Dynamical and Topological methods in Theory of Geodesically Equivalent Metrics.

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Let $g = (g_{ij})$ and $\bar{g} = (\bar{g}_{ij})$ be C^2–smooth metrics on the same manifold M^n.

Definition 1. *The metrics g and $\bar{g}$ are* geodesically equivalent*, if they have the same geodesics (considered as unparameterized curves).*

This is rather classical material. The first nontrivial examples of geodesically equivalent metrics were constructed in 1865 by Beltrami see [2, 3]. In 1869 Dini [6] formulated the problem of local classification of geodesically equivalent metrics, and solved it for dimension two. In 1896 Levi-Civita [7] obtained a local description of geodesically equivalent metrics on manifolds of arbitrary dimension.

Many interesting results in this area were obtained by P. Painlevé, H. Weyl, E. Cartan, P. A. Shirokov, S. Kobayashi, N. S. Sinyukov, A. Z. Petrov, P. Venzi, J. Mikeš, A. V. Aminova, see [1, 11] for references.

The main tool they used was tensor analysis, and the most results were local. The approach we would like to suggest is more global; in particular it helps to find an answer for the following questions:

- What closed manifolds admit geodesically equivalent metrics? [10]
- How many metrics are there geodesically equivalent to a given one? [11]

Our approach is based on the following theorem.

Denote by $G : TM^n \to TM^n$ the fiberwise-linear mapping given by the tensor $g^{-1}\bar{g} = (g^{i\alpha}\bar{g}_{\alpha j})$. In invariant terms, for any $x_0 \in M^n$ the restriction of the mapping G to the tangent space $T_{x_0}M^n$ is the linear transformation of $T_{x_0}M^n$ such that for any vectors $\xi,\ \nu \in T_{x_0}M^n$ the dot product $g(G(\xi), \nu)$ of the vectors $G(\xi)$ and ν in the metric g is equal to the dot product $\bar{g}(\xi, \nu)$ of the vectors ξ and ν in the metric $\bar{g}$. Consider the characteristic polynomial $det(G - \mu E) = c_0\mu^n + c_1\mu^{n-1} + ... + c_n$. The coefficients $c_1, .., c_n$ are smooth functions on the manifold M^n, and $c_0 \equiv (-1)^n$. Consider the fiberwise-linear mappings

$$S_0, S_1, ..., S_{n-1} : TM^n \to TM^n$$

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given by the general formula

$$S_k \stackrel{\text{def}}{=} \left(\frac{det(g)}{det(\bar{g})}\right)^{\frac{k+2}{n+1}} \sum_{i=0}^{k} c_i G^{k-i+1}.$$

Consider the functions $I_k : T^*M^n \to R$, $k = 0, ..., n-1$, given by the formulae

$$I_k(x,p) \stackrel{\text{def}}{=} g^{\alpha i}(S_k)^j_\alpha p_i p_j. \tag{1}$$

In invariant terms the functions I_k are as follows. Let us identify canonically the tangent and the cotangent bundles of M^n by the metric g. Then the value of the function I_k on a vector $\xi \in T_xM^n \cong T^*_xM^n$ is given by

$$I_k(x,\xi) \stackrel{\text{def}}{=} g(S_k(\xi),\xi). \tag{2}$$

Consider the standard symplectic form on T^*M^n. By geodesic flow of the metric g we mean the Hamiltonian system on T^*M^n with the Hamiltonian $H(\xi) = \frac{1}{2}g(\xi,\xi)$.

Remark 1. *The function I_{n-1} is equal to minus twice Hamiltonian of the geodesic flow of the metric g.*

Theorem 1. *If the metrics g and $\bar{g}$ on M^n are geodesically equivalent then the functions I_k pairwise commute. In particular, they are integrals in involution of the geodesic flow of the metric g.*

This theorem allows us to apply the theory of integrable geodesic flows to the theory of geodesically equivalent metrics and vice versa. First of all, the following theorem explaines when the integrals are functionally independent.

Theorem 2 ([16]). *Let metrics g, $\bar{g}$ be geodesically equivalent. If at a point $x_0 \in M^n$ the number of different roots of the polynomial $det(G - \mu E)$ is equal to n_1, then at almost all points of $T_{x_0}M^n$ the dimension of the linear space generated by $dI_0, dI_1, ..., dI_{n-1}$ is greater or equal than n_1.*

In particular, if the characteristic polynomial $det(G - \mu E)$ has no multiple roots at a point $x_0 \in M^n$ then the functions $I_0, I_1, ..., I_{n-1}$ are functionally independent almost everywhere in $TU(x_0)$, where $U(x_0)$ is a sufficiently small neighborhood of x_0.

