,\beta}{}^{\beta}\left(\tfrac{2\lambda}{n} g_{kj} - a_{kj}\right) \\
&- \lambda_i \left(\tfrac{-a_\beta^\alpha R_\alpha^\beta + 4\lambda_{\alpha,}{}^{\alpha}}{n} g_{kj} + a_j^\alpha R_{\alpha k} - 4\lambda_{k,j}\right)
\end{aligned} \tag{23}
$$

We see that $\lambda_{j,k}$ is a linear combination of $a^\alpha_j R_{\alpha k}$, g_{jk}, R_{jk} and a_{kj} as we want. The coefficients in the linear combination are as in the formula below , $\square$

$$4\lambda_{k,j} = a_{\alpha k} R^\alpha_j + \frac{-\lambda_\alpha a^\alpha_\beta \xi^\beta R + 2\lambda \lambda_{\alpha,\beta}{}^\beta \xi^\alpha + a^\alpha_\beta R^\beta_\alpha - 4\lambda^\alpha_\alpha}{n} g_{jk} + \lambda_\alpha a^\alpha_\beta \xi^\beta R_{jk} - \lambda_{\alpha,\beta}{}^\beta \xi^\alpha a_{kj}.$$

Corollary 1. *Assume g is an Einstein metric. Let a_{ij} be a solution of (8). Assume $\lambda_i \neq 0$ at a point p. Then, in a sufficiently small neighborhood of p, $\lambda_{i,j}$ is a linear combination of g_{ij} and a_{ij}:*

$$\lambda_{i,j} = \mu g_{ij} + K a_{ij}, \tag{24}$$

where the coefficients $K := -\frac{R}{n(n-1)}$ and $\mu := \frac{\lambda_{\alpha,}{}^\alpha - 2K\lambda}{n}$.

Proof. By assumption, in a small neighborhood of p we have $\lambda_i \neq 0$; this implies that a_{ij} is not proportional to g_{ij}, because by the result of Weyl [56] if two metrics are geodesically and conformally equivalent, then they are proportional (with a constant coefficient of proportionality).

As we explained in Remark 4, the assumptions of Lemma 2 are fulfilled if the metric is Einstein. Moreover, if the metric is Einstein, then the second term of the right-hand side of (14) is proportional to g, and the last term is proportional to a implying that $\lambda_{i,j}$ is a linear combination of g_{ij} and a_{ij}. We need to calculate the coefficients of the linear combination.

Substituting the condition that the metric is Einstein in (17), we obtain

$$\lambda_\alpha R^\alpha_{ikj} = \tau_j g_{ik} - \tau_k g_{ij}, \tag{25}$$

where

$$\tau_i := \tfrac{1}{4}\left(\lambda_{i,\alpha}{}^\alpha + \tfrac{R}{n}\lambda_i\right). \tag{26}$$

Contracting the equation (25) with g^{ij} we obtain $(n-1)\tau_j = -\frac{R}{n}\lambda_j$ implying

$$\tau_j = -\tfrac{R}{n(n-1)}\lambda_j. \tag{27}$$

Now, since the metric is Einstein, the first bracket in the sum (22) is zero, and the term $a^\alpha_\beta R^\beta_\alpha$ equals $\frac{R}{n}\delta^\beta_\alpha a^\alpha_\beta = 2\frac{R}{n}\lambda$, so the formula (22) reads

$$\lambda_{i,\beta}{}^\beta \left(\frac{2\lambda}{n} g_{kj} - a_{kj}\right) - \lambda_i \left(\frac{-2\lambda\frac{R}{n} + 4\lambda_{\alpha,}{}^\alpha}{n} g_{kj} + \tfrac{R}{n} a_{kj} - 4\lambda_{k,j}\right) = 0 \tag{28}$$

Combining (26) and (27), we obtain

$$\lambda_{i,\beta}{}^\beta = \left(4k - \tfrac{R}{n}\right)\lambda_i. \tag{29}$$

