Vanishing of the entropy pseudonorm for certain integrable systems

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Abstract

We introduce the notion of entropy pseudonorm for an action of $\mathbb{R}^n$ and prove that it vanishes for the group actions associated with a big class of integrable Hamiltonian systems.

1. Entropy pseudonorm

Let W be a smooth manifold and $\Phi : (\mathbb{R}^n, +) \to \text{Diff}(W)$ a smooth action on it. Assume there exists a compact Φ-invariant exhaustion of W. Define the following function on $\mathbb{R}^n$ (where h_{top} is the topological entropy):

$$\rho_\Phi(v) = h_{\text{top}}(\Phi(v)), \quad v \in \mathbb{R}^n.$$

This function is a pseudonorm on $\mathbb{R}^n$ ($\rho_\Phi(v)$ is well-defined because with our hypothesis the entropy h_d of [Bo] does not depend on the distance function d, homogeneity is standard and the triangle inequality follows from the Hu formula [H]). We call ρ_Φ the **entropy pseudonorm**.

We will investigate it in the case of the Poisson action corresponding to an integrable Hamiltonian system on a symplectic manifold (W^{2n}, ω). Namely, let (W^{2n}, ω) possess pair-wise Poisson commuting functions $I_1, I_2, \ldots, I_n$, which are functionally independent almost everywhere.

Denote by φ_i^τ the time τ shift along the Hamiltonian vector field of the function I_i. The maps φ_i^τ commute and therefore generate the Poisson action of the group $(\mathbb{R}^n, +)$,

$$\Phi(\tau_1, \ldots, \tau_n) \stackrel{\text{def}}{=} \varphi_1^{\tau_1} \circ \cdots \circ \varphi_n^{\tau_n} : W^{2n} \to W^{2n},$$

with the corresponding momentum map $\Psi = (I_1, \ldots, I_n) : W^{2n} \to \mathbb{R}^n$, see [A].

The entropy pseudonorm ρ_Φ vanishes in the following important cases:

– Williamson-Vey-Eliasson-Ito non-degenerate singularities [E, I];

– Taimanov non-degeneracy condition [T].

In the first case vanishing of topological entropy of the Hamiltonian flow was proved in [P2], in the second case in [T]. Since there is nothing special about the Hamiltonian in these situations, it can be changed to any of the integrals and $\rho_\Phi \equiv 0$ follows. Also in [P1, BP]) vanishing of h_{top} was proven for the cases:

– Systems integrable with periodic integrals;

– Collectively integrable systems (the definition is in [GS]).

It is not difficult to see that in both cases the entropy pseudonorm ρ_Φ vanishes as well.

Note that Liouville integrability does not imply vanishing of topological entropy, see [BT] (more examples in [Bu]). For these examples the entropy pseudonorm is degenerate, but it is possible to construct integrable examples [K] such that ρ_Φ is a norm.

In the present paper we prove vanishing of the entropy pseudonorm for another class of integrable systems. These systems were recently actively studied in mathematical physics in the framework of the theory of separation of variables. In different contexts they are called Benenti-systems [IMM], L-systems [B2, B3], cofactor systems [LR] or quasi-bi-hamiltonian systems [CST]. Benenti systems are certain integrable Hamiltonian systems on T^*M with the Hamiltonian of the form $H = K_g + V$, where $K_g : T^*M \to \mathbb{R}$, $K_g(x,p) = \langle p,p\rangle_g$, is twice the kinetic energy corresponding to a Riemannian metric g and $V : M \to \mathbb{R}$ is a potential. Important feature of these systems is that every integral is a sum of a function quadratic in momenta and a function on M. Moreover, the quadratic forms corresponding to the quadratic in momenta terms are simultaneously diagonalizable. We will provide precise definitions, the conditions on the metric g and potential V as well as formulas for integrals in Section 2.1, where we also explain how these systems are related to the theory of geodesic equivalence.

Theorem 1. *Let M be a compact connected manifold. Then the entropy pseudonorm of the action Φ associated with any Benenti integrable Hamiltonian system on T^*M vanishes: $\rho_\Phi \equiv 0$.*

For geodesic flows ($V \equiv 0$) degeneracy of $h_{\mathrm{top}}(H)$ was proven in our earlier paper [KM]. Theorem 1 generalizes the result of [KM] in the following two directions. First, it includes the potential energy in the picture. Second, it shows that the topological entropy of the Hamiltonian flow of every integral (not only the Hamiltonian) vanishes.

