# Quantum integrability of Beltrami-Laplace operator as geodesic equivalence.

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**Abstract**

Given two Riemannian metrics on a closed connected manifold $M^n$, we construct self-adjoint differential operators $\mathcal{I}_0$, $\mathcal{I}_1$,..., $\mathcal{I}_{n-1} : C^2(M^n) \to C^0(M^n)$ such that if the metrics have the same geodesics then the operators commute with the Beltrami-Laplace operator of the first metric and pairwise commute. If the operators commute and if they are linearly independent, then the metrics have the same geodesics.

## 1 Introduction.

Let $g$, $\bar{g}$ be two Riemannian metrics on the same manifold $M^n$. They are *geodesically equivalent* if they have the same geodesics considered as unparameterized curves.

Consider the Beltrami-Laplace operator for the metric $g$

$$\Delta := -\frac{1}{\sqrt{det(g)}}\frac{\partial}{\partial x^i}\sqrt{det(g)}g^{ij}\frac{\partial}{\partial x^j},$$

where $det(g)$ denotes the determinant of the matrix corresponding to the metric $g$.

Denote by $G : TM^n \to TM^n$ the fiberwise-linear mapping given by the tensor $g^{-1}\bar{g} = (g^{i\alpha}\bar{g}_{\alpha j})$. In invariant terms, for any $x_0 \in M^n$ the restriction of the mapping $G$ to the tangent space $T_{x_0}M^n$ is the linear transformation of $T_{x_0}M^n$ such that for any vectors $\xi,\ \nu \in T_{x_0}M^n$ the scalar product $g(G(\xi),\nu)$ of the vectors $G(\xi)$ and $\nu$ with respect to the metric $g$ is equal to the scalar product $\bar{g}(\xi,\nu)$ of the vectors $\xi$ and $\nu$ with respect to the metric $\bar{g}$. Consider the characteristic polynomial $det(G - \mu E) = c_0\mu^n + c_1\mu^{n-1} + ... + c_n$. The coefficients $c_1, .., c_n$ are smooth functions on $M^n$, and $c_0 \equiv (-1)^n$. Consider the fiberwise-linear mappings

$$S_0, S_1, ..., S_{n-1} : TM^n \to TM^n$$

given by the general formula

$$S_k \stackrel{\text{def}}{=} \left(\frac{det(g)}{det(\bar g)}\right)^{\frac{k+2}{n+1}} \sum_{i=0}^{k} c_i G^{k-i+1}.$$

We denote by $(S_k)^i_j$ the components of the tensor corresponding to the fiberwise-linear mapping $S_k$.

Consider the operators

$$\mathcal{I}_0,\ \mathcal{I}_1, ..., \mathcal{I}_{n-1} : C^2(M^n) \to C^0(M^n) \tag{1}$$

given by the general formula

$$\mathcal{I}_k := \frac{1}{\sqrt{det(g)}}\frac{\partial}{\partial x^i}(S_k)^i_\alpha \sqrt{det(g)} g^{\alpha j}\frac{\partial}{\partial x^j}. \tag{2}$$

*Remark 1.* In invariant terms the operators $\mathcal{I}_k$ are given by

$$\mathcal{I}_k(f) = \operatorname{div}\left(S_k\left(\operatorname{grad}(f)\right)\right),$$

where grad$(f)$ denotes the gradient $g^{i\alpha}\frac{\partial f}{\partial x^\alpha}$ of the function $f$ and div denotes the divergence.

*Remark 2.* The operator $\mathcal{I}_{n-1}$ is exactly the operator $\Delta$.

**Theorem 1.** *If the metrics $g$ and $\bar g$ on $M^n$ are geodesically equivalent then the operators $\mathcal{I}_k$ pairwise commute. In particular they commute with the Beltrami-Laplace operator $\Delta$.*

Theorem 1 is a 'quantum' version of the following theorem. Consider the functions $I_k : T^*M^n \to R$, $k = 0, 1, ..., n-1$, given by the formulae

$$I_k(x,p) \stackrel{\text{def}}{=} g^{\alpha j}(S_k)^i_\alpha p_i p_j.$$

In invariant terms the functions $I_k$ are as follows. Let us identify canonically the tangent and the cotangent bundles of $M^n$ by the metric $g$. Then the value of the function $I_k$ on a vector $\xi \in T_xM^n \cong T^*_xM^n$ is given by

$$I_k(x,\xi) \stackrel{\text{def}}{=} g(S_k(\xi),\xi). \tag{3}$$

Consider the standard symplectic form on $T^*M^n$. By geodesic flow of the metric $g$ we mean the Hamiltonian system on $T^*M^n$ with the Hamiltonian $H(x,\xi) = \frac{1}{2}g(\xi,\xi)$.

*Remark 3.* The function $\mathcal{I}_{n-1}$ is equal to $-2H$.

**Theorem 2 ([14]).** *If the metrics $g$ and $\bar{g}$ on $M^n$ are geodesically equivalent then the functions $I_k$ pairwise commute. In particular, they are integrals in involution of the geodesic flow of the metric $g$.*

For closed two-dimensional manifolds Theorem 1 was essentially proved in [5]. For two-dimensional manifolds Theorem 2 was proved in [7].

*Remark 4 ([14]).* Let metrics $g$, $\bar{g}$ be geodesically equivalent. If at a point $x_0 \in M^n$ the number of different roots of the polynomial $det(G - \mu E)$ is equal to $n_1$, then at almost all points of $T^*_{x_0}M^n$ the dimension of the linear space generated by $dI_0, dI_1, ..., dI_{n-1}$ is greater or equal than $n_1$.

In particular, if the characteristic polynomial $det(G - \mu E)$ has no multiple roots at the point $x_0 \in M^n$ then the functions $I_0, I_1, ..., I_{n-1}$ are functionally independent almost everywhere in $T^*U(x_0)$, where $U(x_0)$ is a sufficiently small neighborhood of $x_0$.

If at every point of a neighborhood $U(x_0)$ of the point $x_0 \in M^n$ the number of different eigenvalues of the polynomial $det(G - \mu E)$ is less or equal than $n_1$, then at all points of $T^*U(x_0)$ the dimension of the linear space generated by $dI_0, dI_1, ..., dI_{n-1}$ is less or equal than $n_1$.

The metrics $g, \bar{g}$ are *strictly non-proportional* at a point $x_0 \in M^n$ when the characteristic polynomial $det(G - \mu E)$ has no multiple roots at the point $x_0$.

**Corollary 1.** *Suppose $M^n$ is connected. Let metrics $g$, $\bar{g}$ on $M^n$ be geodesically equivalent and strictly non-proportional at least at one point of $M^n$. Then the metrics are strictly non-proportional almost everywhere.*

Corollary 1 follows from the following observation. If we have an integrable Hamiltonian systems then the dimension of the linear space generated by differentials of the integrals is constant along each orbit.

*Proof of Corollary 1.* Identify canonically the tangent and the cotangent bundles of $M^n$ by the metric $g$. Take a geodesic $\gamma : R \to M^n$ and consider the points $x_0 = \gamma(0)$, $x_1 = \gamma(1) \in M^n$. Suppose that the metric $\bar{g}$ is geodesically equivalent to the metric $g$ and is strictly non-proportional with $g$ at the point $x_0$. Let us prove that in each neighborhood of the point $x_1$ there exists a point $x$ such that the metrics are strictly non-proportional at $x$.

The geodesic orbit of the geodesic $\gamma$ is the curve $(\gamma, \dot{\gamma}) : R \to TM^n \cong T^*M^n$ (assuming $\dot{\gamma} = \dot{\gamma}(t) \in T_{\gamma(t)}M^n$ is the velocity vector of $\gamma$.) Take a small neighborhood $O^{2n} \subset TM^n$ of the point $(x_0, \dot{\gamma}(0)) \in TM^n$. Consider the union of all points of all geodesic orbits that start from the points of the neighborhood. Denote it by $U^{2n}$. Since the metrics are strictly non-proportional at $x_0$, the differentials $dI_0, dI_1, ..., dI_{n-1}$ of the integrals $I_0, I_1, I_2, ..., I_{n-1}$ are linearly independent at almost all points of $U^{2n}$. Evidently the set $U^{2n}$ contains a neighborhood of the point $(x_1, \dot{\gamma}(1))$. The integrals $I_0, I_1, ..., I_{n-1}$ are then functionally independent at almost all points of a neighborhood of the point

$(x_1, \dot{\gamma}(1))$. Using Remark 4, we have that in each neighborhood of $x_1$ there exists a point where the metrics are strictly non-proportional. Since each two points of a connected manifold can be joint by a sequence of geodesic segments, in each neighborhood of an arbitrary point of $M^n$ there exist points where the metrics are strictly non-proportional, q. e. d.

More general statement is also true.

**Corollary 2.** *Suppose $M^n$ is connected and the metrics $g,\ \bar{g}$ on $M^n$ are geodesically equivalent. If at every point of a neighborhood $U \subset M^n$ the number of different eigenvalues of the polynomial $det(G - \mu E)$ is less or equal than $n_1$, then at every point of $M^n$ the number of different eigenvalues of the polynomial $det(G - \mu E)$ is less or equal than $n_1$.*

A 'quantum' version of Corollaries 1,2 is as follows.

**Corollary 3.** *Let metrics $g,\ \bar{g}$ on a connected manifold $M^n$ be geodesically equivalent. They are strictly non-proportional at least at one point of $M^n$, if and only if the operators $\mathcal{I}_0, \mathcal{I}_1, ..., \mathcal{I}_{n-1}$ are linearly independent. More generally, the dimension of the space generated by the operators is equal to $n_1$, if and only if the number of different eigenvalues of $G$ is equal to $n_1$ at almost every point of $M^n$.*

Now suppose $M^n$ is closed (= compact + without boundary) and connected. Let metrics $g,\ \bar{g}$ on $M^n$ be geodesically equivalent. Suppose the metrics are strictly non-proportional at least at one point of the manifold. Then we have complete Liouville integrability for the geodesic flow of the metric $g$, and complete quantum integrability for the Beltrami-Laplace operator of the metric $g$. Liouville integrability means in particular that almost all connected components of level submanifolds $\{I_0 = C_0, I_1 = C_1, ..., I_{n-1} = C_{n-1}\}$, where $C_0, C_1, ..., C_{n-1}$ are constants, are $n$-dimensional tori, and orbits on such tori are windings.

Quantum integrability means that there exists a countable basis

$$\Phi = \{f_1, f_2, ..., f_m, ...\}$$

of the space $L_2(M^n)$ such that each $f_m$ is an eigenfunction of each operator $\mathcal{I}_k$.

Moreover, in our case the variables can be separated. More precisely, take any function $f$ from the basis $\Phi$. Since $f$ is an eigenfunction of each operator $\mathcal{I}_k$, we have that $f$ is a solution of the system of $n$ partial differential equations

$$\mathcal{I}_k f = \lambda_k f, \quad k = 0, 1, ..., n-1. \tag{4}$$

The separation of variables means that in a neighborhood of almost any point there exist coordinates $(x^0, x^1, ..., x^{n-1})$ such that in these coordinates the system (4) is equivalent to the system

$$\frac{\partial^2}{(\partial x^k)^2} f = F_k(x^k, \lambda_0, \lambda_1, ..., \lambda_{n-1}) f, \quad k = 0, 1, ..., n-1, \tag{5}$$

where for each $k \in \{0, 1, ..., n-1\}$ the function $F_k$ depends on the variable $x^k$ and on the parameters $\lambda_0, \lambda_1, ..., \lambda_{n-1}$ only. Then $f$ is the product

$$X_0(x^0)X_1(x^1)...X_{n-1}(x^{n-1}),$$

and for each $k \in \{0, 1, ..., n-1\}$ the function $X_k$ is a solution of the $k$-th equation of (5) so that we reduced the system of partial differential equations (4) to the system of ordinary differential equations

$$\frac{\partial^2}{(\partial x^k)^2} X_k(x^k) = F_k(x^k, \lambda_0, \lambda_1, ..., \lambda_{n-1}) X_k(x^k), \quad k = 0, 1, ..., n-1. \qquad (6)$$

Does the Liouville integrability of the geodesic flow imply the existence of a geodesically equivalent metric? Let $g, \bar{g}$ be two metrics on $M^n$. Consider the functions $I_k : T^*M^n \to R$, $k = 0, ..., n-1$, given by (3). Consider the standard symplectic structure on $T^*M^n$.

**Theorem 3.** *Let the functions $I_k$ commute and let them be functionally independent almost everywhere. Then the metrics $g$, $\bar{g}$ are geodesically equivalent.*

Consider the operators $\mathcal{I}_k : C^2(M^n) \to C^0(M^n)$, $k = 0, ..., n-1$, given by (1).

**Corollary 4.** *Suppose the manifold $M^n$ is connected. Let the operators $\mathcal{I}_k$ commute and let them be linearly independent. Then the metrics $g, \bar{g}$ are geodesically equivalent.*

Let the functions $I_k$ commute. If they are not functionally independent then the metrics may be not geodesically equivalent.

