On “All regular Landsberg metrics are always Berwald” by Z. I. Szabó

Vladimir S. Matveev

In his paper [2], Z. I. Szabó claimed (Theorem 3.1) that *all sufficiently smooth Landsberg Finsler metrics are Berwald*; this claim solves the long-standing “unicorn” problem. Unfortunately, as I explain below, the proof of the statement has a gap.

Following [2], let us consider a smooth $n-$dimensional manifold M with a proper Finsler metric $F : TM \to \mathbb{R}$. The second differential of $\frac{1}{2}F^2_{|T_xM}$ will be denoted by $g = g_{(x,y_x)}$ and should be viewed as a Riemannian metric on the punctured tangent space $T_xM - \{0\}$.

For a smooth curve $c(t)$ connecting two points $a, b \in M$, we denote by

$$\tau : T_aM \to T_bM, \quad \tau(a, \underbrace{y_a}_{\in T_aM}) = (b, \underbrace{\phi(y_a)}_{\in T_bM})$$

the Berwald parallel transport along the curve c. Following [1], Z. I. Szabó considers the following Riemannian metric $\mathbf{g}$ on M canonically constructed by F by the formula

$$\mathbf{g}_{(x)}(\xi, \eta) := \int\limits_{\substack{y_x \in T_xM \\ F(x,y_x) \le 1}} g_{(x,y_x)}(\xi, \eta) d\mu_{(x,y_x)} \tag{1}$$

where $\xi, \eta \in T_xM$ are two arbitrary vectors, and the volume form $d\mu$ on T_xM is given by $d\mu_{(x,y_x)} := \sqrt{\det(g_{(x,y_x)})}\, dy_x^1 \wedge \cdots \wedge dy_x^n$.

Z. I. Szabó claims that *if the Finsler metric F is Landsberg, the Berwald parallel transport preserves the Riemannian metric* $\mathbf{g}$. According to the definitions in Section

2 of [2], this claim means that for every $\xi, \eta, \nu \in T_aM$

$$\mathbf{g}_{(a)}(\xi, \eta) = \mathbf{g}_{(b)}(d_\nu \phi(\xi), d_\nu \phi(\eta)). \tag{2}$$

This claim is crucial for the proof; the remaining part of the proof is made of relatively simple standard arguments, and is correct. The claim itself is explained very briefly; basically Z. I. Szabó writes that, for Landsberg metrics, the unit ball $\{y_x \in T_xM \mid F(x, y_x) \leq 1\}$, the volume form $d\mu$, and the metric $g_{(x,y_x)}$ are preserved by the parallel transport, and, therefore, the metric $\mathbf{g}$ given by (1) must be preserved as well.

Indeed, for Landsberg metrics, the unit ball and the volume form $d\mu$ are preserved by the parallel transport. Unfortunately, it seems that the metric g is preserved in a slightly different way one needs to prove the claim.

More precisely, plugging (1) in (2), we obtain

$$\int\limits_{\substack{y_a \in T_aM \\ F(a,y_a) \leq 1}} g_{(a,y_a)}(\xi, \eta) d\mu_{(a,y_a)} = \int\limits_{\substack{y_b \in T_bM \\ F(b,y_b) \leq 1}} g_{(b,y_b)}(d_\nu \phi(\xi), d_\nu \phi(\eta)) d\mu_{(b,y_b)}. \tag{3}$$

As it is explained for example in Section 2 of [2], for every Finsler metric, the parallel transport preserves the unit ball:

$$\phi(\{y_a \in T_aM \mid F(a, y_a) \leq 1\}) = \{y_b \in T_bM \mid F(b, y_b) \leq 1\}. \tag{4}$$

The condition that F is Landsberg implies $\phi_* d\mu_{(a,y_a)} = d\mu_{(b,\phi(y_a))}$. Thus, Szabó's claim is trivially true if at every $y_a \in T_aM$

$$g_{(a,y_a)}(\xi, \eta) = g_{(b,\phi(y_a))}(d_\nu \phi(\xi), d_\nu \phi(\eta)). \tag{5}$$

But the condition that the metric is Landsberg means that

$$g_{(a,y_a)}(\xi, \eta) = g_{(b,\phi(y_a))}(d_{y_a} \phi(\xi), d_{y_a} \phi(\eta)) \tag{6}$$

only, i.e., (5) coincides with the definition of the Landsberg metric at the only point $y_a = \nu \in T_aM$.

Since no explanation why (3) holds is given in the paper, I tend to suppose that Z. I. Szabó oversaw the difference between the formulas (5) and (6); anyway, at the present point, the proof of Theorem 3.1 in [2] is not complete. Unfortunately, I could not get any explanation from Z. I. Szabó by email.

The unicorn problem remains open until somebody closes the gap, or presents another proof, or proves the existence of a counterexample; at the present point I can do neither of these.

Acknowledgement: I thank Deutsche Forschungsgemeinschaft (Priority Program 1154 — Global Differential Geometry) for partial financial support.

Note added in proof: After the paper was submitted, I have known that Z. I. Szabó has written a paper entitled *Correction to "All regular Landsberg metrics are Berwald"*, where he in particular accepts that his proof has a gap (he calls it flaw), precisely at the place I point out in the present paper.

References

[1] Z. I. Szabó: *Berwald metrics constructed by Chevalley's polynomials,* arXiv:math.DG/0601522(2006)

[2] Z. I. Szabó: *All regular Landsberg metrics are always Berwald,* Ann. Glob. Anal. Geom. **34**(2008), n. 4, 381–386

Vladimir S. Matveev, Mathematisches Institut, Friedrich-Schiller Universität Jena
07737 Jena, Germany, `matveev@minet.uni-jena.de`