Outline of the proof will be presented in Section 2.2. We describe singular orbits of the Poisson action and show that the restriction of our integrable system to every singular orbit is a subsystem of a Benenti system on a manifold of smaller dimension. Then we apply induction in the dimension. Again as in the case of geodesic flows the set of singular points can be very complicated: If $n \stackrel{\mathrm{def}}{=} \dim(M) > 2$, then there exists a singular point over every point from M and the set of singular points in $\Psi^{-1}(c)$ can project to a fractal in M^n of Hausdorff dimension $> n-1$.

The class of mechanical systems covered by Theorem 1 contains Lagrange spinning tops, von Neumann system, Braden system, Bogoyavlensky systems, some Manakov systems and many other quadratically integrable (Stäckel) systems. For most of them our vanishing result is new.

Let us also discuss vanishing of other entropies. The well-known entropy for the group action $h_{\mathrm{top}}(\Phi)$ [C] vanishes for Benenti systems by elementary reasons: $h_{\mathrm{top}}(\Phi) = 0$. Actually, if this entropy is positive, then (directly from the definition) all entropies of sub-group actions are infinite, in particular $h_{\mathrm{top}}(H) = +\infty$, which is wrong.

However there is another definition of the entropy for group actions $h_U(\Phi)$, which behaves naturally w.r.t. restrictions to sub-group actions [HS]. Here U is the cube $[-1,1]^n \subset \mathbb{R}^n$ defining the strongly regular system $(U, 2U, 3U, \dots)$ exhausting our group $(\mathbb{R}^n, +)$. With respect to this definition and the action Φ of $\mathbb{R}^n$ associated to a Benenti integrable system we have:

$$h_U(\Phi) = 0.$$

This follows from the following inequalities:

$$\max_{1\le k\le n} h_{\mathrm{top}}(I_k) \le \sup_{v\in U} \rho_\Phi(v) \le h_U(\Phi) \le n \cdot \max_{1\le k\le n} h_{\mathrm{top}}(I_k).$$

Here the first inequality is obvious, the second is Proposition 2.6 from [HS] and the third follows easily from the definition of h_U. Thus vanishing of the entropy pseudonorm ρ_Φ is equivalent to vanishing of the entropy $h_U(\Phi)$.

2. Definitions and sketch of the proof

2.1 Benenti systems and geodesically equivalent metrics

Let g, $\bar g$ be two Riemannian metrics on a connected manifold M^n and $\flat_g : TM \to T^*M$, $\sharp^{\bar g} : T^*M \to TM$ be the corresponding bundle morphisms. We will consider the bundle morphism $\sharp^{\bar g} \circ \flat_g : TM \to TM$ as a (1,1)-tensor.

The metrics $g, \bar{g}$ are called **geodesically equivalent**, if every geodesic of $\bar{g}$, considered as an unparameterized curve, is a geodesic of g. They are said to be **strictly non-proportional** at $P \in M^n$, if the spectrum $\mathrm{Sp}(\sharp^{\bar{g}} \circ \flat_g) \subset \mathbb{R}_+$ is simple at P.

Consider the (1,1)-tensor $L \stackrel{\text{def}}{=} (\sharp^{\bar{g}} \circ \flat_g)/\sqrt[n+1]{\det(\sharp^{\bar{g}} \circ \flat_g)} : TM \to TM$. For every $t \in \mathbb{R}$, consider the $(1,1)$-tensor $S_t \stackrel{\text{def}}{=} \det(L - t\,\mathrm{Id})\,(L - t\,\mathrm{Id})^{-1}$. The family S_t is polynomial in t of degree $n-1$.

We will always identify the tangent and the cotangent bundle of M with the help of $\flat_g$. This identification gives us a symplectic form and a Poisson structure on TM.