Consider the tangent bundle $TM^n$. Consider the fiberwise-linear mapping $G$. The restriction of $G$ to any tangent space $T_{x_0}M^n$ is evidently self-adjoint (with respect to both metrics); then all eigenvalues of $G$ are real, and we have a decomposition of any tangent space $T_xM^n$ to the direct sum $V_x^1 \oplus V_x^2 \oplus ... \oplus V_x^{n_1}$ of eigenspaces.

We will call the set of distributions

$$V_x^1, V_x^2, ..., V_x^{n_1} \subset T_xM^n, \quad V_x^1 \oplus V_x^2 \oplus ... \oplus V_x^{n_1} = T_xM^n$$

*integrable*, if at a neighborhood of almost every points of $M^n$ there exists a coordinate system $x^1, x^2, ..., x^n$ such that the vectors tangent to the coordinate lines lie in the elements of the distribution.

**Theorem 4.** *Let the functions $I_k$ commute and let the set of distributions of the eigenspaces of $G$ be integrable. Then the metrics $g$, $\bar{g}$ are geodesically equivalent.*

Theorems 1,3 show that, if the geodesic flow of a metric $g$ is completely integrable in the classical sense, and if the integrals can be chosen to be in the form (3) then the geodesic flow is also integrable in the quantum sense. In other words, in our case, classical integrability implies quantum integrability. Is it possible to generalize the statement for a bigger class of integrable geodesic flows?

It was observed that there exists an integrable quantum version for every known integrable in the classical sense system. For example, it is possible to extract from the book [6], that if the geodesic flow is completely integrable, and if the integrals are quadratic in momenta and simultaneously diagonalizable then it is possible to find a set of commutative operators such that the symbols of these operators coincide with the integrals. But, in the general situation, these operators do not contain the Beltrami-Laplace operator of the metric $g$.

Are there many interesting examples of geodesically equivalent metrics on closed manifolds? The following theorem (essentially due to N. S. Sinjukov [11]) gives us a construction that, given a pair of geodesically equivalent metrics, produces another pair of geodesically equivalent metrics. Starting from the metric of constant curvature on the sphere, we obtain the metric of the ellipsoid and the metric of the Poisson sphere.

Let $g, \bar g$ be Riemannian metrics on the manifold $M^n$. Consider the fiberwise-linear mapping $B : TM^n \to TM^n$ given by $B^i_j = \left(\frac{det(\bar g)}{det(g)}\right)^{\frac{1}{n+1}} \bar g^{i\alpha} g_{\alpha j}$. By definition, let us put metric $g_B$ equal to $g_{i\alpha}B^\alpha_j$ and put metric $\bar g_B$ equal to $\bar g_{i\alpha}B^\alpha_j$. In invariant terms, the scalar product $g_B(\xi,\nu)$ of arbitrary vectors $\xi,\nu \in T_{x_0}M^n$ is equal to $g(B\xi,\nu)$ and the scalar product $\bar g_B(\xi,\nu)$ is equal to $\bar g(B\xi,\nu)$. Evidently, the restriction of $B$ to any tangent space $T_{x_0}M^n$ is self-adjoint with respect to the metrics $g$ and $\bar g$ and therefore the metrics $g_B$, $\bar g_B$ are well-definite.

**Theorem 5.** *Metrics $g$ and $\bar g$ are geodesically equivalent, if and only if the metrics $g_B$ and $\bar g_B$ are geodesically equivalent.*

Evidently, the metrics $g$ and $\bar g$ are strictly non-proportional at a point $x \in M^n$, if and only if the metrics $g_B$ and $\bar g_B$ are strictly non-proportional at $x$.

Thus if we have a pair of geodesically equivalent metrics $g$, $\bar g$, then we can construct the other pair of geodesically equivalent metrics $g_B$, $\bar g_B$. We can apply the construction once more, the result is another pair of geodesically equivalent metrics. It is natural to denote it by $g_{B^2}$, $\bar g_{B^2}$ since this pair is given by

$$g_{B^2}(\xi,\nu) = g(B^2\xi,\nu), \quad \bar g_{B^2}(\xi,\nu) = \bar g(B^2\xi,\nu).$$

We can go in other direction and consider the metrics $g_{B^{-1}}$, $\bar g_{B^{-1}}$ given by

$$g_{B^{-1}}(\xi,\nu) = g(B^{-1}\xi,\nu), \quad \bar g_{B^{-1}}(\xi,\nu) = \bar g(B^{-1}\xi,\nu).$$

They are geodesically equivalent as well.

To start the process, we need a pair of geodesically equivalent metrics $g$, $\bar{g}$. We take the following one (obtained by E. Beltrami [1], [2]). The metric $g$ is the restriction of the Euclidean metric

$$(dx^1)^2 + (dx^2)^2 + ... + (dx^{n+1})^2$$

to the standard sphere

$$S^n = \left\{(x^1, x^2, ..., x^{n+1}) \in R^{n+1} : (x^1)^2 + (x^2)^2 + ... + (x^{n+1})^2 = 1\right\}.$$

The metric $\bar{g}$ is the pull-back $l^*g$, where the diffeomorphism $l : S^n \to S^n$ is given by

$$l(x) \stackrel{\text{def}}{=} \frac{Ax}{\|Ax\|},$$

where $A$ is an arbitrary non-degenerate linear transformation of $R^{n+1}$, $Ax$ means 'the transformation $A$ applied to the vector $x = (x^1, x^2, ..., x^{n+1})$', and $\|x\|$ is the standard norm $\sqrt{(x^1)^2 + (x^2)^2 + ... + (x^{n+1})^2}$.

Since the mapping $l$ evidently takes the geodesics of the metric $g$ to the geodesics of the metric $g$, the metrics $g$, $\bar{g}$ are geodesically equivalent. For these geodesically equivalent metrics $g$ and $\bar{g}$, the metric $g_B$ is the metric of an ellipsoid, and the metric $\bar{g}_{B^2}$ is the metric of a Poisson sphere. By varying the linear transformation $A$, we can obtain metrics of all possible ellipsoids and all possible Poisson spheres.

Recall that the metric of the ellipsoid

$$E^n \stackrel{\text{def}}{=} \left\{(x^1, x^2, ..., x^{n+1}) \in R^{n+1} : \frac{(x^1)^2}{a_1} + \frac{(x^2)^2}{a_2} + ... + \frac{(x^{n+1})^2}{a_{n+1}} = 1\right\}$$

is the restriction of the metric

$$(dx^1)^2 + (dx^2)^2 + ... + (dx^{n+1})^2$$

to the ellipsoid $E^n$. By the metric of the Poisson sphere we, following [3], mean the restriction of the metric

$$\frac{1}{\frac{(x^1)^2}{a_1^2} + \frac{(x^2)^2}{a_2^2} + ... + \frac{(x^{n+1})^2}{a_{n+1}^2}} \left((dx^1)^2 + (dx^2)^2 + ... + (dx^{n+1})^2\right)$$

to the ellipsoid $E^n$.

The metric of the Poisson sphere has the following mechanical sense. Consider the free motion of an (n+1)-dimensional rigid body in the (n+1)-dimensional space around a fixed point. The configuration space of the corresponding Hamiltonian system is $SO(n+1)$, and the corresponding Hamiltonian is left-invariant. Consider the embedding $SO(n) \to SO(n+1)$ as the stabilizer of a vector $v \in R^{n+1}$. Consider the action of the group $SO(n)$ on $SO(n+1)$ by left translations. The Hamiltonian of the motion is evidently invariant modulo this action, and the reduced system on $T^*SO(n+1)/SO(n) \cong T^*S^n$ is the geodesic flow of the (appropriate) Poisson metric on the sphere $S^n$, see [3] for details.

**Theorem 6 ([14], independently obtained by S. Tabachnikov [12]).** *The restriction of the Euclidean metric*

$$(dx^1)^2 + (dx^2)^2 + ... + (dx^{n+1})^2$$

*to the ellipsoid*

$$E^n = \left\{ (x^1, x^2, ..., x^{n+1}) \in R^{n+1} : \frac{(x^1)^2}{a_1} + \frac{(x^2)^2}{a_2} + ... + \frac{(x^{n+1})^2}{a_{n+1}} = 1 \right\}$$

*is geodesically equivalent to the restriction of the metric*

$$\frac{1}{\left(\frac{x^1}{a_1}\right)^2 + \left(\frac{x^2}{a_2}\right)^2 + ... + \left(\frac{x^{n+1}}{a_{n+1}}\right)^2} \left( \frac{(dx^1)^2}{a_1} + \frac{(dx^2)^2}{a_2} + ... + \frac{(dx^{n+1})^2}{a_{n+1}} \right) \quad (7)$$

*to the same ellipsoid.*

**Theorem 7 (Topalov, [15]).** *The restriction of the metric*

$$\frac{1}{\frac{(x^1)^2}{a_1^2} + \frac{(x^2)^2}{a_2^2} + ... + \frac{(x^{n+1})^2}{a_{n+1}^2}} \left((dx^1)^2 + (dx^2)^2 + ... + (dx^{n+1})^2\right) \quad (8)$$

*to the ellipsoid*

$$E^n = \left\{ (x^1, x^2, ..., x^{n+1}) \in R^{n+1} : \frac{(x^1)^2}{a_1} + \frac{(x^2)^2}{a_2} + ... + \frac{(x^{n+1})^2}{a_{n+1}} = 1 \right\}$$

*is geodesically equivalent to the restriction of the metric*

$$a_1(dx^1)^2 + a_2(dx^2)^2 + ... + a_{n+1}(dx^{n+1})^2 - (x^1dx^1 + x^2dx^2 + ... + x^{n+1}dx^{n+1})^2 \quad (9)$$

*to the same ellipsoid.*

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## 2 Operators $\mathcal{I}_k$ in Levi-Civita coordinates.

Let $g$ and $\bar{g}$ be geodesically equivalent metrics on $M^n$. Consider the fiberwise-linear mapping $G : TM^n \to TM^n$, $G^i_j = g^{i\alpha}\bar{g}_{\alpha j}$. Suppose that the number of different eigenvalues of $G$ is equal to $m \leq n$ at every point of an open domain $D \subset M^n$. Denote by $\rho_1, ..., \rho_m$ the eigenvalues of $G$. Let $k_i$ be the multiplicity of the eigenvalue $\rho_i$.

In the paper [4], Levi-Civita proved that for every point $P \in D$ there is an open neighborhood $U(P) \subset D$ and a coordinate system $\bar{x} = (\bar{x}_1, ..., \bar{x}_m)$ (in $U(P)$), where $\bar{x}_i = (x_i^1, ..., x_i^{k_i})$, $(1 \leq i \leq m,\ k_1 + k_2 + ... + k_m = n)$, such that the quadratic forms of the metrics $g$ and $\bar{g}$ have the following form:

$$\begin{aligned} g(\dot{\bar{x}}, \dot{\bar{x}}) &= \Pi_1 A_1(\bar{x}_1, \dot{\bar{x}}_1) + \ \Pi_2 A_2(\bar{x}_2, \dot{\bar{x}}_2) + \cdots + \\ &+ \ \Pi_m A_m(\bar{x}_m, \dot{\bar{x}}_m), \end{aligned} \tag{10}$$

$$\begin{aligned} \bar{g}(\dot{\bar{x}}, \dot{\bar{x}}) &= \rho_1 \Pi_1 A_1(\bar{x}_1, \dot{\bar{x}}_1) + \rho_2 \Pi_2 A_2(\bar{x}_2, \dot{\bar{x}}_2) + \cdots + \\ &+ \ \rho_m \Pi_m A_m(\bar{x}_m, \dot{\bar{x}}_m), \end{aligned} \tag{11}$$

where $A_i(\bar{x}_i, \dot{\bar{x}}_i)$ are positive-definite quadratic forms in the velocities $\dot{\bar{x}}_i$ with coefficients depending on $\bar{x}_i$, the functions $\Pi_i$ and $\rho_i$ are given by

$$\begin{aligned} \Pi_i &\stackrel{\text{def}}{=} (\phi_i - \phi_1)(\phi_i - \phi_2) \cdots (\phi_i - \phi_{i-1})(\phi_{i+1} - \phi_i) \cdots (\phi_m - \phi_i) \\ \rho_i &\stackrel{\text{def}}{=} \frac{1}{\phi_1 \phi_2 ... \phi_m} \frac{1}{\phi_i}, \end{aligned}$$

where $\phi_1, \phi_2, ..., \phi_m$, $0 < \phi_1 < \phi_2 < ... < \phi_m$, are smooth functions such that

$$\phi_i = \begin{cases} \phi_i(\bar{x}_i), & \text{if} \quad k_i = 1 \\ \text{constant}, & \text{else.} \end{cases} \tag{12}$$

It is easy to see that the functions $\phi_i$ as functions of $\rho_i$ are given by

$$\phi_i = \frac{1}{\rho_i} (\rho_1 \rho_2 ... \rho_m)^{\frac{1}{m+1}}$$

We will call these coordinates *Levi-Civita coordinates.*

It is convenient to put

$$\begin{aligned} \tilde{\Pi}_i &\stackrel{\text{def}}{=} (\phi_i - \phi_1)^{k_1} (\phi_i - \phi_2)^{k_2} \cdots (\phi_i - \phi_{i-1})^{k_{i-1}} (\phi_{i+1} - \phi_i)^{k_{i+1}} \cdots (\phi_m - \phi_i)^{k_m} \\ \tilde{A}_i &\stackrel{\text{def}}{=} \frac{A_i}{(\phi_i - \phi_1)^{k_1 - 1} \cdots (\phi_i - \phi_{i-1})^{k_{i-1} - 1} (\phi_{i+1} - \phi_i)^{k_{i+1} - 1} \cdots (\phi_m - \phi_i)^{k_m - 1}} \end{aligned}$$

so that for each $i \in \{1, 2, ..., m\}$

$$\Pi_i A_i = \tilde{\Pi}_i \tilde{A}_i.$$

Evidently $(\phi_i - \phi_j)^{k_j - 1}$ is independent of $\bar{x}_j$. Hence the coefficients of $\tilde{A}_i$ depend only on $\bar{x}_i$.