If at every point of a neighborhood $U(x_0)$ of a point $x_0 \in M^n$ the number of different eigenvalues of the polynomial $det(G - \mu E)$ is less or equal than n_1, then at all points of $TU(x_0)$ the dimension of the linear space generated by $dI_0, dI_1, ..., dI_{n-1}$ is less or equal than n_1.

The metrics $g, \bar{g}$ are *strictly non-proportional* at a point $x_0 \in M^n$ when the characteristic polynomial $det(G - \mu E)_{|x_0}$ has no multiple roots.

Corollary 1. *Suppose M^n is connected. Let metrics g, $\bar{g}$ on M^n be geodesically equivalent and strictly non-proportional at least at one point on M^n. Then the metrics are strictly non-proportional almost everywhere.*

More generally, suppose M^n is connected and the metrics g, $\bar{g}$ on M^n are geodesically equivalent. If at every point of a neighborhood $U \subset M^n$ the number of different eigenvalues of the polynomial $det(G - \mu E)$ is less or equal than n_1, then at every point of M^n the number of different eigenvalues of the polynomial $det(G - \mu E)$ is less or equal than n_1.

Corollary 1 follows from the following observation. If we have an integrable Hamiltonian systems then the dimension of the linear space generated by differentials of the integrals is constant along each orbit.

Proof of Corollary 1. Identify canonically the tangent and the cotangent bundles of M^n by the metric g. Take a geodesic $\gamma : R \to M^n$ and consider the points $x_0 = \gamma(0),\ x_1 = \gamma(1) \in M^n$. Suppose that the metric $\bar{g}$ is geodesically equivalent to the metric $\bar{g}$ and is strictly non-proportional with g at the point x_0. Let us prove that in each neighborhood of the point x_1 there exists a point x such that the metrics are strictly non-proportional at x.

The geodesic orbit of the geodesic γ is the curve $(\gamma, \dot{\gamma}) : R \to TM^n \cong T^*M^n$ (assuming $\dot{\gamma} = \dot{\gamma}(t) \in T_{\gamma(t)}M^n$ is the velocity vector of γ.) Take a small neighborhood $O^{2n} \subset TM^n$ of the point $(x_0, \dot{\gamma}(0)) \in TM^n$. Consider the union of all points of all geodesic orbits that start from the points of the neighborhood. Denote it by U. Since the metrics are strictly non-proportional at x_0, the differentials $dI_0, dI_1, ..., dI_{n-1}$ of the integrals $I_1, I_2, ..., I_{n-1}$ are linearly independent at almost all points of U. Evidently the set U contains a neighborhood of the point $(x_1, \dot{\gamma}(1))$. The integrals $I_0, I_1, ..., I_{n-1}$ are then functionally independent at almost all points of a neighborhood of the point $(x_1, \dot{\gamma}(1))$. Using Theorem 2, we have that in each neighborhood of x_1 there exists a point where the metrics are strictly non-proportional. Since each two points of a connected manifold can be joint by a sequence of geodesical segments, in each neighborhood of an arbitrary point of M^n there exist points where the metrics are strictly non-proportional, q. e. d.

Let us combine Theorems 1,2 with known results in the theory of integrable geodesic flows.

Theorem 3 (Taimanov, [14]). *If a real-analytic closed manifold M^n with a real-analytic metric satisfies at least one of the conditions:*

a) $\pi_1(M^n)$ is not almost commutative

b) $dim H_1(M^n; \mathbf{Q}) > dim M^n$,

then the geodesic flow on M^n is not analytically integrable.

Corollary 2 ([16]). *Let M^n be a closed real-analytic manifold supplied with two real-analytic metrics $g, \bar{g}$ such that the metrics $g, \bar{g}$ are geodesically equivalent and strictly non-proportional at least at one point. Then the fundamental* *group $\pi_1(M^n)$ of the manifold M^n contains a commutative subgroup of finite index and the dimension of the homology group $H_1(M^n;\mathbf{Q})$ is no greater than n.*

Easy to see that the integrals I_k are quadratic in momenta.