Theorem 2 ([MT, M]). *If g, $\bar{g}$ are geodesically equivalent, then for all $t_1, t_2 \in \mathbb{R}$ the functions*

$$I_{t_i} : TM \to \mathbb{R}, \quad I_{t_i}(v) \stackrel{\text{def}}{=} g(S_{t_i}(v), v)$$

are commuting integrals for the geodesic flow of g.

If, in addition, the metrics are strictly non-proportional at one point, then it is so for almost every point. Consequently for all $t_1 < \cdots < t_n$ the integrals I_{t_i} are functionally independent almost everywhere so that the geodesic flow of g is Liouville integrable.

It is possible to add potential energy to the picture. In local coordinates, it was done in [B1] (see also [B2, B3, BM]); other approaches are in [IMM] and [CST].

Let g and $\bar{g}$ be geodesically equivalent Riemannian metrics on M^n. A smooth function $V : M^n \to \mathbb{R}$ will be called **compatible** with respect to g and $\bar{g}$, if the 1-form

$$dV \circ (L - \mathrm{trace}(L)\mathrm{Id})$$

is exact. For every pair of geodesically equivalent metrics, which are not affine equivalent, we can prove the existence of a nonconstant compatible V (actually of a continuum-dimensional family).

It is possible to show that if there exists a compatible function V, then there exists a family V_t, $t \in \mathbb{R}$, of smooth functions on M^n such that the following two conditions are fulfilled:

$$\begin{cases} V_t & \text{is polynomial in } t \text{ of degree} \le n-1, \\ dV \circ S_t = dV_t & \text{for every } t \in \mathbb{R}. \end{cases} \tag{1}$$

Potential V defines the family V_t up to (addition of) a constant polynomial $P(t)$ of degree $\le n-1$. Note that the family V_t also defines the function V up to a constant. In fact, the function V is the coefficient at t^{n-1}.

Locally, the existence of such V_t was explained in [B3]. From the normal form for the functions V_t, given in Theorem 4 below, it is clear that near generic points we have a lot of freedom in choosing the functions V and V_t: They depend on arbitrary n functions of one variable. Globally on M the existence of such V_t is nontrivial, for instance because the functions X_i from Theorem 4 can have singularities near the bifurcation points of the spectrum of L.

Theorem 3 ([B2, B3, CST, BM]). *Let g, $\bar{g}$ on a connected M^n be geodesically equivalent. Suppose V is compatible with respect to $g, \bar{g}$. Consider a family V_t of functions satisfying conditions (1). Then for all $t_1, t_2 \in \mathbb{R}$ the functions $\hat{I}_{t_i} \stackrel{\text{def}}{=} I_{t_i} + V_{t_i}$ are commuting integrals for the Hamiltonian system with the Hamiltonian $K_g + V$, where K_g is twice the kinetic energy corresponding to g. If, in addition, the metrics are strictly non-proportional at least at one point, then for all $t_1 < \cdots < t_n$ the integrals $\hat{I}_{t_i}$ are functionally independent almost everywhere.*

We will call a **Benenti system** the integrable system on TM^n generated by the integrals $\hat{I}_{t_1}, ..., \hat{I}_{t_n}$ from Theorem 3 assuming that the metrics g, $\bar{g}$ are strictly non-proportional at least at one (and hence at almost every) point.

2.2 Logic of the proof of Theorem 1

We use induction on the dimension. If dimension of the manifold is $n < 2$, Theorem 1 is trivial. Assume that for every dimension less than n Theorem 1 is true and consider $\dim M = n$.

Suppose the topological entropy of the Hamiltonian flow corresponding to an integral $\hat{I}_t$ is not zero. Then, by the variational principle, there exists an ergodic $\hat{I}_t$-invariant Borel probability measure μ such that $h_\mu(g) \neq 0$ [KH]. By ergodicity the support $\mathrm{Supp}(\mu)$ is contained in a connected component of some fiber $\Psi^{-1}(c)$. If $\mathrm{Supp}(\mu)$ contains a point P with $\mathrm{rank}(d\hat{I}_{t_1}, \dots, d\hat{I}_{t_n}) = n$, then the orbit of P is diffeomorphic to a cylinder over torus, $\Phi(\mathbb{R}^n, P) \simeq T^k \times \mathbb{R}^{n-k}$, $0 \leq k \leq n$ ([A]). By the implicit function theorem a small neighborhood of a point P in $\mathrm{Supp}(\mu)$ lies in the orbit $\Phi(\mathbb{R}^n, P)$. Since $\mathrm{Supp}(\mu)$ is a closed invariant subset and its point P cannot be wandering, the support is diffeomorphic to a sub-torus T^l, $l \leq k$, with the flow of $\hat{I}_t$ being conjugated to a standard linear flow. This implies that the entropy h_μ vanishes.