Denote by $(\tilde{A}_i)_{\alpha\beta}$, $\alpha, \beta \in 1, 2, ..., k_i$ the tensor corresponding to the quadratic form $\tilde{A}_i$ and by $(\tilde{A}_i)^{\alpha\beta}$, $\alpha, \beta \in 1, 2, ..., k_i$ the inverse tensor. Denote by

$$\tilde{\mathcal{A}}_i : C^2(M^n) \to C^0(M^n)$$

the differential operator

$$\sum_{\alpha,\beta=1}^{k_i} \frac{1}{\sqrt{det(\tilde{A}_i)}} \frac{\partial}{\partial x_i^\alpha} \sqrt{det(\tilde{A}_i)} (\tilde{A}_i)^{\alpha\beta} \frac{\partial}{\partial x_i^\beta}.$$

Denote by $\sigma_p$ the elementary symmetric polynomial of degree $p$ of $n$ variables

$$\underbrace{\phi_1,...,\phi_1}_{k_1}, \underbrace{\phi_2,...,\phi_2}_{k_2}, ..., \underbrace{\phi_m,...,\phi_m}_{k_m}.$$

Denote by $\sigma_p(\check{\phi}_i)$ the elementary symmetric polynomial of degree $p$ of $n-1$ variables

$$\underbrace{\phi_1,...,\phi_1}_{k_1}, \underbrace{\phi_2,...,\phi_2}_{k_2}, ..., \underbrace{\phi_i,...,\phi_i}_{k_i-1}, ..., \underbrace{\phi_m,...,\phi_m}_{k_m}.$$

In particular, if $m=n$ so that all $k_i$ are equal to one then $\sigma_p(\check{\phi}_i)$ is the elementary symmetric polynomial of degree $p$ of variables $\phi_1, \phi_2, ..., \phi_{i-1}, \phi_{i+1}, ..., \phi_n$.

**Lemma 1.** *In Levi-Civita coordinates the operators $\mathcal{I}_k$ are given by*

$$\mathcal{I}_k = (-1)^{n-k} \sum_{i=1}^{m} \frac{\sigma_{n-k-1}(\check{\phi}_i)}{\tilde{\Pi}_i} \tilde{\mathcal{A}}_i. \tag{13}$$

**Corollary 5.** *In Levi-Civita coordinates the functions $I_k$ (as functions on $TM^n \cong T^*M^n$) are given by*

$$I_k(x,\dot{\bar{x}}) = (-1)^{n-k} \sum_{i=1}^{m} \frac{\sigma_{n-k-1}(\check{\phi}_i)}{\tilde{\Pi}_i} \tilde{A}_i(\bar{x}_i, \dot{\bar{x}}_i). \tag{14}$$

In order to prove Lemma 1, we need the following technical lemma.

Let $G$ be the diagonal matrix $\mathrm{diag}(\rho_1, \rho_2, ..., \rho_n)$, where

$$\rho_i = \frac{1}{\phi_1\phi_2\phi_3...\phi_n} \frac{1}{\phi_i}.$$

Consider the characteristic polynomial $det(G-\mu E) = c_0\mu^n + c_1\mu^{n-1} + ... + c_n$ of the matrix $G$. Consider the matrixes $S_0, S_1, ..., S_{n-1}$ given by the general formula

$$S_k \stackrel{\mathrm{def}}{=} \left(\frac{1}{det(G)}\right)^{\frac{k+2}{n+1}} \sum_{i=0}^{k} c_i G^{k-i+1}.$$

**Lemma 2.** $S_k = (-1)^{n-k}\mathrm{diag}\left(\sigma_{n-k-1}(\check{\phi}_1), \sigma_{n-k-1}(\check{\phi}_2), ..., \sigma_{n-k-1}(\check{\phi}_n)\right).$

*Proof.* It is easy to see that the coefficients $c_k$ are given by

$$c_k = (-1)^{n-k} \frac{\sigma_{n-k}}{(\phi_1 \phi_2 ... \phi_n)^{k+1}}.$$

In particular,

$$det(G) = c_n = \frac{1}{(\phi_1 \phi_2 ... \phi_n)^{n+1}}.$$

Let us check the lemma for $k = 0$. We have

$$\begin{aligned} S_0 &= \left(\frac{1}{det(G)}\right)^{\frac{2}{n+1}} c_0 G \\ &= (-1)^n (\phi_1 \phi_2 ... \phi_n)^2 \mathrm{diag}\left(\frac{1}{\phi_1(\phi_1 \phi_2 ... \phi_n)}, \frac{1}{\phi_2(\phi_1 \phi_2 ... \phi_n)}, ..., \frac{1}{\phi_n(\phi_1 \phi_2 ... \phi_n)}\right) \\ &= (-1)^n \mathrm{diag}(\phi_2 \phi_3 ... \phi_n, \phi_1 \phi_3 ... \phi_n, ..., \phi_1 \phi_2 ... \phi_{n-2} \phi_n, \phi_1 \phi_2 ... \phi_{n-2} \phi_{n-1})) \\ &= (-1)^n \mathrm{diag}\left(\sigma_{n-1}(\check{\phi}_1), \sigma_{n-1}(\check{\phi}_2), ..., \sigma_{n-1}(\check{\phi}_n)\right). \end{aligned}$$

Suppose that the lemma is true for $S_{k-1}$. Then for $S_k$ we have

$$\begin{aligned} S_k &= \left(\frac{1}{det(G)}\right)^{\frac{1}{n+1}} G \left(S_{k-1} + \left(\frac{1}{det(G)}\right)^{\frac{k+1}{n+1}} c_k E\right) \\ &= \mathrm{diag}\left(\frac{1}{\phi_1}, \frac{1}{\phi_2}, ..., \frac{1}{\phi_n}\right) \mathrm{diag}\left((-1)^{n-k}(\sigma_{n-k} - \sigma_{n-k}(\check{\phi}_1)),\right. \\ &\quad \left.(-1)^{n-k}(\sigma_{n-k} - \sigma_{n-k}(\check{\phi}_2)), ..., (-1)^{n-k}(\sigma_{n-k} - \sigma_{n-k}(\check{\phi}_n))\right). \end{aligned}$$

Using that $(\sigma_l - \sigma_l(\check{\phi}_i)) = \phi_i \sigma_{l-1}(\check{\phi}_i)$, we obtain that $S_k$ is equal to

$$(-1)^{n-k} \mathrm{diag}\left(\sigma_{n-k-1}(\check{\phi}_1), \sigma_{n-k-1}(\check{\phi}_2), ..., \sigma_{n-k-1}(\check{\phi}_n)\right),$$

q. e. d.

*Proof of Lemma 1.* We have

$$\mathcal{I}_k = \frac{1}{\sqrt{det(g)}} \frac{\partial}{\partial x^i} (S_k)^i_\alpha \sqrt{det(g)} g^{\alpha j} \frac{\partial}{\partial x^j}.$$

In Levi-Civita coordinates the matrix of $G$ is diagonal

$$\mathrm{diag}\left(\underbrace{\rho_1, ..., \rho_1}_{k_1}, \underbrace{\rho_2, ..., \rho_2}_{k_2}, ..., \underbrace{\rho_m, ..., \rho_m}_{k_m}\right).$$

Using Lemma 2, we have that the matrix of $S_k$ is diagonal

$$S_k = (-1)^{n-k} \mathrm{diag}\left(\underbrace{\sigma_{n-k-1}(\check{\phi}_1), ..., \sigma_{n-k-1}(\check{\phi}_1)}_{k_1}, ..., \underbrace{\sigma_{n-k-1}(\check{\phi}_m), ..., \sigma_{n-k-1}(\check{\phi}_m)}_{k_m}\right).$$

In view of (12), $\sigma_{n-k-1}(\check{\phi}_\alpha)$ is independent of $\bar{x}_\alpha$ so that

$$\mathcal{I}_k = (-1)^{n-k}\sum_{p=1}^{m}\left(\sum_{\alpha,\beta=1}^{k_p}\frac{1}{\sqrt{det(g)}}\sigma_{n-k-1}(\check{\phi}_p)\frac{\partial}{\partial x_p^\alpha}\sqrt{det(g)}\frac{1}{\tilde{\Pi}_p}(\tilde{A}_p)^{\alpha\beta}\frac{\partial}{\partial x_p^\beta}\right).$$

Let us calculate the contribution

$$\sum_{\alpha,\beta=1}^{k_p}\frac{1}{\sqrt{det(g)}}\sigma_{n-k-1}(\check{\phi}_p)\frac{\partial}{\partial x_p^\alpha}\sqrt{det(g)}\frac{1}{\tilde{\Pi}_p}(\tilde{A}_p)^{\alpha\beta}\frac{\partial}{\partial x_p^\beta} \tag{15}$$

of the p-th block to $\mathcal{I}_k$. Let us show that

$$\frac{\sqrt{det(g)}}{\tilde{\Pi}_p}\frac{1}{\sqrt{det(\tilde{A}_p)}} \tag{16}$$

does not depend on $\bar{x}_p$. Evidently,

$$det(g) = det(\tilde{A}_1)det(\tilde{A}_2)...det(\tilde{A}_m)\tilde{\Pi}_1^{k_1}\tilde{\Pi}_2^{k_2}...\tilde{\Pi}_m^{k_m}.$$

Since $\tilde{A}_q$ is independent of $\bar{x}_p$ for $p \neq q$, we must demonstrate that

$$\frac{\sqrt{\tilde{\Pi}_1^{k_1}\tilde{\Pi}_2^{k_2}...\tilde{\Pi}_m^{k_m}}}{\tilde{\Pi}_p} \tag{17}$$

is independent of $\bar{x}_p$.

If $k_p$ is greater than one then $\phi_p$ is constant and the statement is trivial. Let $k_p$ be one. Then for $q \neq p$ the only term in $\tilde{\Pi}_q$ depending on $\bar{x}_p$ is $\pm(\phi_p - \phi_q)$. Using that

$$\tilde{\Pi}_p = (\phi_p - \phi_1)^{k_1}\cdots(\phi_p - \phi_{p-1})^{k_{p-1}}(\phi_{p+1} - \phi_p)^{k_{p+1}}\cdots(\phi_m - \phi_p)^{k_m}$$

we have that the terms in $\tilde{\Pi}_1^{k_1}\tilde{\Pi}_2^{k_2}...\tilde{\Pi}_m^{k_m}$ which depend on $\bar{x}_p$ are

$$(\phi_p - \phi_1)^{2k_1}\cdots(\phi_p - \phi_{p-1})^{2k_{p-1}}(\phi_{p+1} - \phi_p)^{2k_{p+1}}\cdots(\phi_m - \phi_p)^{2k_m}.$$

Thus we can cancel all terms depending on $\bar{x}_p$ in (17).

Using that (16) is independent of $\bar{x}_p$, we see that (15) is equal to

$$\frac{\sigma_{n-k-1}(\check{\phi}_p)}{\tilde{\Pi}_p}\tilde{\mathcal{A}}_p = \frac{\sigma_{n-k-1}(\check{\phi}_p)}{\tilde{\Pi}_p}\sum_{\alpha,\beta=1}^{k_p}\frac{1}{\sqrt{det(\tilde{A}_p)}}\frac{\partial}{\partial x_p^\alpha}\sqrt{det(\tilde{A}_p)}(\tilde{A}_p)^{\alpha\beta}\frac{\partial}{\partial x_p^\beta}$$

and $\mathcal{I}_k$ is giving by (13), q. e. d.

**Lemma 3.** *If the metrics $g$, $\bar{g}$ are geodesically equivalent then the operators $\mathcal{I}_k$ commute.*

In order to prove Lemma 3, we need the following technical lemmas.

Denote by $N(x)$ the number of different eigenvalues of $G$ at the point $x$. By definition, the eigenvalues of $G$ at $x \in M^n$ are roots of the characteristic polynomial $P_x(t) = \det{(G - tE)}_{|x}$. Since each two positive-definite quadratic forms can be simultaneously diagonalized, all roots of $P_x(t)$ are real.

**Lemma 4.** *For a sufficiently small neighborhood of an arbitrary point $x \in M^n$, for any $y$ from the neighborhood the number $N(y)$ is no less than $N(x)$.*

*Proof.* Take a small $\epsilon > 0$ and an arbitrary root $\rho$ of $P_x(t)$. Let us prove that for a sufficiently small neighborhood $U(x) \subset M^n$, for any $y \in U(x)$ there is a root $\rho_y$, $\rho - \epsilon < \rho_y < \rho + \epsilon$, of the polynomial $P_y(t)$.