Substituting this in (28), we obtain

$$\left(4k - \tfrac{R}{n}\right)\left(\frac{2\lambda}{n} g_{kj} - a_{kj}\right)\lambda_i - \left(\frac{-2\lambda\frac{R}{n} + 4\lambda_{\alpha,}{}^{\alpha}}{n} g_{kj} + \tfrac{R}{n} a_{kj} - 4\lambda_{k,j}\right)\lambda_i = 0. \tag{30}$$

Since by assumption $\lambda_i \neq 0$, we obtain (24), □

Remark 6. Assume g is an Einstein metric. Let a_{ij} be a solution of (8). Then,

$$\lambda_\alpha Y^\alpha_{ijk} = 0, \tag{31}$$

where $Y^h_{ijk} := R^h_{ijk} - \frac{R}{n(n-1)}\left(\delta^h_j g_{ik} - \delta^h_k g_{ij}\right)$ is the so-called concircular curvature of g introduced by Yano [58].

Proof. Substituting (27) in (25), we obtain the claim, □

Corollary 2. *Assume g is an Einstein metric. Let a_{ij} be a solution of* (8). *Consider $K := -\frac{R}{n(n-1)}$ and the function $\mu := \frac{\lambda_{\alpha,}{}^{\alpha} - 2K\lambda}{n}$. Then, the function μ satisfies the equation*

$$\mu_{,i} = 2K\lambda_i. \tag{32}$$

Remark 7. In particular, under the assumptions of Corollary 2, for a certain const $\in \mathbb{R}$, the function $\lambda + \text{const}$ is an eigenfunction of the laplacian of g.

Proof of Corollary 2. If λ is constant in a neighborhood of a point, the equation (32) is automatically fulfilled. Below we will assume that λ is not constant. Differentiating the definition of μ and multiplying by n for cosmetic reasons, we obtain

$$n\mu_{,i} = 2\lambda_{\alpha,}{}^{\alpha}{}_{i} - 2K\lambda_i. \tag{33}$$

By definition of curvature we have $\lambda_{i,jk} - \lambda_{i,kj} = \lambda_\alpha R^\alpha_{ijk}$. Contracting this with g^{ij}, and using $R_{ij} = \frac{R}{n} g_{ij}$, we obtain

$$\lambda_{\alpha,}{}^{\alpha}{}_{k} - \lambda_{k,\alpha}{}^{\alpha} = -\tfrac{R}{n}\lambda_k.$$

The formula (29) gives us $\lambda_{k,\alpha}{}^{\alpha}$, whose substitution gives

$$\lambda_{\alpha,}{}^{\alpha}{}_{k} = \left(-2\tfrac{R}{n} + 4K\right)\lambda_k.$$

Substituting this in (33), we obtain $\mu_{,i} = -\frac{2R}{n(n-1)}\lambda_i = 2K\lambda_i$, □

Corollary 3. *Let g and $\bar g$ be geodesically equivalent metrics, assume g is an Einstein metric. Then, the function λ given by* (9) *satisfies*

$$\lambda_{,ijk} - K\cdot\left(2\lambda_{,k} g_{ij} + \lambda_{,j} g_{ik} + \lambda_{,i} g_{jk}\right) = 0, \tag{34}$$

where $K := -\frac{R}{n(n-1)}$.

Proof. If λ is constant in a neighborhood of p, the equation is automatically fulfilled. Then, it is sufficient to prove Corollary 3 at points p such that $\lambda_i(p) \neq 0$.

Covariantly differentiating (24), we obtain $\lambda_{i,jk} = \mu_{,k} g_{ij} + K a_{ij,k}$. Substituting $\mu_{,k}$ by (32), and $a_{ij,k}$ by (8), we obtain the claim, $\square$

Lemma 3. *Let g and $\bar{g}$ be geodesically equivalent. Assume g is Einstein, and assume that $\lambda_i \neq 0$ at a point p.*

Then, the restriction of $\bar{g}$ to a sufficiently small neighborhood $U(p)$ is Einstein as well. Moreover, the following formula holds (at every point of $U(p)$).

$$\phi_{i,j} - \phi_i\phi_j = \tfrac{R}{n(n-1)} g_{ij} - \tfrac{\bar{R}}{n(n-1)} \bar{g}_{ij}, \tag{35}$$

where $\bar{R}$ is the scalar curvature of the metric $\bar{g}$.