Now suppose that every point of $\mathrm{Supp}(\mu)$ is **singular**, so that $\mathrm{rank}(d\hat{I}_{t_1}, \dots, d\hat{I}_{t_n}) \leq k < n$ and rank $= k$ on the support μ-a.e. In this case, we can reduce the dimension. Namely, there exists a closed proper submanifold $N^k \subset M^n$ with the induced Benenti system and a subgroup $\mathbb{R}^k \subset \mathbb{R}^n$ with $\tilde{\Phi} = \Phi_{|\mathbb{R}^k}$ such that $\mathrm{Supp}(\mu)$ is $\tilde{\Phi}$-invariant and $\tilde{\Phi}_{|\,\mathrm{Supp}(\mu)}$ is a sub-system of the Poisson action corresponding to the Benenti system on N^k. Then $h_\mu = 0$ by the inductional assumption.

This is actually the main point of the proof. Precisely the same logic was used in [KM]. To a certain extent not only the statement, but also most of the proofs from [KM] can be generalized for our more general setting. In Sections 2.3, 2.4 we will explain how to construct these closed submanifolds under the additional assumption that all eigenvalues of L are non-constant. This additional assumption makes the proof much shorter (for instance, because in this case we can take $k = n-1$; in the paper [KM] the biggest part was dedicated to deal with constant eigenvalues of L), so that we can hope to make the main ideas of the proof clear to everyone.

2.3 Benenti systems and singular points in Levi-Civita coordinates

Theorem 4 (follows from [LC], [BM], [B3]). *Let g and $\bar{g}$ be geodesically equivalent Riemannian metrics on M^n. Suppose they are strictly non-proportional at $P \in M^n$. Let the function V be compatible with respect to g and $\bar{g}$, and suppose the functions V_t satisfy conditions (1).*

Then in a small neighborhood $U \subset M^n$ of P there exist coordinates (called Levi-Civita coordinates) such that the metrics g, $\bar{g}$ and the functions V_t are given by the formulas

$$ds_g^2 = \sum_{i=1}^{n} (-1)^{i-1} \prod_{j \neq i} (\lambda_j - \lambda_i)\, dx_i^2, \tag{2}$$

$$ds_{\bar{g}}^2 = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{\lambda_i \prod_\alpha \lambda_\alpha} \prod_{j \neq i} (\lambda_j - \lambda_i)\, dx_i^2, \tag{3}$$

$$V_t = \sum_{i=1}^{n} (-1)^{i-1} X_i \prod_{j \neq i} \frac{\lambda_j - t}{\lambda_j - \lambda_i}\,. \tag{4}$$

where, for every i, λ_i and X_i are functions of one variable x_i. If in a neigborhood of almost every point the metrics g, $\bar{g}$ and the functions V_t are given by (2,3,4), then the metrics are geodesically equivalent and the functions satisfy conditions (1) with respect to some compatible function V.

In the Levi-Civita coordinate system L is diagonal $\mathrm{Diag}(\lambda_1, ..., \lambda_n)$. We will always assume that at every point the eigenvalues λ_i of L are indexed according to their value, so that $\lambda_i(P) \leq \lambda_{i+1}(P)$ for every $P \in M^n$ and every $1 \leq i \leq n-1$.