If $\epsilon$ is small, then for a sufficiently small neighborhood $U(x)$ of the point $x$, for any $y \in U(x)$ the numbers $\rho + \epsilon$ and $\rho - \epsilon$ are not roots of $P_y(t)$. Consider the circle $S_\epsilon \stackrel{\text{def}}{=} \{z \in C : \quad |z - \rho| = \epsilon\}$ on the complex plane $C$. Clearly, the number of roots (with multiplicities) of the polynomial $P_y$ inside the circle is equal to

$$\frac{1}{2\pi i}\int_{S_\epsilon} \frac{P'_y(z)}{P_y(z)}dz.$$

Since for any $y \in U(x)$ there are no roots of $P_y$ on the circle $S_\epsilon$, then the function

$$\frac{1}{2\pi i}\int_{S_\epsilon} \frac{P'_y(z)}{P_y(z)}dz$$

continuously depends on $y \in U(x)$, and therefore is a constant. Clearly it is positive. Thus for any $y \in U(x)$ there is at least one root of $P_y$ that lies between $\rho + \epsilon$ and $\rho - \epsilon$. Then for any $y$ from a sufficiently small neighborhood of $x$ we have $N(y) \geq N(x)$, q. e. d.

A point $x \in M^n$ is called *stable*, if in a neighborhood of $x$ the number of different eigenvalues of $G$ does not depend on a point.

Denote by $\mathcal{M}$ the set of stable points of $M^n$. The set $\mathcal{M}$ is an open subset of $M$. Obviously

$$\mathcal{M} = \bigsqcup_{1 \leq m \leq n} \mathcal{M}_m, \tag{18}$$

where $\mathcal{M}_m$ denotes the set of stable points with the number of distinct eigenvalues equalling $m$.

**Lemma 5 ([14]).** *The set $\mathcal{M}$ is everywhere dense in $M^n$.*

*Proof.* Evidently the set $\mathcal{M}$ is an open subset of $M^n$. Then it is sufficient to prove that any open subset $W \subset M^n$ contains a stable point. Suppose otherwise, i.e. let every point of an open subset $W$ be a point of bifurcation. Take a point $y \in W$ with the maximal value of the function $N$ on it. Using Lemma 4, we have that in a sufficiently small neighborhood $U(y) \subset W$ of the point $y$ the function $N$ is constant and is equal to $N(y)$. Then the point $y$ is a stable point, and we get a contradiction, q. e. d.

**Lemma 6.** *Suppose that the functions $\phi_1, \phi_2, ..., \phi_m$ satisfy (12). Then for any $i, j \in \{1, 2, ..., m\}$ and $l, k \in \{0, 1, ..., n-1\}$ the function*

$$\frac{\sigma_l(\check{\phi}_i)\sigma_k(\check{\phi}_j) - \sigma_k(\check{\phi}_i)\sigma_l(\check{\phi}_j)}{\tilde{\Pi}_j} \tag{19}$$

*is independent of $\bar{x}_i$.*

*Proof.* If $i = j$ then $\sigma_l(\check{\phi}_i)\sigma_k(\check{\phi}_j) - \sigma_k(\check{\phi}_i)\sigma_l(\check{\phi}_j)$ is zero and the statement is trivial. If $k_i$ is greater than one then neither $\sigma_k(\check{\phi}_j)$ nor $\sigma_l(\check{\phi}_j)$ nor $\sigma_k(\check{\phi}_i)$ nor $\sigma_l(\check{\phi}_i)$ nor $\Pi_j$ depend on $\bar{x}_i$ and the statement is also trivial.

Now let $k_i$ be equal to one. Without loss of generality, we can assume that $i < j$. Denote by $\sigma_k(\check{\phi}_i, \check{\phi}_j)$ the elementary symmetric polynomial of degree $k$ of $n-2$ variables

$$\underbrace{\phi_1, ..., \phi_1}_{k_1}, \underbrace{\phi_2, ..., \phi_2}_{k_2}, ..., \underbrace{\phi_i, ..., \phi_i}_{k_i - 1}, ..., \underbrace{\phi_j, ..., \phi_j}_{k_j - 1}, ..., \underbrace{\phi_m, ..., \phi_m}_{k_m}.$$

In view of (12), $\sigma_k(\check{\phi}_i, \check{\phi}_j)$ depends neither on $\bar{x}^i$ nor on $\bar{x}^j$. Substitute the trivial identities

$$\begin{aligned}
\sigma_l(\check{\phi}_i) &= \sigma_{l-1}(\check{\phi}_i, \check{\phi}_j)\phi_j + \sigma_l(\check{\phi}_i, \check{\phi}_j)\\
\sigma_k(\check{\phi}_i) &= \sigma_{k-1}(\check{\phi}_i, \check{\phi}_j)\phi_j + \sigma_k(\check{\phi}_i, \check{\phi}_j)\\
\sigma_l(\check{\phi}_j) &= \sigma_{l-1}(\check{\phi}_i, \check{\phi}_j)\phi_i + \sigma_l(\check{\phi}_i, \check{\phi}_j)\\
\sigma_k(\check{\phi}_j) &= \sigma_{k-1}(\check{\phi}_i, \check{\phi}_j)\phi_i + \sigma_k(\check{\phi}_i, \check{\phi}_j)
\end{aligned}$$

in (19). We assume that $\sigma_{-1}(\check{\phi}_i, \check{\phi}_j) = 1$. After a simple calculation, in place of (19) we obtain

$$\frac{\left(\sigma_l(\check{\phi}_i, \check{\phi}_j)\sigma_{k-1}(\check{\phi}_i, \check{\phi}_j) - \sigma_k(\check{\phi}_i, \check{\phi}_j)\sigma_{l-1}(\check{\phi}_i, \check{\phi}_j)\right)(\phi_i - \phi_j)}{\tilde{\Pi}_j},$$

which does not depend on $\bar{x}_i$, q. e. d.

*Proof of Lemma 3.* Lemma 5 shows that it is sufficient to prove the statement of Lemma 3 in the stable points only.

Let us prove that in a stable point the commutator

$$[\mathcal{I}_{n-l-1}, \mathcal{I}_{n-k-1}] = \mathcal{I}_{n-l-1}\left(\mathcal{I}_{n-k-1}\right) - \mathcal{I}_{n-k-1}\left(\mathcal{I}_{n-l-1}\right)$$

of the operators $\mathcal{I}_{n-l-1}$, $\mathcal{I}_{n-k-1}$ is zero.

In Levi-Civita coordinates we have

$$\begin{aligned}(-1)^{k+l}[\mathcal{I}_{n-l-1},\mathcal{I}_{n-k-1}] &= \sum_{i=1}^m \left(\tfrac{\sigma_l(\check\phi_i)}{\tilde\Pi_i}\tilde{\mathcal{A}}_i\left(\sum_{j=1}^m\left(\tfrac{\sigma_k(\check\phi_j)}{\tilde\Pi_j}\tilde{\mathcal{A}}_j\right)\right)\right)\\ &- \sum_{i=1}^m \left(\tfrac{\sigma_k(\check\phi_i)}{\tilde\Pi_i}\tilde{\mathcal{A}}_i\left(\sum_{j=1}^m\left(\tfrac{\sigma_l(\check\phi_j)}{\tilde\Pi_j}\tilde{\mathcal{A}}_j\right)\right)\right).\end{aligned} \tag{20}$$

Using that $\sigma_l(\check\phi_i)$ does not depend on $\bar x_i$ and therefore commutes with $\tilde{\mathcal{A}}_i$, we have that (20) is equal to

$$\begin{aligned}&\sum_{i=1}^m\left(\frac{1}{\tilde\Pi_i}\tilde{\mathcal{A}}_i\left(\sum_{j=1}^m\left(\frac{\sigma_l(\check\phi_i)\sigma_k(\check\phi_j)}{\tilde\Pi_j}\tilde{\mathcal{A}}_j\right)\right)-\frac{1}{\tilde\Pi_i}\tilde{\mathcal{A}}_i\left(\sum_{j=1}^m\left(\frac{\sigma_l(\check\phi_j)\sigma_k(\check\phi_i)}{\tilde\Pi_j}\tilde{\mathcal{A}}_j\right)\right)\right)\\ = &\sum_{i=1}^m\left(\frac{1}{\tilde\Pi_i}\tilde{\mathcal{A}}_i\left(\sum_{j=1}^m\left(\frac{\sigma_l(\check\phi_i)\sigma_k(\check\phi_j)-\sigma_k(\check\phi_i)\sigma_l(\check\phi_j)}{\tilde\Pi_j}\tilde{\mathcal{A}}_j\right)\right)\right).\end{aligned}$$

By Lemma 6, we have that

$$\frac{\sigma_l(\check\phi_i)\sigma_k(\check\phi_j)-\sigma_k(\check\phi_i)\sigma_l(\check\phi_j)}{\tilde\Pi_j}$$

is independent of $\bar x_i$. Then the commutator $[\mathcal{I}_{n-l-1},\mathcal{I}_{n-k-1}]$ is equal to

$$\sum_{i,j=1}^m\left(\left(\frac{\sigma_l(\check\phi_i)\sigma_k(\check\phi_j)-\sigma_k(\check\phi_i)\sigma_l(\check\phi_j)}{\tilde\Pi_i\tilde\Pi_j}\right)\tilde{\mathcal{A}}_i\left(\tilde{\mathcal{A}}_j\right)\right). \tag{21}$$

Since for $i\neq j$ the coefficients of $\tilde A_i$ are independent on $\bar x_j$ then for any $i,j$ $\tilde{\mathcal{A}}_i(\tilde{\mathcal{A}}_j)=\tilde{\mathcal{A}}_j(\tilde{\mathcal{A}}_i)$ and the sum (21) is evidently zero, q. e. d.

## 3 When the operators $\mathcal{I}_k$ are independent?

In this section we assume that $g,\ \bar g$ are geodesically equivalent metrics on $M^n$. As in previous sections, let us denote by $G: TM^n\to TM^n$ the fiberwise-linear mapping given by $G^i_j=g^{i\alpha}\bar g_{\alpha j}$, and by $N(x)$ the number of different eigenvalues of $G$ at $x\in M^n$.

**Lemma 7.** *Let $\bar x=(\bar x_1,\bar x_2,...,\bar x_m)$ be a Levi-Civita coordinate system for the metrics $g,\bar g$ at an open disk $D^n\subset M^n$. Consider the restrictions of the functions $I_k$ to $T^*D^n$. Then the dimension of the linear subspace*

$$< I_0, I_1, ...., I_{n-1} >\subset C^2(T^*D^n)$$

*generated by the functions $I_k: T^*D^n\to R$ is equal to $m$.*

By definition, the symbol of the sum of two operators equals the sum of the symbols. Then the sums

$$\sum_{k=0}^{n-1} \lambda_k I_k \quad \text{and} \quad \sum_{k=0}^{n-1} \lambda_k \mathcal{I}_k$$

vanish simultaneously.

**Corollary 6.** *Let $\bar{x} = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_m)$ be a Levi-Civita coordinate system for the metrics $g, \bar{g}$ at an open disk $D^n \subset M^n$. Consider the operators $\mathcal{I}_k$ as acting on $C^2(D^n)$. Then the dimension of the linear subspace $< \mathcal{I}_0, \mathcal{I}_1, ...., \mathcal{I}_{n-1} >$ generated by the operators $\mathcal{I}_k : C^2(D^n) \to C^0(D^n)$ is equal to $m$.*

**Corollary 7.** *Let $M^n$ be connected. Then the dimension of the linear subspace $< I_0, I_1, ..., I_{n-1} > \subset C^2(T^*M^n)$ generated by the functions $I_k : T^*M^n \to R$ is equal to $\max_{x \in M^n} N(x)$ and for almost each $y \in M^n$*

$$N(y) = \max_{x \in M^n} N(x).$$

**Corollary 8.** *Let $M^n$ be connected. Then the dimension of the linear subspace $< \mathcal{I}_0, \mathcal{I}_1, ...., \mathcal{I}_{n-1} >$ generated by the operators $\mathcal{I}_k : C^2(M^n) \to C^0(M^n)$ is equal to $\max_{x \in M^n} N(x)$ and for almost each $y \in M^n$*

$$N(y) = \max_{x \in M^n} N(x).$$

*Remark 5.* Corollary 2 (Corollary 3, respectively) immediately follows from Lemma 7 and Corollary 7 (from Corollaries 6,8, respectively).

*Proof of Corollary 7.* Suppose that for real $\lambda_0, \lambda_1, ..., \lambda_{n-1}$ the sum

$$\sum_{k=0}^{n-1} \lambda_k I_k \tag{22}$$

is zero for all points of $T^*D^n$, where $D^n \subset M^n$ is an open disk. Let us prove that if the manifold is connected then for all points of $T^*M^n$ the sum (22) is zero also. In particular, the dimension of the linear subspace

$$< I_0, I_1, ...., I_{n-1} > \subset C^2(T^*D^n)$$

generated by the functions $I_k : T^*D^n \to R$ is equal to the dimension of the linear subspace

$$< I_0, I_1, ..., I_{n-1} > \subset C^2(T^*M^n)$$

generated by the functions $I_k : T^*M^n \to R$.

