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Strictly non-proportional geodesically equivalent metrics have $h_{\mathbf{top}}(g) = 0$

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Abstract. If a closed manifold M possesses two Riemannian metrics which have the same unparameterized geodesics and are not strictly proportional at each point, then the topological entropy of both geodesic flows is zero. This is the main result of the paper and it has many dynamical and topological corollaries. In particular, such a manifold M should be finitely covered by the product of a rationally elliptic manifold and a torus.

1. *Definition and main results*

Definition 1. Two (C^{∞}-smooth) Riemannian metrics g and $\bar{g}$ on a manifold M^n are said to be *geodesically equivalent* if their geodesics coincide as unparameterized curves. They are *strictly non-proportional at* $x \in M^n$, if the polynomial $\det(g_{|x} - t\bar{g}_{|x})$ has only simple roots.

The question of whether two different metrics can have the same geodesics is natural and classical. The first examples are due to Beltrami [**B**]; a local description of geodesically equivalent metrics was understood by Dini [**Di**] and Levi-Civita [**LC**]. We will recall Levi-Civita's theorem in §2.1. For more historical details, see the surveys [**Mi, Am**], and/or the introductions to the papers [**M1, M4**].

The main result of our paper is the following theorem (for definition and properties of h_{top} we refer to [**Bo, KH, Ma**]).

THEOREM 1. *Suppose the Riemannian metrics g and $\bar{g}$ on a closed connected manifold M^n are geodesically equivalent and strictly non-proportional at least at one point. Then the topological entropy $h_{\mathrm{top}}(g)$ of the geodesic flow of g vanishes.*

The condition that the metrics are strictly non-proportional is important: for example, the product metric on a closed product manifold $M = M_1 \times M_2$ admits a family $g_1 + tg_2$ of non-proportional metrics (but not strictly non-proportional if $\dim M > 2$) with the same geodesics. However, if at least one factor has fundamental group with positive exponential growth (for instance if M_1 is hyperbolic), then by the Dinaburg theorem any geodesic flow on M has $h_{\mathrm{top}}(g) > 0$.

Vanishing of the topological entropy of a C^∞-smooth flow implies a lot of dynamical restrictions. For example, the ball volume grows subexponentially with its radius (Manning's inequality [**Mn**]), the number of geodesic arcs joining two generic points grows subexponentially with its maximal length (Mañé's formula [**Ma**]) and the volume of a compact submanifold propagated by the geodesic flow also changes subexponentially (Yomdin's theorem [**Y**]), see also [**P2**].

Probably even more interesting are topological restrictions implied by $h_{\mathrm{top}}(g) = 0$. The subexponential growth of $\pi_1(M^n)$ (Dinaburg's theorem [**D**]) is not very intriguing under the assumptions of Theorem 1, since it is known [**M3**] that in this case the fundamental group is virtually abelian. However, the restriction coming from the Gromov–Paternain theorem [**G, P1**] and from [**PP1**] are new, non-trivial and interesting: namely in the simply connected case the manifold M^n is *rationally elliptic*, i.e. $\pi_*(M^n) \otimes \mathbb{Q}$ is finite-dimensional. This is a very restrictive property as by the results of [**FHT, Pa**] a rationally elliptic manifold M^n enjoys the following properties:

(1) $\dim \pi_*(M^n) \otimes \mathbb{Q} \leq n$, $\dim H_*(M^n, \mathbb{Q}) \leq 2^{n-1}$, $\dim H_i(M^n, \mathbb{Q}) \leq \frac{1}{2}\binom{n}{i}$ $(i = 1, \dots, n-1)$;

(2) the Euler characteristic $\chi(M^n)$ satisfies $2^n - n + 1 \geq \chi(M^n) \geq 0$. Moreover, $\chi(M^n) > 0$ if and only if $H_{\mathrm{odd}}(M^n, \mathbb{Q}) = 0$.