We see that the metric g and the tensor L define the Levi-Civita coordinate system up to a shift of the origin and change of the direction of coordinate axes. Indeed, the vector $v_i \stackrel{\mathrm{def}}{=} \frac{\partial}{\partial x_i}$ is defined up to a sign by the conditions

$$\left\{ \begin{array}{rcl} L v_i & = & \lambda_i v_i, \\ g(v_i, v_i) & = & (-1)^{i-1} \prod_{j \neq i} (\lambda_j - \lambda_i). \end{array} \right.$$

In this coordinate system the integrals $\hat I_t = I_t + V_t$ (as the functions of the cotangent bundle) are given by

$$\hat I_t = \sum_{i=1}^{n} (-1)^{i-1}(p_i^2 + X_i) \prod_{j\neq i} \frac{\lambda_j - t}{\lambda_j - \lambda_i}. \tag{5}$$

Note that the functions X_i can be invariantly obtained from V_t and λ_i, namely

$$X_i(P) = (-1)^{i-1} V_{\lambda_i(P)}(P).$$

Let $\hat I'_t = \frac{d}{dt}\hat I_t$. For every t the function $\hat I'_t$ is a linear combination of the integrals $I_{t_1}, ..., I_{t_n}$.

Corollary 1. *Let the metrics g and $\bar g$ be geodesically equivalent on closed connected M and strictly non-proportional at $P \in M$. If a point $(P, \xi) \in TM$ is singular w.r.t. the Poisson action Φ, corresponding to the integrals $\hat I_{t_i} = I_{t_i} + V_{t_i}$, then $d_{(P,\xi)}\hat I_{\tilde\lambda_i} = 0$ for some i, where $\tilde\lambda_i \stackrel{\rm def}{=} \lambda_i(P)$. In addition, in Levi-Civita coordinates the $i^{\rm th}$ component of ξ vanishes: $\xi_i = 0$.*

Proof: Consider the function $\hat I_{\lambda_i(x)}$. Expressed in Levi-Civita coordinates on the cotangent bundle T^*M, it equals $(-1)^{i-1}\left(p_i^2 + X_i(x_i)\right)$, so that its differential is $(-1)^{i-1}\left(2p_i\, dp_i + X'_i(x_i)dx_i\right)$. On the other hand,

$$d\hat I_{\tilde\lambda_i} = d\hat I_{\lambda_i(x_i)} - \hat I'_{\tilde\lambda_i} d\lambda_i(x_i) = (-1)^{i-1} 2p_i dp_i + \left((-1)^{i-1}X'(x_i) - \lambda'_i(x_i)\, \hat I'_{\tilde\lambda_i}\right) dx_i.$$

Thus if a linear combination $\sum \mu_i d\hat I_{\tilde\lambda_i}$ vanishes, then for every $\mu_i \neq 0$ the corresponding $d\hat I_{\tilde\lambda_i}$ vanishes. Then its dp_i and dx_i components vanish, yielding $p_i = 0$ (which implies $\xi_i = 0$). □

2.4 Submanifolds $\mathcal{M}_i$ and Sing_i

In this section we assume that g and $\bar g$ are geodesically equivalent metrics on a closed connected M^n, that every eigenvalue λ_i of L is not constant and that the functions V_t satisfy (1) with respect to a compatible V. For every $i = 1, \dots, n-1$ denote

$$\mathrm{Reg}_i = \{x \in M : \ \lambda_i(x) \neq \lambda_{i+1}(x)\}$$

and for $i = 0$ let $\mathrm{Reg}_0 = M^n$. At every point of $\mathrm{Reg}_i \cap \mathrm{Reg}_{i-1}$ the eigenvalue λ_i is simple. In particular, at every point of $\mathrm{Reg} \stackrel{\rm def}{=} \cap_i \mathrm{Reg}_i$ the eigenvalues $\lambda_1, \dots, \lambda_n$ are mutually different.

For every $x \in \mathrm{Reg}_i \cap \mathrm{Reg}_{i-1}$ denote by $\mathcal{D}_i(x) \subset T_xM^n$ the subspace spanned by the eigenspaces corresponding to λ_j, $j \neq i$. The distribution $\mathcal{D}_i$ is smooth. By Theorem 4, it is integrable in $\mathrm{Reg}_i \cap \mathrm{Reg}_{i-1}$. Denote by $\mathcal{M}_i(P)$ its integral manifold containing $P \in M^n$ (beware, in [KM] we used the notations $D_{\mathsf{C}(i)}$ and $M_{\mathsf{C}(i)}$ instead of present $\mathcal{D}_i$ and $\mathcal{M}_i$). By Theorem 4 the functions λ_i and X_i are constant along $\mathcal{M}_i(P)$.