Consider a geodesic $\gamma$ that goes through a point of the disk. Let us prove that for each point $P \in \gamma$ the sum (22) is zero at each point of $T^*_P M^n$. Consider all geodesics that start at the point $P$ and go through at least one point of $D^n$. Denote by $W^n \subset T_P M^n$ the set of initial velocity vectors of these geodesics. Since the solutions of a differential equation continuously depend on initial data, the set $W^n$ contains an open subset of $T_P M^n$.

Identify the tangent and the cotangent bundles of $M^n$ by the metric $g$. Since the integrals $I_k$ are constant on the geodesics, the function (22) is also constant on each geodesic. By assumption, the function (22) is zero for all points of $T^*D^n$. Then it is zero for all points of $W^n$. Using that the functions $I_k$ are polynomial on $T^*_P M^n$, we have that the function (22) is polynomial on $T^*_P M^n$. Since it is zero on an open subset, it is identically zero.

Now if the point $P$ can be joint by a geodesic with a point of the disk, each point of a sufficiently small neighborhood $U^n(P)$ of $P$ can be joint by a geodesic with a point of the disk. Thus the sum (22) is zero at each point of $T^*U^n(P)$. Using that each two points of a connected manifold can be joint by a finite sequence of geodesic segments, we have that the sum (22) is zero for all points of $T^*M^n$.

Now let the metrics $g,\ \bar g$ be geodesically equivalent on $M^n$. Denote by $m$ the maximum

$$\max_{x \in M^n} N(x).$$

Take a point $P \in M^n$ such that $N(P) = m$. By Lemma 4, we have that the point $P$ is stable. Therefore there exist Levi-Civita coordinates $\bar x = (\bar x_1, \bar x_2, ..., \bar x_m)$ in a small disk $D^n \subset M^n,\ \ P \in D^n$. Using Lemma 7, we have then that the dimension of the linear subspace $< I_0, I_1, ...., I_{n-1} > \subset C^2(T^*D^n)$ generated by the functions $I_k : T^*D^n \to R$ is equal to $m$. If in addition the manifold is connected, the dimension of the linear subspace $< I_0, I_1, ..., I_{n-1} > \subset C^2(T^*M^n)$ generated by the functions $I_k : T^*M^n \to R$ is equal to $m$. Combining this with Lemma 7, we have that the value of the function $N$ at any stable point is $m$. Therefore the value of the function $N$ at almost every point of $M^n$ is $m$, q. e. d.

*Proof of Lemma 7.* Let the metrics $g, \bar g$ be geodesically equivalent on $D^n$. For each $t \in R$ consider the function $F_t : T^*D^n \to R$ given by

$$I_{n-1} t^{n-1} + I_{n-2} t^{n-2} + ... + I_0.$$

By definition, $F_t$ is a linear combination of the integrals $I_k$.

Take different $t_0, t_1, ..., t_{n-1}$. Easy to demonstrate that each $I_k$ is a linear combination of the functions $F_{t_0}, F_{t_1}, ..., F_{t_{n-1}}$. Indeed,

$$\begin{pmatrix} F_{t_0} \\ F_{t_1} \\ \vdots \\ F_{t_{n-1}} \end{pmatrix} = \begin{pmatrix} I_{n-1} t_0^{n-1} + I_{n-2} t_0^{n-2} + ... + I_0 \\ I_{n-1} t_1^{n-1} + I_{n-2} t_1^{n-2} + ... + I_0 \\ \vdots \\ I_{n-1} t_{n-1}^{n-1} + I_{n-2} t_{n-1}^{n-2} + ... + I_0 \end{pmatrix} = \begin{pmatrix} t_0^{n-1} & t_0^{n-2} & ... & 1 \\ t_1^{n-1} & t_1^{n-2} & ... & 1 \\ & \vdots & & \\ t_{n-1}^{n-1} & t_{n-1}^{n-2} & ... & 1 \end{pmatrix} \begin{pmatrix} I_{n-1} \\ I_{n-2} \\ \vdots \\ I_0 \end{pmatrix}.$$

The Vandermonde matrix

$$\begin{pmatrix} t_0^{n-1} & t_0^{n-2} & ... & 1 \\ t_1^{n-1} & t_1^{n-2} & ... & 1 \\ & \vdots & & \\ t_{n-1}^{n-1} & t_{n-1}^{n-2} & ... & 1 \end{pmatrix}$$

is non-degenerate. Therefore the functions $I_k$ are linear combinations of the functions $F_{t_0}, F_{t_1}, ..., F_{t_{n-1}}$. Thus the linear span $< F_t,\ t \in R >$ of the family $F_t$ coincides with the linear span $< I_0, I_1, ..., I_{n-1} >$, and our goal is to prove that the dimension of $< F_t, t \in R >$ is $m$.

Using that

$$\sum_{k=0}^{n-1} t^k \sigma_{n-k-1}(\check{\phi}_i)(-1)^{n-k-1} = (t-\phi_1)^{k_1}(t-\phi_2)^{k_2}...(t-\phi_i)^{k_i-1}...(t-\phi_m)^{k_m}$$

we have

$$\begin{aligned} F_t &= \sum_{k=0}^{n-1}\left[(-1)^{n-k} t^k \sum_{i=1}^{m} \frac{\sigma_{n-k-1}(\check{\phi}_i)}{\tilde{\Pi}_i}\tilde{A}_i.\right] \\ &= -\sum_{i=1}^{m}\left[(t-\phi_1)^{k_1}(t-\phi_2)^{k_2}...(t-\phi_i)^{k_i-1}...(t-\phi_m)^{k_m}\frac{\tilde{A}_i}{\tilde{\Pi}_i}\right] \\ &= (t-\phi_1)^{k_1-1}(t-\phi_2)^{k_2-1}...(t-\phi_m)^{k_m-1}\sum_{k=0}^{m-1} t^k \tilde{I}_k, \end{aligned}$$

where the functions $\tilde{I}_k : T^*D^n \to R$ are given by

$$\tilde{I}_k = (-1)^{m-k}\sum_{i=1}^{m}\frac{\tilde{\sigma}_{m-k-1}(\check{\phi}_i)}{\tilde{\Pi}_i}\tilde{A}_i,$$

where $\tilde{\sigma}_l(\check{\phi}_i)$ denotes the elementary symmetric polynomial of degree $l$ of $m-1$ variables

$$\phi_1, \phi_2, ..., \phi_{i-1}, \phi_{i+1}, ..., \phi_m.$$

If $k_i > 1$ then $\phi_i$ is constant. Then for any $i \in \{0, 1, ..., m\},\ t \in R$, the function $(t-\phi_i)^{k_i-1}$ is constant also. Hence each function $F_t$ is a linear combination of the functions $\tilde{I}_0, \tilde{I}_1, ..., \tilde{I}_{m-1}$. Therefore the dimension of the linear subspace

$$< I_0, I_1, ..., I_{n-1} >$$

generated by the functions $I_k : T^*D^n \to R$ is no greater than $m$.

Let us prove that the functions $\tilde{I}_0, \tilde{I}_1, ..., \tilde{I}_{m-1}$ are linearly independent so that the dimension of the linear subspace $< I_0, I_1, ..., I_{n-1} >$ generated by the functions $I_k : T^*D^n \to R$ is precisely $m$. Evidently the functions $\tilde{A}_i$ are linearly independent so that it is sufficient to show that the following $(m \times m)$ matrix

$$C = \begin{pmatrix} (-1)^m \sigma_{m-1}(\check{\phi}_1) & (-1)^m \sigma_{m-1}(\check{\phi}_2) & ... & (-1)^m \sigma_{m-1}(\check{\phi}_m) \\ (-1)^{m-1} \sigma_{m-2}(\check{\phi}_1) & (-1)^{m-1} \sigma_{m-2}(\check{\phi}_2) & ... & (-1)^{m-1} \sigma_{m-2}(\check{\phi}_m) \\ & & \vdots & \\ -\sigma_0(\check{\phi}_1) & -\sigma_0(\check{\phi}_2) & ... & -\sigma_0(\check{\phi}_m) \end{pmatrix}$$

is non-degenerate. To prove this, let us multiply it (from left) by the Vandermonde $(m \times m)$ matrix

$$V = \begin{pmatrix} 1 & \phi_1 & \phi_1^2 & ... & \phi_1^{m-1} \\ 1 & \phi_2 & \phi_2^2 & ... & \phi_2^{m-1} \\ & & & \vdots & \\ 1 & \phi_m & \phi_m^2 & ... & \phi_m^{m-1} \end{pmatrix}.$$

Using that

$$\sum_{k=0}^{m-1} t^k \tilde{\sigma}_{m-k-1}(\check{\phi}_i)(-1)^{m-k-1} = (t-\phi_1)(t-\phi_2)...(t-\phi_{i-1})(t-\phi_{i+1})...(t-\phi_m)$$

we have that the sum

$$\sum_{k=0}^{m-1} \phi_j^k \tilde{\sigma}_{m-k-1}(\check{\phi}_i)(-1)^{m-k}$$

is zero for $i \neq j$ and is equal to $(-1)^{m-i+1}\Pi_i$ for $i = j$. Then the product $VC$ is the diagonal matrix

$$\mathrm{diag}\left((-1)^m \Pi_1, (-1)^{m-1}\Pi_2, ..., -\Pi_m\right),$$

which is clearly non-degenerate, q. e. d.

# 4 If the manifold is closed then the operators $\mathcal{I}_k$ are self-adjoint.

In this section we assume that the manifold $M^n$ is closed. Let $g$ and $\bar{g}$ be Riemannian metrics on $M^n$. Consider the operators $\mathcal{I}_k$ given by (2). Recall that $L_2$ is a Hilbert space with the following scalar product:

$$< \phi, \psi >= \int_{M^n} \phi\psi dV,$$

where $dV$ denotes the standard volume form $\sqrt{det(g)}dx^1dx^2...dx^n$ on $M^n$.

**Lemma 8.** *The operators $\mathcal{I}_k$ are self-adjoint.*

In other words, $\quad < \phi, \mathcal{I}_k(\psi) > = < \mathcal{I}_k(\phi), \psi >$ for any $\phi, \psi \in C^2(M^n)$.

*Proof.*

$$\begin{aligned}
< \phi, \mathcal{I}_k(\psi) > &= \int_{M^n} \phi \mathcal{I}_k(\psi) dV \\
&= \int_{M^n} \phi \frac{1}{\sqrt{det(g)}} \frac{\partial}{\partial x^i} (S_k)^i_\alpha \sqrt{det(g)} g^{\alpha j} \frac{\partial \psi}{\partial x^j} dV \\
&= \int_{M^n} \frac{1}{\sqrt{det(g)}} \frac{\partial}{\partial x^i} \sqrt{det(g)} (S_k)^i_\alpha g^{\alpha j} \phi \frac{\partial \psi}{\partial x^j} dV \\
&- \int_{M^n} (S_k)^i_\alpha g^{\alpha j} \left( \frac{\partial \phi}{\partial x^i} \right) \left( \frac{\partial \psi}{\partial x^j} \right) dV.
\end{aligned}$$

Since the manifold is closed, its boundary is empty. Using divergence theorem we have that the integral

$$\int_{M^n} \frac{1}{\sqrt{det(g)}} \frac{\partial}{\partial x^i} \sqrt{det(g)} \xi^i dV$$

vanishes for any smooth vector field $\xi$. Then $< \phi, \mathcal{I}_k(\psi) >$ is equal to

$$- \int_{M^n} (S_k)^i_\alpha g^{\alpha j} \left( \frac{\partial \phi}{\partial x^i} \right) \left( \frac{\partial \psi}{\partial x^j} \right) dV.$$

Using that the restriction of the fiberwise-linear mapping $S_k$ to any tangent space $T_{x_0} M^n$ is self-adjoint with respect to the metric $g$, we have that the two-form $g^{i\alpha} (S_k)^j_\alpha$ is symmetric, and therefore, the operator $\mathcal{I}_k$ is self-adjoint, q. e. d.

# 5 Separation of variables.

In this section we assume that the manifold $M^n$ is closed and connected, the metrics $g, \bar{g}$ on $M^n$ are geodesically equivalent and are strictly non-proportional at least at one point. Our goal is to show that we can separate the variables in the equation

$$\Delta(f) = \lambda f.$$

Since the operators $\mathcal{I}_k$ are self-adjoint and commute, it is possible to diagonalize them simultaneously. That is, there exist a countable basis

$$\Phi = \{ f_1, f_2, ..., f_m, ... \}$$

of the space $L_2(M^n)$ such that each $f_m$ is an eigenfunction of each operator $\mathcal{I}_k$. Since $\mathcal{I}_{n-1}$ is precisely the Beltrami-Laplace operator $\Delta$, each $f_m$ is an eigenfunction of $\Delta$.

It is known that the dimension of any eigenspace of $\Delta$ is finite. Therefore any eigenfunction of $\Delta$ is a finite linear combination of functions from $\Phi$, and all functions from the combination have the same eigenvalue. In particular, if all eigenvalues of the functions $f_1, f_2, ..., f_m, ...$ are different, then each eigenspace of $\Delta$ has dimension one. Then if $f$ is an eigenfunction of $\Delta$ then $f = Cf_m$ for the appropriate $C \in R$ and $m \in N$.