Theorem 4 ([8]). *Let g be a Riemannian metric on a closed surface M^2. Suppose that the geodesic flow of g admits an integral that is quadratic in momenta and functionally independent of the Hamiltonian. Then the surface M^2 is homeomorphic either to the torus or to the sphere or to the Klein bottle or to the projective plane.*

Corollary 3 ([16]). *Let metrics $g, \bar{g}$ on a closed surface of negative Euler characteristic be geodesically equivalent. Then $g = C\bar{g}$, where C is a constant.*

We say that the Hamiltonian and an integral of a geodesic flow are *proportional* at a point $x \in M^n$, if for a constant c for each point $\xi \in T^*_x M^n$ we have $I(\xi) = cH(\xi)$.

Recall that a vector field on M^n is *Killing* (with respect to a metric), if the flow of the field preserves the metric.

Theorem 5 ([8]). *Let the geodesic flow of a metric g on the sphere S^2 admits an integral I quadratic in velocities and functionally independent of the Hamiltonian H of the geodesic flow. Then there are only three possibilities.*

1. *The Hamiltonian and the integral are proportional at exactly two points.*
2. *The Hamiltonian and the integral are proportional at exactly four points.*
3. *The Hamiltonian and the integral are completely proportional, i.e. $cH = I$, where c is a constant.*

In the first case the metric admits a Killing vector field.

Easy to see that the metrics g, $\bar{g}$ are proportional at a point if and only if the Hamiltonian of the geodesic flow of g and the integral I_0 are proportional at the point.

Corollary 4 ([16]). *Let metrics g, $\bar{g}$ on the sphere S^2 be geodesically equivalent. Then there are only three possibilities.*

1. *The metrics are proportional at exactly two points.*
2. *The metrics are proportional at exactly four points.*
3. *The metrics are completely proportional, i.e. $g = C\bar{g}$, where C is a positive constant.*

In the first case the metrics admit a Killing vector field.

Corollary 5. *Let $g, \bar{g}$ be geodesically equivalent metrics on M^n. If g admits a non-trivial Killing vector field then $\bar{g}$ also admits a non-trivial Killing vector field.*

Proof. Let g, $\bar{g}$ be geodesically equivalent metrics on M^n. By Noether's theorem, if a metric admits a (non-trivial) Killing vector field, then the geodesic flow of the metric admits a (non-trivial) integral, linear in velocities, and vice versa. Then it is sufficient to prove that, given an integral linear in velocities for the geodesic flow of $\bar{g}$, we can construct an integral linear in velocities for the geodesic flow of g. Since the metrics are geodesically equivalent, the fucntion $\phi : T^*M^n \cong TM^n \to TM^n \cong T^*M^n$ given by

$$\phi(x, \xi) = \left(x, \frac{||\xi||_g}{||\xi||_{\bar{g}}}\xi\right) \tag{3}$$

takes the orbits of the geodesic flow of g to the orbits of the geodesic flow of $\bar{g}$. Suppose the function

$$F_1 = \sum_{i=1}^{n} a_i(x)\xi^i$$

is constant on the trajectories of the geodesic flow of the metric $\bar{g}$. Then the function

$$\phi^* F_1 = \frac{||\xi||_g}{||\xi||_{\bar{g}}} \sum_{i=1}^{n} a_i(x)\xi^i$$

is constant on the trajectories of the geodesic flow of the metric g. Since the function $I_0 = \left(\frac{\det(g)}{\det(\bar{g})}\right)^{\frac{2}{n+1}} \bar{g}(\xi, \xi)$ is an integral of the geodesic flow of the metric g, and since the function $||\xi||_g = \sqrt{g(\xi, \xi)}$ is also an integral of the geodesic flow of the metric g, then the function

$$\frac{\sqrt{g(\xi, \xi)}}{\sqrt{I_0}}\phi^* F_1 = \left(\frac{\det(g)}{\det(\bar{g})}\right)^{\frac{1}{n+1}} \sum_{i=1}^{n} a_i(x)\xi^i,$$

linear in velocities, is also an integral of the geodesic flow of the metric g, q. e. d.

Does the integrability of a geodesic flow implies the existence of geodesically equivalent metric? Theorem 1 shows that we should have integrals in the form (1).