For every $i = 1, ..., n-1$ let $\bar\lambda_i = \frac{1}{2}(\max_{x\in M}\lambda_i + \min_{x\in M}\lambda_{i+1})$. By Corollary 1 from [M] for every point $P \in M^n$ we have $\lambda_i(P) \leq \bar\lambda_i \leq \lambda_{i+1}(P)$. Consider

$$\mathrm{Sing}_i \stackrel{\rm def}{=} \{P \in M^n : \ (\lambda_i(P) - \bar\lambda_i)(\lambda_{i+1}(P) - \bar\lambda_i) = 0\}.$$

In [M] (see Theorem 5 there) it was proven that if Sing_i is non-empty, then, under the assumption that all λ_i are not constant, it is a connected submanifold of codimension 1. Moreover, almost all points of Sing_i belong to Reg and the intersection $\mathrm{Sing}_i \cap \mathrm{Reg}_i$ ($\mathrm{Sing}_i \cap \mathrm{Reg}_{i+1}$ respectively) is a finite union of leaves $\mathcal{M}_i$ ($\mathcal{M}_{i+1}$ respectively).

In [KM] (see Lemma 2 there) we proved that if the function λ_i is not constant, then every $\mathcal{M}_i(P)$ is a closed submanifold or is a part of Sing_i or Sing_{i+1}. Combining this observation and Corollary 1, we obtain that the projection of every singular orbit belongs to some compact $\mathcal{M}_i$ or to one of Sing_i. Since our measure μ from §2.2 is ergodic, $\mathrm{Supp}(\mu)$ belongs to the closure of an orbit. Then the projection of $\mathrm{Supp}(\mu)$ belongs to a compact submanifold of smaller dimension.

The last step is to explain that the dynamics on $\mathrm{Supp}(\mu)$ is a subsystem of a certain Benenti system on this submanifold. Since almost every point of Sing_i belongs to $\mathcal{M}_i \cup \mathcal{M}_{i+1}$, it is sufficient to consider only $\mathcal{M}_i$. $\mathcal{M}_i \cap \mathrm{Reg}$ is dense in $\mathcal{M}_i$, so we can use Levi-Civita coordinates. Since $T^*\mathcal{M}_i$ is a symplectic submanifold of T^*M^n, the claim follows from the fact that restrictions of the integrals $\hat{I}_t$ to $T^*\mathcal{M}_i$ are linear combinations of the integrals of the induced Benenti system on $\mathcal{M}_i$. This latter can be checked in Levi-Civita coordinates using formula (5).

In fact, the family V_t is defined up to a constant polynomial of degree $n-1$. Since the function X_i is constant on $\mathcal{M}_i(P)$, without loss of generality we can assume that $X_i = 0$ on $\mathcal{M}_i(P)$. Since the coordinate p_i vanishes along $T^*\mathcal{M}_i(P)$ and the function λ_i is constant on $\mathcal{M}_i(P)$, the restriction of the integral $\hat{I}_t = I_t + V_t$ to $T^*\mathcal{M}_i(P)$ is equal to

$$\hat{I}_t = \sum_{k\neq i} (-1)^{k-1}(p_k^2 + X_k) \prod_{j\neq k} \frac{\lambda_j - t}{\lambda_j - \lambda_k} = (\lambda_i - t)\cdot \left[\sum_{k\neq i} \frac{(-1)^{k-1-\theta(k-i)}(p_k^2+X_k)}{|\lambda_i - \lambda_k|} \prod_{j\neq k,i} \frac{\lambda_j - t}{\lambda_j - \lambda_k}\right],$$

where $\theta(x)$ is the Heaviside function, so that $\psi(k) = k - \theta(k-i)$ enumerates $\{1,\dots,n\}\setminus\{i\}$.