A manifold M with finite $\pi_1(M)$ is said to be *rationally hyperbolic*, if its universal cover is not rationally elliptic. Thus, as a consequence of Theorem 1, we have the following corollary.

COROLLARY 1. *A rationally hyperbolic closed manifold M^n does not admit two geodesically equivalent Riemannian metrics g and $\bar{g}$ which are strictly non-proportional at least at one point.*

Rational hyperbolicity means nothing in dimensions less than four, as all closed 4-manifolds with finite fundamental group are rational-elliptic. Note that the topology of closed 2- and 3-manifolds admitting non-proportional geodesically equivalent metrics is completely understood: in dimension two, such manifolds are homeomorphic to the sphere, the projective plane, the torus or the Klein bottle [**MT2, BMF**]. In dimension three, such manifolds are homeomorphic to lens spaces or to Seifert manifolds with zero Euler number [**M2**].

Starting from dimension four, almost all simply-connected manifolds are rationally hyperbolic. For example, in dimension four, up to homeomorphism, there exist infinitely many simply-connected closed manifolds, and only five of them are rationally elliptic: S^4, $S^2 \times S^2$, $\mathbb{C}P^2$, $\mathbb{C}P^2 \# \mathbb{C}P^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. It is possible to construct geodesically equivalent metrics on S^4 and $S^2 \times S^2$ that are strictly non-proportional at least at one point. We conjecture here that these two are the only closed simply-connected 4-manifolds admitting strictly non-proportional geodesically equivalent metrics. In dimension five, a closed rational-elliptic manifold has rational homotopy type of $S^2 \times S^3$ or S^5 (there are infinitely many homotopy types for simply-connected 5-manifolds). By recent results of [**PP1**, Theorem E], a closed manifold admitting a metric with zero topological entropy is S^5, $S^3 \times S^2$, $SU(3)/SO(3)$ or the non-trivial S^3-bundle over S^2. We conjecture that $S^3 \times S^2$ and S^5 are the only closed simply-connected 5-manifolds admitting geodesically equivalent metrics which are strictly non-proportional at least at one point.

In §5 we announce the restrictions on the topology of non-simply-connected manifolds (admitting geodesically equivalent metrics which are strictly non-proportional at least at one point) that follow from Corollary 1.

Now let us comment the proof of Theorem 1. The main ingredients are Theorems 2, 3 and Corollary 2, which imply that the geodesic flow of g is Liouville-integrable.

Precisely the same integrable systems were recently actively studied in mathematical physics, in the framework of the theory of separation of variables. Depending on the school, they are called L-systems [**Be**], Benenti-systems [**IMM**] and quasi-bi-Hamiltonian systems [**CST**].

However, Liouville integrability does not immediately imply a vanishing of the topological entropy; counterexamples can be found in [**BT1, BT2, Bu1, Bu2, K, KT**]. If the singularities of the integrable system behave sufficiently well (non-degenerate in the sense of Williamson–Vey–Eliasson–Ito [**E, I**], see [**P1**], or the Taimanov conditions [**T**]), or if the system has a lot of symmetries (for example, as in collective integrability [**BP, P1**]), then $h_{\text{top}}(g) = 0$. However, for other situations nothing is known, even if the integrals are real-analytic or polynomial in momenta (if $n > 2$, but vanishing of $h_{\text{top}}(g)$ for $n = 2$ and analytically integrable metrics is proved in [**P0**]; also this follows from a description of singularities [**FM**] in the analytic case).

It is worth mentioning that geodesically equivalent metrics are usually not real-analytic: Levi-Civita's theorem from §2.1 shows the existence of an infinite-dimensional space of non-analytic C^∞-perturbations in the class of geodesically-equivalent metrics. Also the set of singular points of the constructed integrals for the corresponding Hamiltonian system can be quite complicated. For instance, the projection of the singularities in TM^n to the base M^n is surjective for $n > 2$ and its restriction to a singular Liouville leaf can have image which is locally the product of the Cantor set and the $(n - 1)$-dimensional disk (see Remark 3).