Remark 8. The first statement of the lemma easily follows from certain formulas obtained in [42]. In dimension 4, under additional assumptions ($R = 0$ and Lorentz signature), the first statement was proved in [20].

Proof of Lemma 3. We covariantly differentiate (7) (the index of differentiation is "j"); then we substitute the expression (5) for $\bar{g}_{ij,k}$ to obtain

$$\begin{array}{rcl} \lambda_{i,j} & = & -2e^{2\phi}\phi_j\phi_\alpha \bar{g}^{\alpha\beta} g_{\beta i} - e^{2\phi}\phi_{\alpha,j}\bar{g}^{\alpha\beta} g_{\beta i} + e^{2\phi}\phi_\alpha \bar{g}^{\alpha\gamma}\bar{g}_{\gamma l,j}\bar{g}^{l\beta} g_{\beta i} \\ & = & -e^{2\phi}\phi_{\alpha,j}\bar{g}^{\alpha\beta} g_{\beta i} + e^{2\phi}\phi_\alpha\phi_\gamma \bar{g}^{\alpha\gamma} g_{ij} + e^{2\phi}\phi_j\phi_l \bar{g}^{l\beta} g_{\beta i} \end{array}, \tag{36}$$

where $\bar{g}^{\alpha\beta}$ is the tensor dual to $\bar{g}_{\alpha\beta}$. We now substitute $\lambda_{i,j}$ from (24), use that a_{ij} is given by (6), and divide by $e^{2\phi}$ for cosmetic reasons to obtain

$$e^{-2\phi}\mu g_{ij} + K\bar{g}^{\alpha\beta} g_{\alpha j} g_{\beta_i} = -\phi_{\alpha,j}\bar{g}^{\alpha\beta} g_{\beta i} + \phi_\alpha\phi_\gamma \bar{g}^{\alpha\gamma}\bar{g}_{ij} + \phi_j\phi_l \bar{g}^{l\beta} g_{\beta i}. \tag{37}$$

Multiplying with $g^{i\xi}\bar{g}_{\xi k}$, we obtain

$$\phi_{k,j} - \phi_k\phi_j = (\phi_\alpha\phi_\beta \bar{g}^{\alpha\beta} - e^{-2\phi}\mu)\bar{g}_{kj} - K g_{kj}. \tag{38}$$

Let us now show that the coefficient $\bar{K} := -\frac{\phi_\alpha\phi_\beta\bar{g}^{\alpha\beta} - e^{-2\phi}\mu}{n-1}$ is constant. Substituting (38) in (11), and using $R_{ij} = \frac{R}{n} g_{ij}$, we obtain

$$\bar{R}_{ij} = \tfrac{R}{n} g_{ij} - \tfrac{R}{n} g_{ij} - (\phi_\alpha\phi_\beta \bar{g}^{\alpha\beta} - e^{-2\phi}\mu)\bar{g}_{ij}.$$

We see that $\bar{R}_{ij}$ is proportional to $\bar{g}_{ij}$. Then, $\bar{g}$ is an Einstein metric; in particular, $\bar{K} := -\frac{\phi_\alpha\phi_\beta\bar{g}^{\alpha\beta} - e^{-2\phi}\mu}{n-1}$ is a constant equal to $-\frac{\bar{R}}{n(n-1)}$, and (38) gives us the formula

$$\bar{K}\bar{g}_{ij} = K g_{ij} + \phi_{i,j} - \phi_i\phi_j, \tag{39}$$

which is evidently equivalent to (35), $\square$

Corollary 4. *Let g and $\bar{g}$ be geodesically equivalent metrics, assume g is an Einstein metric. Consider a (parametrized) geodesic γ of the metric g, and denote by $\dot{\phi}$, $\ddot{\phi}$ and $\dddot{\phi}$ the first, second and third derivatives of the function ϕ given by (4) along the geodesic. Then, along the geodesic, the following ordinary differential equation holds:*

$$\dddot{\phi} \quad = \quad 4Kg(\dot{\gamma}, \dot{\gamma})\dot{\phi} + 6\phi\dot{\phi}\ddot{\phi} - 4(\dot{\phi})^3 \ , \tag{40}$$

where $g(\dot{\gamma}, \dot{\gamma}) := g_{ij}\dot{\gamma}^i\dot{\gamma}^j$.