Theorem 6. *Let g, $\bar{g}$ be metrics on M^n. Consider the functions $I_0, I_1, ..., I_{n-1}$. Let they be functionally independent almost everywhere and let they commute. Then the metrics g, $\bar{g}$ are geodesically equivalent.*

Dynamical Background:
We suggest a construction that, given an orbital diffeomorphism between two Hamiltonian systems, produces integrals of them.

Let v and $\bar v$ be Hamiltonian systems on symplectic manifolds (N,ω) and $(\bar N,\bar\omega)$ with Hamiltonians H and $\bar H$, respectively.

Consider the isoenergy surfaces

$$Q \stackrel{\mathrm{def}}{=} \{x \in N : H(x) = h\}\,, \quad \bar Q \stackrel{\mathrm{def}}{=} \left\{x \in \bar N : \bar H(x) = \bar h\right\},$$

where h and $\bar h$ are regular values of the functions H, $\bar H$, respectively.

Definition 2. *A diffeomorphism $\phi : Q \longrightarrow \bar Q$ is said to be* orbital*, if it takes the orbits of the system v to the orbits of the system $\bar v$.*

Given orbital diffeomorphism, we can invariantly construct integrals. Denote by σ, $\bar\sigma$ the restrictions of the forms $\omega,\bar\omega$ to Q, $\bar Q$ respectively. Consider the form $\phi^*\bar\sigma$ on Q.

Lemma 1 (Topalov, [15]). *The flow v preserves the form $\phi^*\bar\sigma$.*

Proof. The Lie derivative L_v of the form $\phi^*\bar\sigma$ along the vector field v satisfies

$$L_v\phi^*\bar\sigma = \mathrm{d}\left[\imath_v\phi^*\bar\sigma\right] + \imath_v\mathrm{d}\left[\phi^*\bar\sigma\right].$$

On the right-hand side both terms vanish. Since the form $\bar\omega$ is closed, the form $\bar\sigma$ is also closed and $\mathrm{d}\left[\phi^*\bar\sigma\right] = \phi^*(\mathrm{d}\bar\sigma) = 0$. Since the diffeomorphism takes the orbits to orbits, it takes the kernel of the form σ to the kernel of the form $\bar\sigma$, so that $\imath_v\phi^*\bar\sigma$ is equal to zero, q. e. d.

It is obvious that the kernels of the forms σ and $\phi^*\bar\sigma$ coincide (in the space $\mathcal{T}_xQ$ at each point $x \in Q$) with the linear span of the vector v. Therefore these forms induce two non-degenerate tensor fields on the quotient bundle $\mathcal{T}Q/\langle v\rangle$. We shall denote the corresponding forms on $\mathcal{T}Q/\langle v\rangle$ also by the letters $\sigma,\bar\sigma$.

Lemma 2. *The characteristic polynomial of $(\sigma)^{-1}(\phi^*\bar\sigma)$ on $\mathcal{T}Q/\langle v\rangle$ is preserved by the flow v.*

Proof. Since the flow v preserves the Hamiltonian H and the form ω, the flow v preserves the form σ. Since the flow v preserves both forms, it preserves the characteristic polynomial of $(\sigma)^{-1}(\phi^*\bar\sigma)$, q. e. d.

Since both forms are skew-symmetric, each root of the characteristic polynomial $(\sigma)^{-1}(\phi^*\bar\sigma)$ has an even multiplicity. Then the characteristic polynomial is the square of a polynomial $\delta^{n-1}(t)$ of degree $n-1$. Hence the polynomial $\delta^{n-1}(t)$ is also preserved by the flow v. Therefore the coefficients of the polynomial $\delta^{n-1}(t)$ are integrals of the system v.

Geodesic flows of geodesically equivalent metrics can be considered as orbitally equivalent systems. The manifold $N = \bar{N} = TM^n$, the forms ω, $\bar{\omega}$ are given by

$$\omega \stackrel{\text{def}}{=} \mathrm{d}[g_{ij}\xi^j dx^i], \quad \bar{\omega} \stackrel{\text{def}}{=} \mathrm{d}[\bar{g}_{ij}\xi^j dx^i]$$

and the orbital diffeomorphism ϕ is given by (3). Direct calculations give us the formulae for the integrals I_k from Theorem 1.

Are there interesting examples of geodesically equivalent metrics on closed manifolds?