The logic of our proof for Theorem 1 is as follows.

(1) We show that the topological entropy is supported on the singularities, which we describe.

(2) We show that dynamics on them can be considered as a subsystem of the geodesic flow:
   - on a lower-dimensional closed submanifold;
   - admitting geodesically equivalent metrics which are strictly non-proportional at least at one point.

   Therefore we can apply induction in the dimension.

2. *Geometry behind the geodesic equivalence*

In the following argument we always assume that the manifold M^n is connected and that the Riemannian metrics g and $\bar g$ on M^n are geodesically equivalent and strictly non-proportional at least at one point.

2.1. *Integrability and Levi-Civita's theorem.*

A Riemannian metric g determines the map $\flat_g : TM \to T^*M$ with the inverse $\sharp^g : T^*M \to TM$. Consider the (1,1)-tensor (automorphism field) $L : TM \to TM$ given by the formula

$$L = (\det(\sharp^{\bar g} \circ \flat_g))^{-1/(n+1)} \cdot (\sharp^{\bar g} \circ \flat_g). \tag{1}$$

In local coordinates, $L_i^j = (\det(\bar g)/\det(g))^{1/(n+1)} g_{i\alpha}\bar g^{\alpha j}$. This tensor L determines the family $S_t \in C^\infty(T^*M \otimes TM)$, $t \in \mathbb{R}$, of $(1,1)$-tensors

$$S_t := \det(L - t\ \mathrm{Id}) \cdot (L - t\ \mathrm{Id})^{-1}. \tag{2}$$

Remark 1. Although $(L - t\ \mathrm{Id})^{-1}$ is not defined for eigenvalues $t \in \mathrm{Sp}(L)$, the tensor S_t is well-defined for every $t \in \mathbb{R}$. In fact, it is the adjunct matrix of $(L - t\ \mathrm{Id})$. Thus, by the Laplace main minors formula, S_t is a polynomial in t of degree $(n-1)$ with coefficients being $(1,1)$-tensors.

The isomorphism $\flat^g$ allows us to identify the tangent and cotangent bundles of M^n. This identification allows us to transfer the natural Poisson structure and the Hamiltonian system $H(x, p) = \frac{1}{2} p \cdot \sharp^g(p)$ from T^*M^n to TM^n.

THEOREM 2. [**MT1**] *If g, $\bar g$ are geodesically equivalent, then, for every $t_1, t_2 \in \mathbb{R}$, the functions*

$$I_{t_i} : TM^n \to \mathbb{R}, \quad I_{t_i}(v) := g(S_{t_i}(v), v) \tag{3}$$

are commuting integrals for the geodesic flow of g.

As L is self-adjoint with respect to both g and $\bar g$, the spectrum $\mathrm{Sp}(L)$ is real at every point $x \in M^n$. Denote it by $\lambda_1(x) \le \cdots \le \lambda_n(x)$. Every eigenvalue $\lambda_i(x)$ is at least a continuous function on M^n, and is smooth near the points where it is a simple eigenvalue.

THEOREM 3. [**M1**] *Let (M^n, g) be a geodesically complete connected Riemannian manifold. Let a Riemannian metric $\bar g$ on M^n be geodesically equivalent to g. Then, for every $i \in \{1, \dots, n-1\}$ and for all $x, y \in M^n$, the following hold:*

(1) $\lambda_i(x) \le \lambda_{i+1}(y)$;
(2) *if $\lambda_i(x) < \lambda_{i+1}(x)$, then $\lambda_i(z) < \lambda_{i+1}(z)$ for almost every point $z \in M^n$;*
(3) *if $\lambda_i(x) = \lambda_j(y)$ for a certain $j \ne i$, then there exists $z \in M^n$ such that $\lambda_i(z) = \lambda_j(z)$.*