Proof. If $\phi_i \equiv 0$ in a neighborhood U, the equation is automatically fulfilled. Then, it is sufficient to prove Corollary 4 assuming ϕ is not constant.

The formula (35) is evidently equivalent to (39), which is evidently equivalent to

$$\phi_{i,j} = \bar{K}\bar{g}_{ij} - Kg_{ij} + \phi_i\phi_j. \tag{41}$$

Taking covariant derivative of (41), we obtain

$$\phi_{i,jk} = \bar{K}\bar{g}_{ij,k} + 2\phi_{i,k}\phi_j + 2\phi_{j,k}\phi_i. \tag{42}$$

Substituting the expression for $\bar{g}_{ij,k}$ from (5), and substituting $\bar{K}\bar{g}_{ij}$ given by (39), we obtain

$$\begin{aligned} \phi_{i,jk} &= \bar{K}(2\bar{g}_{ij}\phi_k + \bar{g}_{ik}\phi_j + \bar{g}_{jk}\phi_i) + 2\phi_{i,k}\phi_j + 2\phi_{j,k}\phi_i \\ &= K(2g_{ij}\phi_k + g_{ik}\phi_j + g_{jk}\phi_i) + 2(\phi_k\phi_{i,j} + \phi_i\phi_{j,k} + \phi_j\phi_{k,i}) - 4\phi_i\phi_j\phi_k \end{aligned} \tag{43}$$

Contracting with $\dot{\gamma}^i\dot{\gamma}^j\dot{\gamma}^k$ and using that ϕ_i is the differential of the function (4) we obtain the desired ODE (40), □

Corollary 5. *Let $\bar{g}$ (on a connected $M^{n\geq 3}$) be geodesically equivalent to an Einstein metric g, but is not affine equivalent to g. Then, the restrictions of g and $\bar{g}$ to any neighborhood are also not affine equivalent.*

Remark 9. The assumption that g is Einstein is important: Levi-Civita's description of geodesically equivalent metrics [26] immediately gives counterexamples.

Proof of Corollary 5. We consider the function ϕ given by (4). Suppose $\phi_i \neq 0$ at a point p. Consider a geodesic $\gamma(t)$ such that $\gamma(0) = p$, $\dot{\gamma}^\alpha(0)\phi_\alpha(p) \neq 0$. Note that almost every geodesic with $\gamma(0) = p$ satisfies $\dot{\gamma}^\alpha(0)\phi_\alpha(p) \neq 0$.

By Corollary 4, the function $\phi(t) = \phi(\gamma(t))$ (whose $t-$derivative is $\dot{\phi}(t) = \dot{\gamma}^i(t)\,\phi_i(\gamma(t))$) satisfies equation (40) along the geodesic. Clearly, every constant is a solution of the equation. Since $\dot{\phi}(0) \neq 0$, by uniqueness of the solutions of ODE, the restriction $\phi(t)$ to every open interval can not be constant. Hence, the subset of t such that $\dot{\phi}(t) \neq 0$ is everywhere dense. Since as we mentioned above, $\dot{\phi}(0) \neq 0$ for almost every geodesic γ (with $\gamma(0) = p$), we have that for every point p_0 of every geodesic passing through p there exists a sequence $p_k \in M^n$ such that $\phi_i(p_k) \neq 0$ and such that $p_k \overset{k\to\infty}{\longrightarrow} p_0$. Since every point can be reached from the point p by a sequence of geodesics, we have that $\phi_i \neq 0$ at every point of an open everywhere dense subset of M^n, □

2.4 Proof of Theorem 1 for Riemannian metrics of nonpositive scalar curvature, and for pseudo-Riemannian metrics

Assume the metric g on a connected $M^{n\ge 3}$ is Einstein and is either Riemannian (i.e., positive definite) with nonpositive scalar curvature, or there exist light-like vectors (i.e., for no constant $c \ne 0$ the metric $c \cdot g$ is Riemannian). Let $\bar g$ be geodesically equivalent to g. Assume both metrics are complete. Our goal is to show that ϕ given by (4) is constant, because in view of (1) this implies that the metrics are affine equivalent.