Suppose $f$ is an eigenfunction of each operator $\mathcal{I}_k$. Denote by $\lambda_k$ the corresponding eigenvalue. Then $f$ is a solution of the following system of partial differential equations.

$$\left\{\begin{array}{rcl} \mathcal{I}_0(f) & = & \lambda_0 f \\ \mathcal{I}_1(f) & = & \lambda_1 f \\ & \vdots & \\ \mathcal{I}_{n-1}(f) & = & \lambda_{n-1} f \end{array}\right. \tag{23}$$

By Corollary 1, the metrics are strictly non-proportional at almost all points of $M^n$. By Lemma 4, if the metrics are strictly non-proportional at a point $P \in M^n$ then $P$ is a stable point. Therefore in a neighborhood of $P$ there exist Levi-Civita coordinates $(y^0, y^1, ..., y^{n-1})$. In these coordinates the metrics have the form

$$\begin{array}{lll} \Pi_0 A_0(y^0)(dy^0)^2 + & \Pi_1 A_1(y^1)(dy^1)^2 + \cdots + & \Pi_{n-1} A_{n-1}(y^{n-1})(dy^{n-1})^2 \\ \rho_0 \Pi_0 A_0(y^0)(dy^0)^2 + & \rho_1 \Pi_1 A_1(y^1)(dy^1)^2 + \cdots + & \rho_{n-1} \Pi_{n-1} A_{n-1}(y^{n-1})(dy^{n-1})^2, \end{array}$$

respectively. Here $A_i$ are positive smooth functions of one variable,

$$\Pi_i \stackrel{\text{def}}{=} (\phi_i - \phi_0)(\phi_i - \phi_1) \cdots (\phi_i - \phi_{i-1})(\phi_{i+1} - \phi_i) \cdots (\phi_{n-1} - \phi_i) \tag{24}$$

and $\phi_0, \phi_1, ..., \phi_{n-1}$, $0 < \phi_0 < \phi_1 < ... < \phi_{n-1}$, are smooth functions such that for any $i$ the function $\phi_i$ depends on $y^i$ only.

Consider the coordinate system $(x^0, x^1, ..., x^{n-1})$ given by $x^i(y^i) = \int_{y_0^i}^{y^i} \sqrt{A_i(\xi)} d\xi$. In other words, the coordinate lines of the coordinate systems $(x^0, x^1, ..., x^{n-1})$ and $(y^0, y^1, ..., y^{n-1})$ coincide and for each $i$ we have $(dx^i)^2 = A_i(y^i)(dy^i)^2$. In new coordinates the operators $\tilde{\mathcal{A}}_i$ are simply $\frac{\partial^2}{(\partial x^i)^2}$ and the operators $\mathcal{I}_k$ are given by

$$\mathcal{I}_k = (-1)^{n-k} \sum_{i=0}^{n-1} \frac{\sigma_{n-k-1}(\check{\phi}_i)}{\Pi_i} \frac{\partial^2}{(\partial x^i)^2}.$$

Then in place of (23) we have

$$\begin{pmatrix} \mathcal{I}_0(f) \\ \mathcal{I}_1(f) \\ \vdots \\ \mathcal{I}_{n-1}(f) \end{pmatrix} =$$

$$\begin{pmatrix} (-1)^n \frac{\sigma_{n-1}(\check\phi_0)}{\Pi_0} & (-1)^n \frac{\sigma_{n-1}(\check\phi_1)}{\Pi_1} & ... & (-1)^n \frac{\sigma_{n-1}(\check\phi_{n-1})}{\Pi_{n-1}} \\ (-1)^{n-1} \frac{\sigma_{n-2}(\check\phi_0)}{\Pi_0} & (-1)^{n-1} \frac{\sigma_{n-2}(\check\phi_1)}{\Pi_1} & ... & (-1)^{n-1} \frac{\sigma_{n-2}(\check\phi_{n-1})}{\Pi_{n-1}} \\ & & \vdots & \\ -\frac{1}{\Pi_0} & -\frac{1}{\Pi_1} & ... & -\frac{1}{\Pi_{n-1}} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 f}{(\partial x^0)^2} \\ \frac{\partial^2 f}{(\partial x^1)^2} \\ \vdots \\ \frac{\partial^2 f}{(\partial x^{n-1})^2} \end{pmatrix} =$$

$$\begin{pmatrix} \lambda_0 f \\ \lambda_1 f \\ \vdots \\ \lambda_{n-1} f \end{pmatrix}$$

Using that

$$\sum_{i=0}^{n-1} (-1)^{i+1} t^{n-1-i} \sigma_i(\check\phi_k) = -(t-\phi_0)(t-\phi_1)...(t-\phi_{k-1})(t-\phi_{k+1})...(t-\phi_{n-1})$$

we have that for each $k, m \in \{0, ..., n-1\}$ the sum

$$\sum_{i=0}^{n-1} (-1)^{i+1} \phi_m^{n-1-i} \frac{\sigma_i(\check\phi_k)}{\Pi_k}$$

is zero for $k \neq m$ and is equal to $(-1)^{n-k}$ for $m = k$. Then the system (23) is equivalent to

$$\begin{array}{rclc} \frac{\partial^2 f}{(\partial x_0)^2} & = & (-1)^n & (\lambda_0 + \lambda_1\phi_0 + \lambda_2\phi_0^2 + ... + \lambda_{n-1}\phi_0^{n-1}) f \\ \frac{\partial^2 f}{(\partial x_1)^2} & = & (-1)^{n-1} & (\lambda_0 + \lambda_1\phi_1 + \lambda_2\phi_1^2 + ... + \lambda_{n-1}\phi_1^{n-1}) f \\ & \vdots & & \\ \frac{\partial^2 f}{(\partial x_{n-1})^2} & = & - & (\lambda_0 + \lambda_1\phi_{n-1} + \lambda_2\phi_{n-1}^2 + ... + \lambda_{n-1}\phi_{n-1}^{n-1}) f. \end{array} \tag{25}$$

The right-hand side of the k-th equation of (25) depends on $x^k$ only, q. e. d.

# 6 Inverse theorems.

Let $g$ be a Riemannian metric on $M^n$. Let us identify canonically the tangent and the cotangent bundles of $M^n$ by the metric $g$.

**Lemma 9.** *Let $M^n$ be connected. Suppose that for a function $f : M^n \to R$ the function*

$$F : T^*M^n \to R, F(x, \xi) \stackrel{\text{def}}{=} f(x) g(\xi, \xi)$$

*(where $(x, \xi) \in T^*M^n \cong TM^n$, $x \in M^n$ and $\xi \in T_xM^n$) be an integral of the geodesic flow of the metric $g$. Then the function $f$ is constant.*

*Proof.* It is known that the Hamiltonian $H : T^*M^n \to R$, $H(x, \xi) \stackrel{\text{def}}{=} \frac{1}{2} g(\xi, \xi)$ is an integral for the geodesic flow of the metric $g$. Then the ratio $\frac{F}{H} = 2f$ is also an integral. Using that each two points of $M^n$ can be joint by a sequence of geodesic segments, we have that $f$ is constant, q. e. d.

*Proof of Theorems 3,4.* Let $g$, $\bar{g}$ be Riemannian metrics on $M^n$. Consider the fiberwise-linear mapping $G$ given by the tensor $g^{i\alpha}\bar{g}_{\alpha j}$. From the results of [6], it follows that if the functions $I_k$ given by (3) are in involution and are functionally independent then the set of distributions of the eigenspaces of $G$ is integrable; hence Theorem 3 follows from Theorem 4.

Suppose the set of the distributions of the eigenspaces of $G$ is integrable, and the functions $I_k$ given by (3) are integrals in involution of the geodesic flow of $g$.

By definition, we have that in a neighborhood of almost each point of $M^n$ there exists a coordinate system $\bar{x} = (\bar{x}_1, ..., \bar{x}_m)$ (where $\bar{x}_i = (x_i^1, ..., x_i^{k_i})$, $(1 \le i \le m)$), such that the matrix of $G$ is diagonal

$$\mathrm{diag}\left(\underbrace{\rho_1, ..., \rho_1}_{k_1}, \underbrace{\rho_2, ..., \rho_2}_{k_2}, ..., \underbrace{\rho_m, ..., \rho_m}_{k_m}\right).$$

In this coordinate system, the matrixes of the metrics $g$, $\bar{g}$ are block-diagonal. Without loss of generality we can assume that $\rho_1 > \rho_2 > ... > \rho_m > 0$. By definition, put $\phi_i$ equal to $\frac{1}{\rho_i}(\rho_1\rho_2...\rho_m)^{\frac{1}{m+1}}$. By definition, put $\tilde{\Pi}_i$ equal to

$$(\phi_i - \phi_1)^{k_1}(\phi_i - \phi_2)^{k_2} \cdots (\phi_i - \phi_{i-1})^{k_{i-1}}(\phi_{i+1} - \phi_i)^{k_{i+1}} \cdots (\phi_m - \phi_i)^{k_m}.$$

Then the quadratic forms of the metrics $g$ and $\bar{g}$ are as follows:

$$\begin{aligned} g(\dot{\bar{x}}, \dot{\bar{x}}) &= \tilde{\Pi}_1 \tilde{A}_1(\bar{x}, \dot{\bar{x}}_1) + \tilde{\Pi}_2 \tilde{A}_2(\bar{x}, \dot{\bar{x}}_2) + \cdots + \\ &+ \tilde{\Pi}_m \tilde{A}_m(\bar{x}, \dot{\bar{x}}_m), \\ \bar{g}(\dot{\bar{x}}, \dot{\bar{x}}) &= \rho_1 \tilde{\Pi}_1 \tilde{A}_1(\bar{x}, \dot{\bar{x}}_1) + \rho_2 \tilde{\Pi}_2 \tilde{A}_2(\bar{x}, \dot{\bar{x}}_2) + \cdots + \\ &+ \rho_m \tilde{\Pi}_m \tilde{A}_m(\bar{x}, \dot{\bar{x}}_m), \end{aligned}$$

where $\tilde{A}_i(\bar{x}, \dot{\bar{x}}_i)$ are positive-definite quadratic forms in the velocities $\dot{\bar{x}}_i$ with coefficients depending on $\bar{x}$.

Our goal is to prove that the metrics are geodesically equivalent. It is sufficient to prove that the coordinate system $\bar{x} = (\bar{x}_1, ..., \bar{x}_m)$ is a Levi-Civita coordinate system for the metrics. That is we must prove that for each $i$ the coefficients of $\tilde{A}_i$ depend on $\bar{x}_i$ only, and that for each $i$ the function $\phi_i$ is constant for $k_i > 1$ and depends only on $\bar{x}_i$ for $k_i = 1$.

By Lemma 2, the integrals $I_k$ are given by

$$I_k(\bar{x}, \dot{\bar{x}}) = (-1)^{n-k} \sum_{i=1}^{m} \frac{\sigma_{n-k-1}(\check{\phi}_i)}{\tilde{\Pi}_i} \tilde{A}_i(\bar{x}, \dot{\bar{x}}_i).$$

Denote by $(\tilde{A}_i)_{\alpha\beta}$, $\alpha, \beta \in 1, 2, ..., k_i$, the tensor corresponding to the quadratic form $\tilde{A}_i$ and by $(\tilde{A}_i)^{\alpha\beta}$, $\alpha, \beta \in 1, 2, ..., k_i$, the inverse tensor. On the cotangent bundle $T^*M^n$ the integrals $I_k$ are given by

$$I_k(\bar{x}, \bar{p}) = (-1)^{n-k} \sum_{i=1}^{m} \left[ \sum_{\alpha,\beta=1}^{k_i} \frac{\sigma_{n-k-1}(\check{\phi}_i)}{\tilde{\Pi}_i} (\tilde{A})_i^{\alpha\beta} p_{x_i^\alpha} p_{x_i^\beta} \right].$$

By definition, put

$$F_i = \frac{1}{\tilde{\Pi}_i} \sum_{\alpha,\beta=1}^{k_i} (\tilde{A})_i^{\alpha\beta} p_{x_i^\alpha} p_{x_i^\beta}.$$

Then the integrals $I_k$ have the form

$$I_k(\bar{x}, \bar{p}) = (-1)^{n-k} \sum_{i=1}^{m} \sigma_{n-k-1}(\check{\phi}_i) F_i.$$

**Lemma 10.** *Suppose that the functions $I_k$ are in involution. Then for each $l_1, l_2 \in \{0, 1, ..., n-1\}$ and $i, j \in \{1, 2, ..., m\}$ the functions*

$$\sigma_{n-l_1-1}(\check{\phi}_i)F_i + \sigma_{n-l_1-1}(\check{\phi}_j)F_j \quad \text{and} \quad \sigma_{n-l_2-1}(\check{\phi}_i)F_i + \sigma_{n-l_2-1}(\check{\phi}_j)F_j \qquad (26)$$

*are in involution. In particular for each $l_1, l_2 \in \{0, 1, ..., n-1\}$ and $i \in \{1, 2, ..., m\}$ the functions*

$$\sigma_{n-l_1-1}(\check{\phi}_i)F_i \qquad \text{and} \qquad \sigma_{n-l_2-1}(\check{\phi}_i)F_i$$

*are in involution.*

*Proof.* We have

$$\{I_{l_1}, I_{l_2}\} = (-1)^{l_1+l_2} \sum_{\alpha,\beta=1}^{m} \left\{\sigma_{n-l_1-1}(\check{\phi}_\alpha)F_\alpha, \sigma_{n-l_2-1}(\check{\phi}_\beta)F_\beta\right\} = 0. \qquad (27)$$

Evidently $\{I_{l_1}, I_{l_2}\}$ is polynomial of third degree in momenta (in other words the restriction of $\{I_{l_1}, I_{l_2}\}$ to each cotangent space is a polynomial of degree three). Then all coefficients of this polynomial are zero. Note that the terms $p_{x_i^r} p_{x_i^s} p_{x_i^t}$ (where $r = 1, ..., k_i$, $s = 1, ..., k_i$, $t = 1, ..., k_i$), $p_{x_i^r} p_{x_i^s} p_{x_j^t}$ (where

$r = 1,...,k_i,\ s = 1,...,k_i,\ t = 1,...,k_j)$, $p_{x_i^r}p_{x_j^s}p_{x_j^t}$ (where $r = 1,...,k_i,\ s = 1,...,k_j,\ t = 1,...,k_j$) and $p_{x_j^r}p_{x_j^s}p_{x_j^t}$ (where $r = 1,...,k_j,\ s = 1,...,k_j,\ t = 1,...,k_j$) are found only in

$$\left\{\sigma_{n-l_1-1}(\check{\phi}_i)F_i + \sigma_{n-l_1-1}(\check{\phi}_j)F_j, \sigma_{n-l_2-1}(\check{\phi}_i)F_i + \sigma_{n-l_2-1}(\check{\phi}_j)F_j\right\} \tag{28}$$

and all terms in (28) are of this type. Since all coefficients by these terms are zero, (28) is zero and the functions (26) are in involution, q. e. d.