COROLLARY 2. [**MT3**] *Let (M^n, g) be a connected Riemannian manifold. Suppose a Riemannian metric $\bar g$ on M^n is geodesically equivalent to g and is strictly non-proportional to g at least at one point. Then, for every mutually-different $t_1, t_2, \dots, t_n \in \mathbb{R}$, the integrals I_{t_i} are functionally independent almost everywhere, i.e. the differentials dI_{t_i} are linearly independent almost everywhere in TM.*

Let us describe the local form of the integrals I_t. For every $x \in M^n$ consider coordinates in $T_x M^n$ such that the metric g is given by the diagonal matrix $\mathrm{diag}(1, 1, \ldots, 1)$ and the tensor L is given by the diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Then the tensor (2) reads

$$\begin{aligned} S_t &= \det(L - t\ \mathrm{Id})(L - t\ \mathrm{Id})^{(-1)} \\ &= \mathrm{diag}(\Pi_1(t), \Pi_2(t), \ldots, \Pi_n(t)), \end{aligned}$$

where the polynomials $\Pi_i(t)$ are given by the formula

$$\Pi_i(t) \stackrel{\mathrm{def}}{=} \prod_{j \neq i} (\lambda_j - t).$$

Hence, for every $\xi = (\xi_1, \ldots, \xi_n) \in T_x M^n$, the polynomial $I_t(x, \xi)$ is given by

$$I_t = \xi_1^2 \Pi_1(t) + \xi_2^2 \Pi_2(t) + \cdots + \xi_n^2 \Pi_n(t). \tag{4}$$

For further use, let us consider the one-parameter family of functions

$$I_t' \stackrel{\mathrm{def}}{=} \frac{d}{dt}\big(I_t\big).$$

For every fixed $t \in \mathbb{R}$ this function is an integral of the geodesic flow for g.

Let us now formulate (a weaker version of) the classical Levi-Civita's theorem.

THEOREM 4. (Levi-Civita [**LC**]) *Consider two Riemannian metrics on an open subset $U^n \subset M^n$ and the tensor L given by formula (1). Suppose the spectrum* $\mathrm{Sp}(L)$ *is simple at every point $x \in U^n$.*

Then the metrics are geodesically equivalent on U^n if and only if around each point $x \in U^n$ there exist coordinates $x_1, x_2, \ldots, x_n$ in which the metrics have the following model form:

$$ds_g^2 = |\Pi_1(\lambda_1)|\, dx_1^2 + |\Pi_2(\lambda_2)|\, dx_2^2\ + \cdots + |\Pi_n(\lambda_n)|\, dx_n^2, \tag{5}$$

$$ds_{\bar g}^2 = \rho_1 |\Pi_1(\lambda_1)|\, dx_1^2 + \rho_2 |\Pi_2(\lambda_2)|\, dx_2^2 + \cdots + \rho_n |\Pi_n(\lambda_n)|\, dx_n^2, \tag{6}$$

where the functions ρ_i are given by

$$\rho_i \stackrel{\mathrm{def}}{=} \frac{1}{\lambda_1 \lambda_2 \cdots \lambda_n} \frac{1}{\lambda_i}$$

and $\lambda_i = \lambda_i(x_i)$ are smooth functions of one variable.

Definition 2. The above coordinates will be called *Levi-Civita coordinates* and the neighborhoods where the coordinates are defined will be called Levi-Civita charts.

In Levi-Civita coordinates the tensor L is diagonal $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, so the notation in the Levi-Civita theorem is compatible with that at the beginning of the section.

COROLLARY 3. [**M1, BM**] *Suppose the Riemannian metrics g, $\bar g$ are geodesically equivalent on M. Then, the Nijenhuis torsion of the tensor L given by (1) vanishes: $N_L = 0$.*

If the metrics are strictly non-proportional at least at one point, Corollary 3 follows from the above version of Levi-Civita's theorem. In the general case, Corollary 3 follows from the original version of Levi-Civita's theorem [**LC**] and was proven in [**M1**] and [**BM**].