By the direct calculation we check that for every t the above expression in square brackets is a linear combinations of the integrals $\hat{I}^{\mathrm{new}}_{\tau_1},\dots,\hat{I}^{\mathrm{new}}_{\tau_{n-1}}$ of the Benenti system corresponding to geodesically equivalent metrics $g^{\mathrm{new}} \overset{\mathrm{def}}{=} g_{|\mathcal{M}_i(P)}$, $\bar{g}^{\mathrm{new}} \overset{\mathrm{def}}{=} \lambda_i \bar{g}_{|\mathcal{M}_i(P)}$ on $\mathcal{M}_i(P)$ and the family $V^{\mathrm{new}}_t : \mathcal{M}_i(P) \to \mathbb{R}$, $V^{\mathrm{new}}_t \overset{\mathrm{def}}{=} \frac{1}{\lambda_i - t} V_{t|\mathcal{M}_i(P)}$. The facts that the metrics g^{new}, $\bar{g}^{\mathrm{new}}$ are geodesically equivalent and strictly non-proportional at least at one point and that the family V_t satisfies condition (1) follow from Theorem 4, because on $\mathcal{M}_i(P)$ the coefficient $|\lambda_k - \lambda_i|$ depends on x_k only and therefore can be "hidden" in the corresponding dx_k and X_k. Thus our system is a subsystem of a Benenti system and the induction hypothesis finishes the proof of Theorem 1.

3. Discussion

It is clear that all the difficulties with vanishing of entropies for integrable systems are due to a complicated singularity set (we explained essentially in §2.2 that the set of regular points bears no entropy), as positive-entropy examples of [BT, K, BP, Bu] demonstrate. In all good cases, where vanishing of the entropy was proven, some stratification of singularities was achieved, see e.g. [P1, P2, T, BP, KM]. We formulate here a scheme for most vanishing results.

We will consider systems on non-compact W^{2n}, but such that the variation principle holds. This is, for instance, the case when W^{2n} admits an exhaustion by compact invariant sets (other cases are discussed in [Pe]). Define the following Φ-invariant subsets of W^{2n}:

$$\Sigma_k = \{x \in W^{2n} \mid \mathrm{rank}(d_x\Phi) = k\}.$$

Theorem 5. *Suppose for every $k < n$ we can decompose $\Sigma_k = \Sigma_k^+ \cup \Sigma_k^-$, where Σ_k^+ is a closed invariant subset of Σ_k and Σ_k^- consists of non-recurrent points of $\Phi(v)$ for a.e. $v \in \mathbb{R}^n$. Let also the momentum map $\Psi : \Sigma_k^+ \to \mathbb{R}^n$ can be factorized to the composition of continuous maps $\pi_k : \Sigma_k^+ \to A_k$ to a Hausdorff space A_k and $\sigma_k : A_k \to \mathbb{R}^n$, such that each fiber $\Sigma_k^\alpha = \pi_k^{-1}(\alpha)$ is a Φ-invariant k-dimensional submanifold of W^{2n}. Then the entropy pseudonorm vanishes: $\rho_\Phi \equiv 0$.*

Proof: By the variational principle it suffices to prove $h_\mu(\Phi(v)) = 0$ for a.e. v and every $\Phi(v)$-invariant ergodic measure μ. By ergodicity Ψ is constant on the support of μ. Consequently it suffices to prove that for every k the system has zero entropy on $\Psi^{-1}(c)\cap\Sigma_k$. Since $\mathcal{R}(v)\cap\Sigma_k \subset \Sigma_k^+$, where $\mathcal{R}(v)$ denotes the set of $\Phi(v)$-recurrent points, it is enough to show vanishing of entropy on the set $\Sigma_k^+ \cap \Psi^{-1}(c)$.

But this set is foliated by strata Σ_k^α and hence μ should be supported on one connected component of it only. This components possesses a transitive Poisson $\mathbb{R}^k$-action and so is isomorphic to a torus T^k (it cannot be a cylinder $T^{k-l}\times\mathbb{R}^l$ because consists of recurrent points) with quasi-periodic dynamics and hence $h_\mu(\Phi(v)|\Sigma_k^+ \cap \Psi^{-1}(c)) = 0$ implying the claim. □

The hypotheses of Theorem 5 are satisfied for integrable systems with Williamson-Vey-Eliasson-Ito non-degenerate singularities or with Taimanov non-degeneracy condition. Our induction approach implies that the singularities of Benenti systems are also stratified in the manner of the theorem. It is feasible that a kind of good stratification is necessary for vanishing of the entropies.

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