Consider a parameterized geodesic $\gamma(t)$ of g. If the metric g is pseudo-Riemannian, we additionally assume that γ is a light-like geodesic i.e., $\dot\gamma^i\dot\gamma^j g_{ij} = 0$. Since the metrics are geodesically equivalent, for a certain function $\tau : \mathbb{R} \to \mathbb{R}$ the curve $\gamma(\tau)$ is a geodesic of $\bar g$. Since the metrics are complete, the reparameterization $\tau(t)$ is a diffeomorphism $\tau : \mathbb{R} \to \mathbb{R}$. Without loss of generality we can think that $\dot\tau := \frac{d}{dt}\tau$ is positive, otherwise we replace t by $-t$. Then, the equation (2) along the geodesic reads

$$\phi(t) = \tfrac{1}{2}\log(\dot\tau(t)) + \mathrm{const}_0. \tag{44}$$

Now let us consider the equation (40). Substituting

$$\phi(t) = -\tfrac{1}{2}\log(p(t)) + \mathrm{const}_0 \tag{45}$$

in it (since $\dot\tau > 0$, the substitution is global), we obtain

$$\dddot p = 4Kg(\dot\gamma,\dot\gamma)\dot p. \tag{46}$$

Since the length of the tangent vector is preserved along a geodesic, $g(\dot\gamma,\dot\gamma)$, and therefore $4Kg(\dot\gamma,\dot\gamma)$ is a constant. The assumptions above imply that this constant is nonnegative.

Indeed, if $\dot\gamma$ is a light-like vector, this constant is zero, since γ is an light-like geodesic. If the metric is Riemannian of nonpositive curvature, $g(\dot\gamma,\dot\gamma) \ge 0$, and $K \ge 0$, so their product is nonnegative.

The equation (46) can be solved. We will first consider the case $Kg(\dot\gamma,\dot\gamma) = 0$. In this case, the solution of (46) is $p(t) = C_2t^2 + C_1t + C_0$. Combining (45) with (44), we see that $\dot\tau = \frac{1}{C_2t^2+C_1t+C_0}$. Then

$$\tau(t) = \int_{t_0}^{t} \frac{d\xi}{C_2\xi^2 + C_1\xi + C_0} + \mathrm{const}. \tag{47}$$

We see that if the polynomial $C_2t^2 + C_1t + C_0$ has real roots (which is always the case if $C_2 = 0$, $C_1 \ne 0$), then the integral diverges (goes to infinity in finite time). If the polynomial has no real roots, but $C_2 \ne 0$, the function τ is bounded. Thus, the only possibility for τ to be a diffeomorphism is $C_2 = C_3 = 0$ implying $\tau(t) = \frac{1}{C_0}t + \mathrm{const}_1$ implying $\dot\tau = \frac{1}{C_0}$ implying ϕ is constant along the geodesic.

Now, let us consider the case $Kg(\dot\gamma,\dot\gamma) > 0$. In this case, the general solution of the equation (46) is

$$C + C_+ e^{2\sqrt{Kg(\dot\gamma,\dot\gamma)}t} + C_- e^{-2\sqrt{Kg(\dot\gamma,\dot\gamma)}t}. \tag{48}$$

Then, the function τ satisfies the ODE $\dot\tau = \frac{1}{C + C_+ e^{2\sqrt{Kg(\dot\gamma,\dot\gamma)}t} + C_- e^{-2\sqrt{Kg(\dot\gamma,\dot\gamma)}t}}$ implying

$$\tau(t) = \int_{t_0}^{t} \frac{d\xi}{C + C_+ e^{2\sqrt{Kg(\dot\gamma,\dot\gamma)}\xi} + C_- e^{-2\sqrt{Kg(\dot\gamma,\dot\gamma)}\xi}} + \text{const}. \tag{49}$$

If one of the constants C_+, C_- is not zero, the integral (49) is bounded from one side, or diverges (goes to infinity in finite time). Thus, the only possibility for τ to be a diffeomorphism of $\mathbb{R}$ on itself is $C_+ = C_- = 0$. Finally, ϕ is a constant along the geodesic γ.