Combining formulae (5) and (4), we see that in the Levi-Civita coordinates the function I_t is given by

$$I_t = \sum_i |\Pi_i(\lambda_i(x))|\Pi_i(t)\xi_i^2. \tag{7}$$

In particular, the function $I_{\lambda_i(x)}$ as the function on the cotangent bundle is equal to $(-1)^{i-1}p_i^2$.

2.2. *Distributions of eigenvectors: submanifolds M_A.* We begin with investigation of the set of points from the Levi-Civita charts, the union of which is the open dense set

$$\mathrm{Reg}(M) = \{x \in M \mid \lambda_i(x) \neq \lambda_j(x) \text{ for } i \neq j\}.$$

This set can be represented as the intersection $\mathrm{Reg}(M) = \cap_A \mathrm{Reg}_A(M)$ over all (proper) subsets $A \subset \{1, 2, \dots, n\}$, where we denote

$$\mathrm{Reg}_A(M) = \{x \in M \mid \forall i \in A, \forall j \notin A,\ \lambda_i(x) \neq \lambda_j(x)\}.$$

Notice that if at some point the metrics are strictly non-proportional and $n > 3$ (or $n > 2$ and no constant eigenvalues exist), then $\bigcup_A \mathrm{Reg}_A(M) = M$ (union by proper subsets).

For every point $x \in \mathrm{Reg}_A(M)$ denote by $D_A(x)$ the subspace of T_xM^n spanned by the eigenspaces with the eigenvalues λ_i, where $i \in A$. As the eigenvalues λ_i for $i \in A$ do not bifurcate with the eigenvalues λ_j for $j \notin A$, D_A is a smooth distribution on $\mathrm{Reg}_A(M)$. By Corollary 3 it is integrable. We will denote by $M_A(x)$ its integral submanifold containing $x \in \mathrm{Reg}_A(x) \subset M^n$.

LEMMA 1. *For $x \in \mathrm{Reg}_A(M)$ the following statements hold.*

(1) *The restrictions of g and $\bar g$ to $M_A(x)$ are geodesically equivalent.*

(2) *$g|_{M_A(x)}$ and $\bar g|_{M_A(x)}$ are strictly non-proportional at least at one point.*

(3) *For $i \in A$ the ith eigenspace of the operator L (corresponding to λ_i) coincides with the respective eigenspace of the operator $L_A : TM_A \to TM_A$, constructed via formula (1) for the restricted to $M_A(x)$ metrics $g|_{M_A(x)}$ and $\bar g|_{M_A(x)}$.*

(4) *There exists a constant c (which depends on $M_A(x)$ only and is explicitly calculated in the proof) such that the part of $c \cdot \mathrm{Sp}(L)$, corresponding to A, coincides with the spectrum $\mathrm{Sp}(L_A)$.*

(5) *In particular, if an eigenvalue λ_i, $i \in A$, is constant, then the corresponding eigenvalue of the operator L_A, constructed for the restrictions of g and $\bar g$ to $M_A(x)$, is constant on $M_A(x)$.*

Proof. The distribution D_A defines a foliation on $\mathrm{Reg}_A(M)$ and on its open dense subset $\mathrm{Reg}(M)$. Then it is sufficient to prove the first, third and fourth statements of the lemma at the points of this subset. By Theorems 3 and 4 in a neighborhood of every point $x \in \mathrm{Reg}(M)$, there exist Levi-Civita coordinates such that the metrics g, $\bar g$ are given by formulas (5) and (6). In these coordinates, $M_A(x)$ is the coordinate plaque of the coordinate collection x_α with $\alpha \in A = \{\alpha_1, \dots, \alpha_m\}$. Then the restrictions of the metrics to $M_A(x)$ are given by