Combining Lemma 9 and Lemma 10, we have that for any $i \in \{1, 2, ..., m\}$ and $l \in \{0, 1, ..., n-1\}$ the function $\sigma_l(\check{\phi}_i)$ does not depend on $\bar{x}_i$. Consider the polynomial

$$\begin{aligned} & (t-\phi_1)^{k_1}(t-\phi_2)^{k_2}...(t-\phi_i)^{k_i-1}...(t-\phi_m)^{k_m} \\ = \ & t^{n-1}\sigma_0(\check{\phi}_i) - t^{n-2}\sigma_1(\check{\phi}_i) + ... + (-1)^{n-2}t\sigma_{n-2}(\check{\phi}_i) + (-1)^{n-1}\sigma_{n-1}(\check{\phi}_i). \end{aligned}$$

Since the coefficients of the polynomial are independent of $\bar{x}_i$, its roots are also independent of $\bar{x}_i$. Then for each $i \in \{1, 2, ..., m\}$ we have that $\phi_i$ is constant for $k_i > 1$ and depends only on $\bar{x}_i$ for $k_i = 1$.

The last step is to prove that the coefficients of $\tilde{A}_i$ are independent of $\bar{x}_j$ for $j \neq i$. It is sufficient to show that for each $i \neq j$ the functions

$$F_i\tilde{\Pi}_i, \quad F_j\tilde{\Pi}_j \tag{29}$$

are in involution. More precisely, suppose that (29) are in involution. Using that

$$\begin{aligned} \{F_i\tilde{\Pi}_i, F_j\tilde{\Pi}_j\} &= \left\{ \sum_{\alpha_1,\beta_1=1}^{k_i} (\tilde{A}_i)^{\alpha_1\beta_1} p_{x_i^{\alpha_1}} p_{x_i^{\beta_1}}, \sum_{\alpha_2,\beta_2=1}^{k_j} (\tilde{A}_j)^{\alpha_2\beta_2} p_{x_j^{\alpha_2}} p_{x_j^{\beta_2}} \right\} \\ &= 2\sum_{\alpha_1,\beta_1=1}^{k_i} \sum_{\alpha_2,\beta_2=1}^{k_j} \left[ (\tilde{A}_i)^{\alpha_1\beta_1} \frac{\partial(\tilde{A}_j)^{\alpha_2\beta_2}}{\partial x_i^{\alpha_1}} p_{x_i^{\beta_1}} p_{x_j^{\alpha_2}} p_{x_j^{\beta_2}} \right] \\ &- 2\sum_{\alpha_1,\beta_1=1}^{k_i} \sum_{\alpha_2,\beta_2=1}^{k_j} \left[ (\tilde{A}_j)^{\alpha_2\beta_2} \frac{\partial(\tilde{A}_i)^{\alpha_1\beta_1}}{\partial x_j^{\alpha_2}} p_{x_j^{\beta_2}} p_{x_i^{\alpha_1}} p_{x_i^{\beta_1}} \right]. \end{aligned} \tag{30}$$

we have that all coefficients of the polynomial in momenta (30) are zero. Since the form $\tilde{A}_j$ is non-degenerate, the sum

$$\sum_{\beta_2=1} (\tilde{A}_j)^{\alpha_2\beta_2} p_{x_j^{\beta_2}}$$

is zero, if and only if $\bar{p}_j = 0$. Then for any $\alpha_2 \in \{1, ..., k_j\}, \alpha_1 \in \{1, ..., k_i\}, \beta_1 \in \{1, ..., k_i\}$ we have that

$$\frac{\partial(\tilde{A}_i)^{\alpha_1\beta_1}}{\partial x_j^{\alpha_2}}$$

is zero. Thus the coefficients of $\tilde{A}_i$ depend on $\bar{x}_i$ only.

Let us demonstrate that the functions (29) are in involution. Using Lemma 10, we have that

$$\{F_i + F_j, \sigma_1(\check{\phi}_i)F_i + \sigma_1(\check{\phi}_j)F_j\} = 0. \tag{31}$$

By Lemma 10, $\{F_i, \sigma_1(\check{\phi}_i)F_i\} = \{F_j, \sigma_1(\check{\phi}_j)F_j\} = 0$. Then in place of (31) we have

$$\begin{aligned} 0 &= \{F_i, \sigma_1(\check{\phi}_j)F_j\} + \{F_j, \sigma_1(\check{\phi}_i)F_i\} \\ &= \sum_{\alpha=1}^{k_i} \left( \frac{\partial F_i}{\partial p_{x_i^\alpha}} \frac{\partial \sigma_1(\check{\phi}_j)F_j}{\partial x_i^\alpha} \right) + \sum_{\beta=1}^{k_j} \left( \frac{\partial F_j}{\partial p_{x_j^\beta}} \frac{\partial \sigma_1(\check{\phi}_i)F_i}{\partial x_j^\beta} \right) \\ &- \sum_{\alpha=1}^{k_i} \left( \frac{\partial \sigma_1(\check{\phi}_i)F_i}{\partial p_{x_i^\alpha}} \frac{\partial F_j}{\partial x_i^\alpha} \right) - \sum_{\beta=1}^{k_j} \left( \frac{\partial \sigma_1(\check{\phi}_j)F_j}{\partial p_{x_j^\beta}} \frac{\partial F_i}{\partial x_j^\beta} \right) \end{aligned}$$

Since $\sigma_1(\check{\phi}_i)$ does not depend on $\bar{x}_i$, we have that

$$\left( \frac{\partial F_i}{\partial p_{x_i^\alpha}} \frac{\partial \sigma_1(\check{\phi}_j)F_j}{\partial x_i^\alpha} \right) - \left( \frac{\partial \sigma_1(\check{\phi}_i)F_i}{\partial p_{x_i^\alpha}} \frac{\partial F_j}{\partial x_i^\alpha} \right) = \left( \frac{\partial F_i}{\partial p_{x_i^\alpha}} \frac{\partial (\sigma_1(\check{\phi}_j) - \sigma_1(\check{\phi}_i))F_j}{\partial x_i^\alpha} \right)$$

and similarly

$$\left( \frac{\partial F_j}{\partial p_{x_j^\beta}} \frac{\partial \sigma_1(\check{\phi}_i)F_i}{\partial x_j^\beta} \right) - \left( \frac{\partial \sigma_1(\check{\phi}_j)F_j}{\partial p_{x_j^\beta}} \frac{\partial F_i}{\partial x_j^\beta} \right) = \left( \frac{\partial F_j}{\partial p_{x_j^\beta}} \frac{\partial (\sigma_1(\check{\phi}_i) - \sigma_1(\check{\phi}_j))F_i}{\partial x_j^\beta} \right).$$

By Lemma 6,

$$\frac{(\sigma_1(\check{\phi}_i) - \sigma_1(\check{\phi}_j))}{\tilde{\Pi}_i} \qquad \left( \frac{(\sigma_1(\check{\phi}_i) - \sigma_1(\check{\phi}_j))}{\tilde{\Pi}_j}, \quad \text{respectively} \right)$$

is independent of $\bar{x}_j$ (of $\bar{x}_i$, respectively). Then the left-hand side of (31) is equal to

$$\frac{(\sigma_1(\check{\phi}_i) - \sigma_1(\check{\phi}_j))}{\tilde{\Pi}_i \tilde{\Pi}_j} \left\{ F_i \tilde{\Pi}_i, F_j \tilde{\Pi}_j \right\},$$

and therefore the functions (29) are in involution. Thus the coordinates $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_m)$ are Levi-Civita coordinates for the metrics $g, \bar{g}$ and the metrics are geodesically equivalent, q. e. d.

# 7 Proof of Theorems 6,7.

Consider the space $R^{n+1}$ with the standard coordinates $x = (x^1, x^2, ..., x^{n+1})$. Consider the standard sphere

$$S^n \subset R^{n+1}, \quad S^n = \{x \in R^{n+1} : (x^1)^2 + (x^2)^2 + ... + (x^{n+1})^2 = 1\}.$$

Denote by $g_{euclid}$ the Euclidean metrics

$$(dx^1)^2 + (dx^2)^2 + ... + (dx^{n+1})^2$$

on $R^{n+1}$. Denote by $g$ the restriction of $g_{euclid}$ to $S^n$. Let $A$ be the diagonal matrix diag$(\frac{1}{\sqrt{a_1}}, \frac{1}{\sqrt{a_2}}, ..., \frac{1}{\sqrt{a_{n+1}}})$. We assume that all $a_i$ are positive.

In order to prove Theorems 6,7, we consider the linear transformation $x \longmapsto Ax$ of the space $R^{n+1}$ and construct the corresponding transformation $l : x \mapsto \frac{Ax}{\|Ax\|}$ of the sphere $S^n$ and the metric $\bar{g} = l^* g$ on the sphere. Then we construct the fiberwise-linear mapping $B$ given by $B^i_j = \left(\frac{det(\bar{g})}{det(g)}\right)^{\frac{1}{n+1}} \bar{g}^{i\alpha} g_{\alpha j}$, and show that the metrics $g_B$, $\bar{g}_B$ are essentially the metrics from Theorem 6 and the metrics $g_{B^2}$, $\bar{g}_{B^2}$ are essentially the metrics from Theorem 7. Therefore Theorems 6,7 follow from Theorem 5.

Take an arbitrary point $x = (x^1, x^2, ..., x^{n+1}) \in S^n$. Denote by $T_x R^{n+1}$ and $T_x S^n$ the tangent spaces at the point $x$ to $R^{n+1}$ and to $S^n$, respectively. Identify the tangent space $T_x R^{n+1}$ with $R^{n+1}$ by moving the origin of the coordinate system to the point $x$. Consider the matrix

$$\bar{A} = \frac{1}{\sqrt{\frac{(x^1)^2}{a_1} + \frac{(x^2)^2}{a_2} + ... + \frac{(x^{n+1})^2}{a_{n+1}}}} A$$

and the matrix $P$ with the general element $P_{ij} = x^i x^j$. The matrixes $\bar{A}$, $P$ depend on the choice of the point $x$. It is easy to see that for each vector $v \in T_x R^{n+1}$, the vector $Pv$ is the orthogonal projection of the vector $v$ on the vector normal to the sphere. Then

$$PP = P, \tag{32}$$

since the square of the projection is the projection. Similarly,

$$P\bar{A}^2 P = P, \tag{33}$$

since in view of (32), the matrix $P\bar{A}^2P$ commutes with $P$, takes $T_x S^n$ to zero and is therefore proportional to the projection $P$. It is easy to see that the coefficient of proportionality is 1, since for the normal vector $\eta = (x^1, x^2, ..., x^{n+1})$ to the sphere we have $P\bar{A}^2 P\eta = \eta$.

Consider the mapping

$$L : R^{n+1} \to R^{n+1}, \ L(x) = \frac{\sqrt{(x^1)^2 + (x^2)^2 + ... + (x^{n+1})^2}}{\sqrt{\frac{(x^1)^2}{a_1} + \frac{(x^2)^2}{a_2} + ... + \frac{(x^{n+1})^2}{a_{n+1}}}} Ax.$$

The mapping $L$ is a diffeomorphism on $R^{n+1} \setminus 0$, takes the sphere $S^n$ to itself and coincides with the mapping $l : x \longmapsto \frac{Ax}{\|Ax\|}$ on the sphere.