Since every point of a connected manifold can be reached by a sequence of light like geodesics in the pseudo-Riemannian case, or by a sequence of geodesics in the Riemannian case, ϕ is a constant, so that $\phi_i \equiv 0$, and the metrics are affine equivalent by (1), □

Remark 10. A similar idea was used by Couty [11] in an investigation of projective transformations of Einstein manifolds, and by Shen [51] in an investigation of Finsler Einstein geodesically equivalent metrics.

2.5 Proof of Theorem 1 for Riemannian metrics of positive scalar curvature

We assume that g is a complete Einstein Riemannian metric of positive scalar curvature on a connected manifold (we do not need that the second metric is complete). Then, by Corollary 3, λ is a solution of (34). If the metrics are not affine equivalent, λ is not identically constant.

The equation (34) was studied by Obata and Tanno in [45, 54] in a completely different geometrical context. They proved (actually, Tanno [54], because the proof of Obata [45] has a mistake) that a complete Riemannian g such that there exists a nonconstant function λ satisfying (34) must have a constant positive curvature. Applying this result in our situation, we obtain the claim, □

3 Proof of Theorem 2

It is sufficient to prove Theorem 2 in a neighborhood of a point p such that λ_i given by (7) does not vanish. Indeed, by Corollary 5, either such points are everywhere dense, or the metrics are affine equivalent. We will first formulate two simple lemmas from Linear Algebra, then prove a simple Lemma 6 which generalizes certain result of Levi-Civita [26], and then obtain Theorem 2 as an easy corollary.

3.1 Two simple lemmas from Linear Algebra

We say that the vector v^i lies in *kernel* of the tensor Z_{ijkl}, if $v^i Z_{ijkl} = 0$.

Lemma 4. *Assume the tensor Z_{ijkl} on $\mathbb{R}^4$ has the following symmetries:*

$$Z_{ijkl} = Z_{klij} \ , \ Z_{ijkl} = -Z_{jikl}, \tag{50}$$

and satisfies $Z_{ijkl}g^{ik} = 0$. Suppose the vector v^i such that $g(v,v) := v^i v^j g_{ij} \neq 0$ lies in the kernel of Z_{ijkl}. Then, $Z = 0$.

Remark 11. The assumption $g(v,v) \neq 0$ is important: one immediately constructs a counterexample. The dimension is also important: the claim fails for dimensions ≥ 5.

Proof of Lemma 4 is an easy exercise and will be left to the reader. We recommend to consider a basis such that the first vector is v and the metric is given by the matrix

$$\begin{pmatrix} \varepsilon_1 & & & \\ & \varepsilon_2 & & \\ & & \varepsilon_3 & \\ & & & \varepsilon_4 \end{pmatrix},$$

where all $\varepsilon_i \neq 0$. Then, the conditions $v^i Z_{ijkl} = 0$ and $Z_{ijkl}g^{ik} = 0$ are a system of homogeneous linear equations on the components of Z which admits only trivial solution implying the claim, □

Lemma 5. *Let a and Z be $n \times n$ matrices over $\mathbb{C}$ such that Z is skew-symmetric and such that their product aZ is symmetric. Let the geometric multiplicity of the eigenvalue $\rho \in \mathbb{C}$ of the matrix a be 1. Then, every vector v from the generalized eigenspace of ρ lies in the kernel of the matrix Z.*

(Recall that *geometric multiplicity* of ρ is the dimension of the kernel of $(a - \rho \cdot \mathbf{1})$, and the generalized eigenspace of ρ is the kernel of $(a - \rho \cdot \mathbf{1})^n$.)