$$g_{|M_A} = |\Pi_{\alpha_1}(\lambda_{\alpha_1})|\, dx_{\alpha_1}^2 + |\Pi_{\alpha_2}(\lambda_{\alpha_2})|\, dx_{\alpha_2}^2 + \dots + |\Pi_{\alpha_m}(\lambda_{\alpha_m})|\, dx_{\alpha_m}^2,$$
$$\bar g_{|M_A} = \rho_{\alpha_1}|\Pi_{\alpha_1}(\lambda_{\alpha_1})|\, dx_{\alpha_1}^2 + \rho_{\alpha_2}|\Pi_{\alpha_2}\lambda_{\alpha_2}|\, dx_{\alpha_2}^2 + \dots + \rho_{\alpha_m}|\Pi_{\alpha_m}(\lambda_{\alpha_m})|\, dx_{\alpha_m}^2.$$

As λ_j is constant on $M_A(x)$ for every $j \notin A$, every factor of Π_{α_i} of the form $\lambda_j - \lambda_{\alpha_i}$ can be 'hidden' in $dx_{\alpha_i}^2$. We see that then the first metric is already in the Levi-Civita form, and the second metric becomes in the Levi-Civita form after multiplication by

$$C \stackrel{\text{def}}{=} \prod_{j \notin A} \lambda_j, \tag{8}$$

which is constant on $M_A(x)$. Hence, by Levi-Civita's theorem, the restrictions of the metrics to M_A are geodesically equivalent.

Direct calculations show that in local coordinates the tensor L_A is given by:

$$C^{1/(m+1)} \operatorname{diag}(\lambda_{\alpha_1}, \dots, \lambda_{\alpha_m}). \tag{9}$$

The third and fourth statements of the lemma follow.

Now let us prove the second statement. Suppose the restriction of the metrics are not strictly non-proportional at every point of a certain $M_A(x)$. Then, by Theorem 3, there exist $\alpha_1, \alpha_2 \in A$ such that $\lambda_{\alpha_1} \equiv \lambda_{\alpha_2}$ on $M_A(x)$. Consider the set $B := \{1, \dots, n\} \setminus A$. Take the union of all leaves M_B containing at least one point of $M_A(x)$. Clearly, this union contains an open subset of M^n. As the eigenvalues $\lambda_{\alpha_1},\ \lambda_{\alpha_2}$ are constant along M_B, in view of formula (9) and Theorem 3, at every point of this open subset we have $\lambda_{\alpha_1} = \lambda_{\alpha_2}$, which contradicts Theorem 3. □

LEMMA 2. *Suppose the eigenvalue λ_i is not a constant. Take a point $y \in M^n$ such that*

$$\max_{x \in M} \lambda_{i-1}(x) < \lambda_i(y) < \min_{x \in M} \lambda_{i+1}(x).$$

(We assume by definition that $\min_{x\in M} \lambda_{n+1}(x) = \infty$ *and* $\max_{x \in M} \lambda_0(x) = -\infty$.*)*

Let $\mathsf{C}(i) := \{1, 2, \dots, n\} \setminus \{i\}$. Then, $M_{\mathsf{C}(i)}(y)$ is a closed submanifold.

The conditions that the eigenvalue is not constant and that λ_i is neither maximum nor minimum are important: one can construct counterexamples, if either of these conditions is omitted.

Proof. Since $\max_{x\in M} \lambda_{i-1}(x) < \lambda_i(y) < \min_{x \in M} \lambda_{i+1}(x)$, there exist $c_{\text{small}}, c_{\text{big}} \in \mathbb{R}$ such that:

- $c_{\text{small}} < \lambda_i(y) < c_{\text{big}}$;
- at least one of the numbers $c_{\text{small}}, c_{\text{big}}$ is a regular value of the function λ_i;
- the other number is not a critical value of λ_i (i.e. it is either a regular value or is equal to λ_i at no point).