The Jacobian $J$ of the mapping $L$ at the point $x \in S^n$ is equal to

$$\bar{A} - \bar{A}P\bar{A}^2 + \bar{A}P.$$

Then the pull-back $L^* g_{euclid}$ of the Euclidean metric $g_{euclid}$ is given by

$$\begin{array}{rcl} J^{\mathrm{T}}J & = & (\bar{A} - \bar{A}^2P\bar{A} + P\bar{A})(\bar{A} - \bar{A}P\bar{A}^2 + \bar{A}P) \\ & = & \bar{A}^2 - \bar{A}^2P\bar{A}^2 + \bar{A}^2P \\ & - & \bar{A}^2P\bar{A}^2 + \bar{A}^2P\bar{A}^2P\bar{A}^2 - \bar{A}^2P\bar{A}^2P \\ & + & P\bar{A}^2 - P\bar{A}^2P\bar{A}^2 + P\bar{A}^2P \\ & = & \bar{A}^2 - \bar{A}^2P\bar{A}^2 + P \end{array}$$

and therefore the metric $\bar{g}$ is the restriction of the metric given by the matrix $\bar{A}^2 - \bar{A}^2P\bar{A}^2 + P$ to the sphere.

Let us demonstrate that the matrix

$$G^{-1} = \bar{A}^{-2} - \bar{A}^{-2}P - P\bar{A}^{-2} + P\bar{A}^{-2}P + P$$

is inverse to the matrix $G = \bar{A}^2 - \bar{A}^2P\bar{A}^2 + P$.

Denote by $\mathbf{1}$ the identity $(n+1) \times (n+1)$ matrix $\mathrm{diag}(\underbrace{1, 1, ..., 1}_{n+1})$. We have

$$\begin{array}{rcl} G^{-1}G & = & (\bar{A}^{-2} - \bar{A}^{-2}P - P\bar{A}^{-2} + P\bar{A}^{-2}P + P)(\bar{A}^2 - \bar{A}^2P\bar{A}^2 + P) \\ & = & \mathbf{1} - \bar{A}^{-2}P\bar{A}^2 - P + P\bar{A}^{-2}P\bar{A}^2 + P\bar{A}^2 \\ & - & P\bar{A}^2 + \bar{A}^{-2}P\bar{A}^2P\bar{A}^2 + PP\bar{A}^2 - P\bar{A}^{-2}P\bar{A}^2P\bar{A}^2 - P\bar{A}^2P\bar{A}^2 \\ & + & \bar{A}^{-2}P - \bar{A}^{-2}PP - P\bar{A}^{-2}P + P\bar{A}^{-2}PP + PP \\ & = & \mathbf{1}. \end{array}$$

The Euclidean metric $g_{euclid}$ is given by the matrix $\mathbf{1}$. Then for any vectors $u, v \in T_xR^{n+1}$ we have $L^* g_{euclid}(G^{-1}u, v) = g_{euclid}(u, v)$. Let us show that $G^{-1}$ takes $T_xS^n$ to $T_xS^n$ and therefore coincides with the fiberwise-linear mapping given by $\bar{g}^{i\alpha}g_{\alpha j}$ on $T_xS^n$. Evidently, $T_xS^n$ is the eigenspace of the matrix $P$ (with the eigenvalue 0). Then it is sufficient to show that the matrix $P$ commutes with $G^{-1}$. We have

$$\begin{array}{rcl} G^{-1}P & = & (\bar{A}^{-2} - \bar{A}^{-2}P - P\bar{A}^{-2} + P\bar{A}^{-2}P + P)P \\ & = & \bar{A}^{-2}P - \bar{A}^{-2}PP - P\bar{A}^{-2}P + P\bar{A}^{-2}PP + PP \\ & = & \bar{A}^{-2}P - \bar{A}^{-2}P - P\bar{A}^{-2}P + P\bar{A}^{-2}P + P \\ & = & P\bar{A}^{-2} - P\bar{A}^{-2} - P\bar{A}^{-2}P + P\bar{A}^{-2}P + P \\ & = & P(\bar{A}^{-2} - \bar{A}^{-2}P - P\bar{A}^{-2} + P\bar{A}^{-2}P + P) \\ & = & PG^{-1}. \end{array}$$

Thus the fiberwise-linear mapping $B$ and the linear mapping given by $G^{-1}$ are proportional at $T_xS^n$. Let us find the coefficient of proportionality. By definition, it is equal to $\left(\frac{det(\bar{g})}{det(g)}\right)^{\frac{1}{n+1}}$. Let us demonstrate that $\frac{det(\bar{g})}{det(g)}$ is equal to $det(G)$. Indeed, the matrix $P$ commutes with the matrix $G^{-1}$. Hence the matrix $P$ commutes with the matrix $G$. Then $T_xS^n$ is invariant under multiplication by the matrix $G$. Using that the restriction of the linear mapping given by $G$ to $T_xS^n$ is exactly the restriction of the fiberwise-linear mapping given by the tensor $g^{i\alpha}\bar{g}_{\alpha j}$ to $T_xS^n$, we have that at the point $x$, all eigenvalues and eigenvectors of $g^{i\alpha}\bar{g}_{\alpha j}$ are eigenvalues and eigenvectors of $G$. In particular, $n$ eigenvectors of $G$ lie in $T_xS^n$. The only eigenvector, which does not lie in $T_xS^n$, is the normal vector $\eta = (x^1, x^2, ..., x^{n+1})$ to the sphere. It is easy to check by direct calculation that the corresponding eigenvalue is equal to 1. Using that the determinant is the product of all eigenvalues, we obtain

$$\frac{det(\bar{g})}{det(g)} = det(g^{i\alpha}\bar{g}_{\alpha j}) = det(G).$$

Let us find the determinant of the matrix $G$. We have

$$\begin{aligned} det(G) &= det^2(J) \\ &= det^2(\bar{A} - \bar{A}P\bar{A}^2 + \bar{A}P) \\ &= \frac{1}{det^2(\bar{A})} det^2(\bar{A}^2 - \bar{A}^2P\bar{A}^2 + \bar{A}^2P) \end{aligned}$$

Let us show that

$$det(\bar{A}^2 - \bar{A}^2P\bar{A}^2 + \bar{A}^2P) = det(\bar{A}^2 - \bar{A}^2P\bar{A}^2 + P). \tag{34}$$

In view of $P(\mathbf{1} - P) = 0$, we have

$$(\bar{A}^2 - \bar{A}^2P\bar{A}^2 + \bar{A}^2P)(\mathbf{1} - P) = (\bar{A}^2 - \bar{A}^2P\bar{A}^2 + P)(\mathbf{1} - P).$$

Since the projection of $T_xR^{n+1}$ on $T_xS^n$ is given by the matrix $\mathbf{1} - P$, the mappings given by $(\bar{A}^2 - \bar{A}^2P\bar{A}^2 + \bar{A}^2P)$ and $(\bar{A}^2 - \bar{A}^2P\bar{A}^2 + P)$ coincide on $T_xS^n$. Then there exists precisely one eigenvector $v \notin T_xS^n$ of the matrix $(\bar{A}^2 - \bar{A}^2P\bar{A}^2 + \bar{A}^2P)$. The corresponding eigenvalue is equal to 1, since $P(\bar{A}^2 - \bar{A}^2P\bar{A}^2 + \bar{A}^2P) = P$. Using that the determinant is the product of all eigenvalues, we have that the equality (34) is true. Thus

$$\frac{1}{det^2(\bar{A})} det^2(G) = det(G)$$

and therefore

$$det(G) = det^2(\bar{A}) = \frac{1}{a_1a_2...a_{n+1}} \left( \frac{1}{\frac{(x^1)^2}{a_1} + \frac{(x^2)^2}{a_2} + ... + \frac{(x^{n+1})^2}{a_{n+1}}} \right)^{n+1}.$$

Finally, the restriction of the fiberwise-linear mapping $B$ to $T_xS^n$ coincides with the restriction of the linear transformation of $T_xR^{n+1}$ given by the matrix

$$\left(\frac{C}{\frac{(x^1)^2}{a_1}+\frac{(x^2)^2}{a_2}+...+\frac{(x^{n+1})^2}{a_{n+1}}}\right)G^{-1}$$

to $T_xS^n$, where $C=(a_1a_2...a_{n+1})^{-n-1}$. Then the metrics $g_B$, $\bar{g}_B$ are the restriction of the metrics given by the matrixes

$$\left(\frac{C}{\frac{(x^1)^2}{a_1}+\frac{(x^2)^2}{a_2}+...+\frac{(x^{n+1})^2}{a_{n+1}}}\right)G^{-1}, \qquad \left(\frac{C}{\frac{(x^1)^2}{a_1}+\frac{(x^2)^2}{a_2}+...+\frac{(x^{n+1})^2}{a_{n+1}}}\right)\mathbf{1},$$

respectively, to $T_xS^n$. Let us demonstrate that the metrics $g_B$, $\bar{g}_B$ are isometric (up to a scaling) to the metrics from Theorem 6. First of all, the metrics given by the matrixes

$$\left(\frac{1}{\frac{(x^1)^2}{a_1}+\frac{(x^2)^2}{a_2}+...+\frac{(x^{n+1})^2}{a_{n+1}}}\right)G^{-1} \quad \text{and} \quad A^{-2}$$

coincide on $T_xS^n$. Indeed, since the projection on $T_xS^n$ is given by the matrix $(\mathbf{1}-P)$, it is sufficient to check that

$$(\mathbf{1}-P)G^{-1}(\mathbf{1}-P)=(\mathbf{1}-P)\bar{A}^{-2}(\mathbf{1}-P)$$

which is trivial in view of $(\mathbf{1}-P)P=P(\mathbf{1}-P)=0$.

Let us show that the restriction of the metric given by the matrix $A^{-2}$ to the sphere is isometric to the metric of the ellipsoid. Consider the ellipsoid

$$\left\{\left(x^1,x^2,...,x^{n+1}\right)\in R^{n+1}:\ \frac{(x^1)^2}{a_1}+\frac{(x^2)^2}{a_2}+...+\frac{(x^{n+1})^2}{a_{n+1}}=1\right\}.$$

Consider the mapping

$$m:R^{n+1}\to R^{n+1},\quad m\left(x^1,x^2,...,x^{n+1}\right)=\left(\frac{x^1}{\sqrt{a_1}},\frac{x^2}{\sqrt{a_2}},...,\frac{x^{n+1}}{\sqrt{a_{n+1}}}\right).$$

The mapping $m$ takes the ellipsoid to the sphere. Its Jacobian is equal to $A$. Then the pull-back of the metric with the matrix $A^{-2}$ is given by the matrix

$$A^TA^{-2}A=\mathbf{1},$$

and therefore the metric $m^*g_B$ is (up to a scaling) the metric of the ellipsoid.

Similarly, the metric $m^*\bar{g}_B$ coincides (up to a scaling) with the restriction of the metric given by (7) to the ellipsoid. More precisely, the pull-back of the metric with the matrix

$$\left(\frac{1}{\frac{(x^1)^2}{a_1}+\frac{(x^2)^2}{a_2}+...+\frac{(x^{n+1})^2}{a_{n+1}}}\right)\mathbf{1}$$

is given by the matrix

$$\left(\frac{1}{\frac{(x^1)^2}{a_1^2}+\frac{(x^2)^2}{a_2^2}+...+\frac{(x^{n+1})^2}{a_{n+1}^2}}\right)A^2,$$

which is exactly the matrix of the metric (7). Thus the Euclidean metric and the metric (7) induce geodesically equivalent metrics on the ellipsoid. Theorem 6 is proved.

Let us show that the metrics $m^*g_{B^2}$, $m^*\bar{g}_{B^2}$ coincide (up to a scaling) with the metrics from Theorem 7. By definition, the metrics $g_{B^2}$, $\bar{g}_{B^2}$ are the restriction of the metrics given by the matrixes

$$\left(\frac{C}{\frac{(x^1)^2}{a_1}+\frac{(x^2)^2}{a_2}+...+\frac{(x^{n+1})^2}{a_{n+1}}}\right)^2(G^{-1})^2,\quad \left(\frac{C}{\frac{(x^1)^2}{a_1}+\frac{(x^2)^2}{a_2}+...+\frac{(x^{n+1})^2}{a_{n+1}}}\right)^2 G^{-1},$$

respectively, to the sphere. It is easy to check by direct calculations that the matrix $(\mathbf{1}-P)(G^{-1})^2(\mathbf{1}-P)=(\mathbf{1}-P)\left(\bar{A}^{-4}-\bar{A}^{-2}P\bar{A}^{-2}\right)(\mathbf{1}-P)$ so that the matrixes $(G^{-1})^2$ and $\bar{A}^{-4}-\bar{A}^{-2}P\bar{A}^{-2}$ give us the same metric on the sphere. Then the pull-back $m^*g_{B^2}$ is (up to a scaling) the restriction of the metric given by the matrix $A^{-2}-P$ to the ellipsoid. Similarly, the pull-back $m^*\bar{g}_{B^2}$ is (up to a scaling) the restriction of the metric given by

$$\left(\frac{1}{\frac{(x^1)^2}{a_1^2}+\frac{(x^2)^2}{a_2^2}+...+\frac{(x^{n+1})^2}{a_{n+1}^2}}\right)\mathbf{1},$$

to the ellipsoid. Finally, the metrics (8), (9) induce geodesically equivalent metrics on the ellipsoid, q. e. d.

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