The following theorem (essentially due to N. S. Sinjukov [12]) gives us a construction that, given a pair of geodesically equivalent metrics, produces another pair of geodesically equivalent metrics. Starting from the metric of constant curvature on the sphere, we obtain the metric of the ellipsoid and the metric of the Poisson sphere.

Let $g, \bar{g}$ be Riemannian metrics on M^n. Consider the fiberwise-linear mapping $B : TM^n \to TM^n$ given by $B^i_j = \left(\frac{det(\bar{g})}{det(g)}\right)^{\frac{1}{n+1}} \bar{g}^{i\alpha} g_{\alpha j}$. By definition, let us put metric g_B equal to $g_{ip}B^p_j$ and put metric $\bar{g}_B$ equal to $\bar{g}_{ip}B^p_j$. In invariant terms, the dot product $g_B(\xi, \nu)$ of arbitrary vectors $\xi, \nu \in T_{x_0}M^n$ is equal to $g(B\xi, \nu)$ and the dot product $\bar{g}_B(\xi, \nu)$ is equal to $\bar{g}(B\xi, \nu)$. Evidently, the restriction of B to any tangent space $T_{x_0}M^n$ is self-adjoint with respect to the metrics g and $\bar{g}$ and the metrics g_B, $\bar{g}_B$ are well-definite.

Theorem 7. *The metrics g and $\bar{g}$ are geodesically equivalent if and only if the metrics g_B and $\bar{g}_B$ are geodesically equivalent.*

Evidently, the metrics g and $\bar{g}$ are strictly non-proportional at a point $x \in M^n$ if and only if the metrics g_B and $\bar{g}_B$ are strictly non-proportional at the point x.

Thus if we have a pair of geodesically equivalent metrics g, $\bar{g}$, then we can construct the other pair of geodesically equivalent metrics g_B, $\bar{g}_B$. We can apply the construction once more, the result is another pair of geodesically equivalent metrics. It is natural to denote it by g_{B^2}, $\bar{g}_{B^2}$ since this pair is given by

$$g_{B^2}(\xi, \nu) = g(B^2\xi, \nu), \quad \bar{g}_{B^2}(\xi, \nu) = \bar{g}(B^2\xi, \nu).$$

We can go in other direction and consider the metrics $g_{B^{-1}}$, $\bar{g}_{B^{-1}}$ given by

$$g_{B^{-1}}(\xi, \nu) = g(B^{-1}\xi, \nu), \quad \bar{g}_{B^{-1}}(\xi, \nu) = \bar{g}(B^{-1}\xi, \nu).$$

They are geodesically equivalent as well.

To start the process, we need a pair of geodesically equivalent metrics g, $\bar{g}$. We take the following one (obtained by E. Beltrami [2], [3]). The metric g is the restriction of the Euclidean metric

$$(dx^1)^2 + (dx^2)^2 + ... + (dx^{n+1})^2$$

to the standard sphere

$$S^n = \{(x^1, x^2, ..., x^{n+1}) \in R^{n+1} : (x^1)^2 + (x^2)^2 + ... + (x^{n+1})^2 = 1\}.$$

The metric $\bar{g}$ is the pull-back $l^* g$, where the diffeomorphism $l : S^n \to S^n$ is given by

$$l(x) \stackrel{\text{def}}{=} \frac{Ax}{\|Ax\|},$$

where A is an arbitrary non-degenerate linear transformation of R^{n+1}, Ax means 'the transformation A applied to the vector $x = (x^1, x^2, ..., x^{n+1})$,' and $\|x\|$ is the standard norm $\sqrt{(x^1)^2 + (x^2)^2 + ... + (x^{n+1})^2}$.

Easy to see that the mapping l preserves the geodesics of the sphere. More precisely, the geodesics of the sphere are intersections of the planes, which go through the origin, with the sphere. The linear transformation A takes planes to planes, therefore the mapping l takes geodesics to geodesics. Then the standard metric g of the sphere and the metric $l^* g$ are geodesically equivalent.

For these geodesically equivalent metrics the metric g_B is the metric of an ellipsoid, and the metric $\bar{g}_{B^2}$ is the metric of a Poisson sphere. By varying the linear operator A, we can obtain metrics of all possible ellipsoids and all possible Poisson spheres.