Denote by N the connected component of the set

$$\{x \in M^n \mid c_{\text{small}} \le \lambda_i(x) \le c_{\text{big}}\},$$

containing the point y. Then $N \subset \mathrm{Reg}_{\mathsf{C}(i)}(M)$ is a connected manifold with boundary. Therefore, $D_{\mathsf{C}(i)}$ is a smooth distribution on N. As it is integrable by Corollary 3, it defines a foliation. By Corollary 3, the function λ_i is constant on the leaves of the foliation. Then, every connected component of the boundary of N is a leaf of the foliation.

At every $x \in M^n$, consider the vector v_i satisfying

$$\begin{cases} L(v_i) = \lambda_i(x)v_i, \\ g(v_i, v_i) = |\Pi_i(\lambda_i)|. \end{cases} \tag{10}$$

By definition of N, the function $|\Pi_i(\lambda_i)|$ is non-zero and smooth at every point of N. Thus v_i vanishes nowhere in N. Hence, at least on the double cover of N, it is defined globally up to a sign and is smooth. The double cover projection maps closed submanifolds into closed ones. Therefore, without loss of generality, we can assume that the vector field v_i is globally defined already on N.

Consider the flow of the vector field v_i. It takes leaves to leaves. Indeed, it is sufficient to prove this almost everywhere, for instance in Levi-Civita charts. In Levi-Civita coordinates the leaves of the foliation are the plaques of the coordinates x_α, where $\alpha \in \mathsf{C}(i)$, and the vector field v_i is $\pm\partial/\partial x_i$, so the claim is trivial.

As the leaves are $(n-1)$-dimensional and the flow of v_i shuffles them, the flow acts transitively and all leaves are homeomorphic. Every connected component of the boundary of N is compact and is a leaf, hence all leaves are compact. In particular, $M_{\mathsf{C}(i)}(y)$ is compact. □

2.3. *Bifurcation of eigenvalues: submanifolds* Sing_i^j. The spectrum $\mathrm{Sp}(L)$ is simple in $\mathrm{Reg}(M)$, i.e. almost everywhere in M^n. However, at certain points the multiplicity of some λ_i can become greater than one. Such points will be called *the bifurcation points* of λ_i. By Theorem 3 the following types of bifurcations of the eigenvalue λ_i are possible.

Case 1. The eigenvalues λ_i and λ_{i+1} are not constant and there exists $x \in M$ such that $\lambda_i(x) = \lambda_{i+1}(x)$. Denote $\bar\lambda_i = \max \lambda_i(x) = \min \lambda_{i+1}(x)$. Let us consider the set

$$\mathrm{Sing}_i^1 \stackrel{\mathrm{def}}{=} \{x \in M^n \mid (\lambda_i(x) - \bar\lambda_i)(\lambda_{i+1}(x) - \bar\lambda_i) = 0\}.$$

This set was studied in [**M1**, Theorem 6]. It was shown that Sing_i^1 is a connected closed totally geodesic submanifold of codimension one. The restrictions of the metrics to it are strictly non-proportional at least at one point. Note that not all points of Sing_i^1 are points of bifurcation of the eigenvalues λ_i, λ_{i+1}.

Case 2. There exists $x \in M$ and $i \in \{2, \dots, n-1\}$ such that $\lambda_{i-1}(x) = \lambda_{i+1}(x)$. In this case, the eigenvalue λ_i is constant. Let us consider the set

$$\mathrm{Sing}_i^2 \stackrel{\mathrm{def}}{=} \{x \in M^n \mid (\lambda_{i-1}(x) - \lambda_i)(\lambda_{i+1}(x) - \lambda_i) = 0\}.$$

This set was also studied in [**M1**, Theorem 6]. It was shown that Sing_i^2 is a connected closed totally geodesic submanifold of codimension two. The restrictions of the metrics to it are strictly non-proportional at least at one point. Moreover, the set of the points $x \in \mathrm{Sing}_i^2$ such that $\lambda_{i-1}(x) = \lambda_{i+1}(x)$ is nowhere dense in Sing_i^2.