The proof of Lemma 5 is an easy exercise in linear algebra and will be left to the reader. We recommend to consider the basis such that the matrix a is in Jordan form, and then to calculate the matrix aZ. One immediately sees that it is block diagonal, and that if the eigenspace is one dimensional then the corresponding block is trivial, □

Corollary 6. *Suppose Z_{ijkl} is skew-symmetric with respect to indexes i, j. Suppose*

$$a_i^\alpha Z_{\alpha jkl} + a_j^\alpha Z_{\alpha ikl} = 0 \tag{51}$$

for a (1,1)-tensor a satisfying $a_j^\alpha g_{\alpha i} = a_i^\alpha g_{\alpha j}$, where (the metric) g is a symmetric nondegenerate $(0, 2)$-tensor. We assume that all components of Z, g, and a are real. Suppose there exists a (possibly, complex) eigenvalue ρ with geometric multiplicity 1. Then, there exists a vector v such that $g_{ij}v^i v^j \neq 0$ lying in the kernel of Z.

Proof. The condition $a_j^\alpha Z_{\alpha i} + a_i^\alpha Z_{\alpha j} = 0$ precisely means that the matrix aZ is symmetric. We see that this condition is the condition (51) with "forgotten" indexes k and l. Then, by Lemma 5, every vector v from the sum of the generalized eigenspaces of ρ and of its complex-conjugate $\bar{\rho}$ lies in kernel of Z. Since the generalized eigenspaces of ρ and of $\bar{\rho}$ are orthogonal to all other generalized eigenspaces because of the condition $a_j^\alpha g_{\alpha i} = a_i^\alpha g_{\alpha j}$, and because the direct sum of all all generalized eigenspaces coincides with the whole vector space, the sum of the generalized eigenspaces of ρ and of $\bar{\rho}$ contains a (real) vector v such that $g_{ij}v^i v^j \neq 0$, □

3.2 If all eigenspaces are more than one-dimensional, the metrics are affine equivalent.

Lemma 6. *If geometric multiplicity of every eigenvalue of the solution a_{ij} of* (8) *is at least two, then the function λ given by* (9) *is constant.*

Remark 12. For Riemannian metrics, the statement is due to Levi-Civita [26]. The proof for the pseudo-Riemannian case is essentially the same, the additional difficulties are due to possible Jordan blocks. In a certain form, it appears in [1].

Proof of Lemma 6. We prove the lemma assuming every Jordan-Block of a_j^i is as most 3-dimensional, this is sufficient for our four-dimensional goals. The proof for arbitrary dimensions of Jordan blocks can be done by induction.

Let ρ be an eigenvalue of a_j^i; let u^i be an eigenvector corresponding to ρ. In a small neighborhood of almost every point, ρ a smooth (possibly, complex-valued) function. We will show that the differential $\rho_{,i}$ is proportional to u_i. If the eigenspace of ρ is more than one-dimensional, this will imply that $\rho_{,i}$ is constant. This implies that if all eigenspaces are more than one-dimensional, the trace of a_j^i is constant implying the metrics are affine equivalent.

Let u be an eigenvector corresponding to ρ, i.e.,

$$u_\alpha a_i^\alpha = \rho u_i. \tag{52}$$

We take the covariant derivative and use (8). We obtain

$$u_{\alpha,j}a_i^\alpha + u_\alpha \lambda^\alpha g_{ij} + \lambda_i u_j = \rho_{,j} u_i + \rho u_{i,j}. \tag{53}$$

We multiply (53) with u^i and sum over i, to obtain (using (52))

$$2\lambda_\alpha u^\alpha u_j = u_\alpha u^\alpha \rho_{,j}. \tag{54}$$

We see that if $u_\alpha u^\alpha \neq 0$ (which is in particular always the case when the Jordan block corresponding to ρ is 1-dimensional), we are done.

Suppose the Jordan block corresponding to ρ is more than 1-dimensional, i.e., there exists v_i such that

$$v_\alpha a_i^\alpha = \rho v_i + u_i. \tag{55}$$

Then, u_i is automatically a light like vector: indeed, multiplying (55) by u^i, summing over i, and using (52), we obtain

$$u_\alpha u^\alpha = 0. \tag{56}$$

Differentiating (56), we obtain

$$u_{\alpha,i} u^\alpha = 0. \tag{57}$$

Substituting (56) in (54), we obtain $\lambda_\alpha u^\alpha = 0$. Differentiating (55) and using (8), we obtain

$$v_{\alpha,j}a_i^\alpha + v_\alpha \lambda^\alpha g_{ij} + \lambda_i v_j = \rho_{,j} v_i + \rho v_{i,j} + u_{i,j}. \tag{58}$$

Multiplying (58) by u^i and summing over i, we obtain

$$v_\alpha \lambda^\alpha u_j = v_\alpha u^\alpha \rho_{,j}. \tag{59}$$

We see that if $v_\alpha u^\alpha \neq 0$, (which is in particular always the case when the Jordan block corresponding to ρ is 2-dimensional), we are done.