Recall that the metric of the ellipsoid

$$E \stackrel{\text{def}}{=} \left\{ (x^1, x^2, ..., x^{n+1}) \in R^{n+1} : \frac{(x^1)^2}{a_1} + \frac{(x^2)^2}{a_2} + ... + \frac{(x^{n+1})^2}{a_{n+1}} = 1 \right\}$$

is the restriction of the metric

$$(dx^1)^2 + (dx^2)^2 + ... + (dx^{n+1})^2$$

to the ellipsoid E. By the metric of the Poisson sphere we, following [5], mean the restriction of the metric

$$\frac{1}{\frac{(x^1)^2}{a_1^2} + \frac{(x^2)^2}{a_2^2} + ... + \frac{(x^{n+1})^2}{a_{n+1}^2}} \left((dx^1)^2 + (dx^2)^2 + ... + (dx^{n+1})^2 \right)$$

to the ellipsoid E.

The metric of Poisson sphere has the following mechanical sense. Consider the free motion of (n+1)-dimension rigid body in (n+1)-dimensional space around a fixed point. The configuration space of the corresponding Hamiltonian system is $SO(n+1)$, and the corresponding Hamiltonian is left-invariant. Consider the embedding $SO(n)$ into $SO(n+1)$ as the stabilizer of a vector $v \in R^{n+1}$. Consider the action of the group $SO(n)$ on $SO(n+1)$ by left translations. The Hamiltonian of the motion is evidently invariant modulo this action, and the reduced system on $TSO(n+1)/SO(n) \cong TS^n$ is the geodesic flow of the (appropriate) Poisson metric.

Theorem 8 ([16], independently obtained by S. Tabachnikov [13]). *The restriction of the Euclidean metric*

$$(dx^1)^2 + (dx^2)^2 + ... + (dx^{n+1})^2$$

to the standard ellipsoid

$$E^n = \left\{ (x^1, x^2, ..., x^{n+1}) \in R^{n+1} : \frac{(x^1)^2}{a_1} + \frac{(x^2)^2}{a_2} + ... + \frac{(x^{n+1})^2}{a_{n+1}} = 1 \right\}$$

is geodesically equivalent to the restriction of the metric

$$\frac{1}{\left(\frac{x^1}{a_1}\right)^2 + \left(\frac{x^2}{a_2}\right)^2 + ... + \left(\frac{x^{n+1}}{a_{n+1}}\right)^2} \left(\frac{(dx^1)^2}{a_1} + \frac{(dx^2)^2}{a_2} + ... + \frac{(dx^{n+1})^2}{a_{n+1}} \right) \quad (4)$$

to the same ellipsoid.

Theorem 9 (Topalov, [17]). *The restriction of the metric*

$$\frac{1}{\frac{(x^1)^2}{a_1^2} + \frac{(x^2)^2}{a_2^2} + ... + \frac{(x^{n+1})^2}{a_{n+1}^2}} \left((dx^1)^2 + (dx^2)^2 + ... + (dx^{n+1})^2 \right) \quad (5)$$

to the ellipsoid

$$E^n = \left\{ (x^1, x^2, ..., x^{n+1}) \in R^{n+1} : \frac{(x^1)^2}{a_1} + \frac{(x^2)^2}{a_2} + ... + \frac{(x^{n+1})^2}{a_{n+1}} = 1 \right\}$$

is geodesically equivalent to the restriction of the metric

$$a_1(dx^1)^2 + a_2(dx^2)^2 + ... + a_{n+1}(dx^{n+1})^2 - (x^1dx^1 + x^2dx^2 + ... + x^{n+1}dx^{n+1})^2 \quad (6)$$

to the same ellipsoid.

Quantum integrability of the Beltrami-Laplace operator for geodesically equivalent metrics:
Let $g, \bar{g}$ be Riemannian metrics on M^n. Consider the Beltrami-Laplace operator

$$\Delta := -\frac{1}{\sqrt{det(g)}} \frac{\partial}{\partial x^i} \sqrt{det(g)} g^{ij} \frac{\partial}{\partial x^j},$$

where $det(g)$ denotes the determinant of the matrix corresponding to the metric g.

Consider the operators

$$\mathcal{I}_0,\ \mathcal{I}_1, ..., \mathcal{I}_{n-1} : C^2(M^n) \to C^0(M^n) \quad (7)$$

given by the general formula

$$\mathcal{I}_k := \frac{1}{\sqrt{det(g)}} \frac{\partial}{\partial x^i} (S_k)^i_\alpha \sqrt{det(g)} g^{\alpha j} \frac{\partial}{\partial x^j}. \quad (8)$$

Remark 2. *The operator $\mathcal{I}_{n-1}$ is exactly the operator Δ.*

Theorem 10. *If the metrics g and $\bar{g}$ on M^n are geodesically equivalent then the operators $\mathcal{I}_k$ pairwise commute. In particular they commute with the Beltrami-Laplace operator Δ. If the manifold M^n is closed then the operators $\mathcal{I}_k$ are self-adjoint.*

Corollary 6. *Suppose M^n is connected. Suppose the metrics g, $\bar{g}$ on M^n are geodesically equivalent. They are strictly non-proportional at least at one point of M^n if and only if the operators $\mathcal{I}_0, ..., \mathcal{I}_{n-1}$ are linearly independent.*

Corollary 7. *Suppose the manifold M^n is connected. Let g, $\bar{g}$ be metrics on it. Let the operators $\mathcal{I}_k$ commute and let they be linearly independent. Then the metrics $g, \bar{g}$ are geodesically equivalent.*

So that if the manifold is closed and if the metrics g, $\bar{g}$ on it are geodesically equivalent and strictly non-proportional at least at one point then we have the complete quantum integrability of the Beltrami-Laplace operator of the metric g.

Quantum integrability means that there exists a countable basis

$$\Phi = \{\phi_1, \phi_2, ..., \phi_m, ...\}$$

of the space $L_2(M^n)$ such that each ϕ_m is an eigenfunction of each operator $\mathcal{I}_k$.

Moreover, in our case the variables can be separated. More precisely, take any function ϕ from the basis Φ. Since ϕ is an eigenfunction of each operator $\mathcal{I}_k$, we have that ϕ is a solution of the system of n partial differential equations

$$\mathcal{I}_k\phi = \lambda_k\phi, \quad k = 0, 1, ..., n-1. \tag{9}$$

The separation of variables means that in a neighborhood of almost any point there exist coordinates $(x^0, x^1, ..., x^{n-1})$ such that in these coordinates the system (9) is equivalent to the system

$$\frac{\partial^2}{(\partial x^k)^2}\phi = F_k(x^k, \lambda_0, \lambda_1, ..., \lambda_{n-1})\phi, \quad k = 0, 1, ..., n-1, \tag{10}$$

where the function F_k depends on the variable x^k and on the parameters $\lambda_0, \lambda_1, ..., \lambda_{n-1}$. Then ϕ is the product

$$X_0(x^0)X_1(x^1)...X_{n-1}(x^{n-1}),$$

and each X_k is a solution of the k-th equation of (10) so that we reduced the system of partial differential equations (9) to the system of ordinary differential equations

$$\frac{\partial^2}{(\partial x^k)^2}X_k(x^k) = F_k(x^k, \lambda_0, \lambda_1, ..., \lambda_{n-1})X_k(x^k), \quad k = 0, 1, ..., n-1. \tag{11}$$

The space of the metrics, geodesically equivalent to a given one: If two metrics are geodesically equivalent then there exist at least one-parametric family of geodesically equivalent metrics and this family includes these two metrics [9] or [4]. It is possible to show that the set of metrics, geodesically equivalent to a given metric, is a manifold. What is the dimension of this manifold?

Even locally, there exist metrics that admit no (non-trivial) geodesically equivalent [11].

Even locally, the dimension of this space does not exceed $\frac{(n+1)(n+2)}{2}$ and is equal to $\frac{(n+1)(n+2)}{2}$ only for the metrics of constant curvature [11].

Let $g, \bar{g}$ be geodesically equivalent on M^n. Suppose M^n is closed and suppose the metrics are strictly non-proportional almost everywhere. Then the geodesic flow of the metric g is completely integrable, and almost all orbits lie at the corresponding Liouville tori. Suppose that the geodesic flow is non-resonant. Then the Liouville foliation is uniquely definite, and any integral of the geodesic flow commutes with the integrals $I_0, ..., I_{n-1}$. Assume in addition, that there are sufficiently many caustics of the Liouville tori of the geodesic flow: almost each point of the surface is an intersection of n caustics. Then the dimension of the space of the metrics (modulo multiplication by a constant), geodesically equivalent to the metric g, is equal one.

The geodesic flows of the metric of the ellipsoid and of the Poisson sphere (for different $a_1, a_2, ..., a_n$) satisfies all these conditions.

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