Suppose the Jordan block corresponding to ρ is precisely 3-dimensional, i.e., there exists w_i such that

$$w_\alpha u^\alpha \neq 0 \tag{60}$$

and such that

$$w_\alpha a_i^\alpha = \rho w_i + v_i. \tag{61}$$

We multiply (61) with v^i and sum over i, to obtain

$$w_\alpha u^\alpha = v_\alpha v^\alpha. \tag{62}$$

We multiply (61) with u^i and sum over i, to obtain

$$u_\alpha v^\alpha = 0. \tag{63}$$

Differentiating (63), we obtain

$$u_{\alpha,i} v^\alpha = -u^\alpha v_{\alpha,i}. \tag{64}$$

Moreover, combining (63) with (59), we obtain $\lambda_\alpha v^\alpha = 0$. Differentiating (61), we obtain

$$w_{\alpha,j} a_i^\alpha + w_\alpha \lambda^\alpha g_{ij} + \lambda_i w_j = \rho_{,j} w_i + \rho w_{i,j} + v_{i,j}.$$

Contracting this with u^i, we obtain

$$w_\alpha \lambda^\alpha u_j = w_\alpha u^\alpha \rho_{,j} + u^\alpha v_{\alpha,j} \overset{(64)}{=} w_\alpha u^\alpha \rho_{,j} - u_{\alpha,i} v^\alpha. \tag{65}$$

We multiply (58) with v^i and sum over i to obtain

$$v_{\alpha,j} u^\alpha = v_\alpha v^\alpha \rho_{,j} + u_{\alpha,j} v^\alpha. \tag{66}$$

Using (64), we obtain

$$2v_{\alpha,j} u^\alpha = v_\alpha v^\alpha \rho_{,j} \overset{(62)}{=} w_\alpha u^\alpha \rho_{,j}. \tag{67}$$

Combining (67) with (65), we obtain $2w_\alpha \lambda^\alpha u_j = 3u_\alpha w^\alpha \rho_{,j}$. Combining this with (60), we obtain that the differential $\rho_{,i}$ is proportional to the eigenvector u_i. If the eigenspace of ρ is more that one-dimensional, this implies that $\rho_{,i} = 0$, □

3.3 Proof of Theorem 2

If the dimension is 3, Theorem 2 follows from the well-known fact that every Einstein 3-manifold has constant curvature.

We assume that g is an Einstein metric on M^4. Let $\bar{g}$ be geodesically equivalent to g. We consider the solution a_{ij} of (8) given by (6). Assume that the corresponding $\lambda_i \neq 0$ at p. We will show that in a small neighborhood of p the metric g has constant curvature implying the metrics $\bar{g}$ and $\hat{g}$ have constant curvature as well by Beltrami Theorem (see for example [38], or the original papers [4] and [50]).

Substituting equation (24) in (10), we obtain $a_{i\alpha} Z^\alpha_{jkl} + a_{\alpha j} Z^\alpha_{ikl} = 0$, where

$$Z^i_{jkl} = R^i_{jkl} - K \cdot (\delta^i_l g_{jk} - \delta^i_k g_{jl}). \tag{68}$$

We see that by construction the tensor Z_{ijkl} has the symmetries (50). Since g is Einstein, the tensor Z_{ijkl} satisfies $Z_{ijkl} g^{ik} = 0$.

By Lemma 6, at almost every point there exists an eigenvalue of a^i_j with geometric multiplicity one. Then, by Corollary 6, there exists a vector v^i such that $g(v, v) \neq 0$ and such that v^i lies in the kernel of Z. By Lemma 5, the tensor $Z \equiv 0$ implying in view of (68) the claim, □

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