Case 3(a). The eigenvalue λ_i is constant, there exists $x \in M$ such that $\lambda_i = \lambda_{i+1}(x)$ and there exists no y such that $\lambda_{i-1}(y) = \lambda_i$.

Case 3(b). The eigenvalue λ_i is constant, there exists $x \in M$ such that $\lambda_i = \lambda_{i-1}(x)$ and there exists no y such that $\lambda_{i+1}(y) = \lambda_i$.

In Cases 3(a) and 3(b), let us consider respectively the sets

$$\mathrm{Sing}_i^3 = \{x \in M^n \mid \lambda_i = \lambda_{i+1}(x)\} \quad \text{or} \quad \mathrm{Sing}_i^3 = \{x \in M^n \mid \lambda_i = \lambda_{i-1}(x)\}.$$

The next lemma shows that, similar to Cases 1 and 2, Sing_i^3 is a submanifold of codimension two and the restrictions of the metrics to Sing_i^3 are geodesically equivalent and strictly non-proportional at least at one point. Note that, in contrast to the previous cases, the set Sing_i^3 is not necessary connected.

LEMMA 3. *Under assumptions of Cases 3(a) or 3(b), the set* Sing_i^3*:*

(1) *is totally geodesic;*

(2) *is a closed submanifold of codimension two;*

(3) *the restrictions of the metrics g and $\bar{g}$ to it are strictly non-proportional at least at one point.*

Here we will prove that Sing_i^3 is a closed submanifold of codimension two such that the restrictions of the metrics to it are strictly non-proportional at least at one point. The first statement of the lemma, namely that Sing_i^3 is totally geodesic, will follow immediately from Theorem 6, see Remark 4. Before Theorem 6, Lemma 3 will be used only once, namely in the proof of Theorem 5. As the proof of Theorem 6 does not require Theorem 5, no logical loop appears.

Proof of statements (2) and (3) of Lemma 3. We consider Case 3(a), the other case is completely analogous. By definition, the set Sing_i^3 is closed and, therefore, compact.

Let us show that locally Sing_i^3 is a submanifold of codimension two. Let $A = \{i, i+1\}$. Take a point x_0 such that $\lambda_i = \lambda_{i+1}(x_0)$. Then $x_0 \in \mathrm{Reg}_A(M)$ and we can consider the set $M_A(x_0)$. By Lemma 1, the restrictions of the metrics to $M_A(x_0)$ are geodesically equivalent and strictly non-proportional at least at one point. Since $M_A(x_0)$ is two-dimensional, the set of points where these restrictions are proportional is discrete [**MT2, BF**]. In view of Lemma 1, the restrictions of the metrics are proportional at x_0. Then in a small neighborhood of x_0, there exists no other point $x \in M_A(x_0)$ such that $\lambda_i = \lambda_{i+1}(x)$. Denote by B the set $\{1, 2, \ldots, n\} \setminus A$. For every point x of a small neighborhood of x_0 in $M_A(x_0)$, consider the set $M_B(x)$. It is a submanifold of codimension two. As the eigenvalues λ_i, λ_{i+1} are constant along M_B, in a small neighborhood of x_0 the set Sing_i^3 coincides with $M_B(x_0)$. Thus it is a submanifold of codimension two.

By the second statement of Lemma 1, the restrictions of the metrics to Sing_i^3 are strictly non-proportional at least at one point. $\square$

Remark 2. We will not prove or use it, but for general understanding let us note that the sets $T\,\mathrm{Sing}_i^j$ consist of singular points which are not removable (the definitions are in §3).

Moreover,

$$T\,\mathrm{Sing}_i^1 = \{(x,\xi)\in TM \mid I_{\bar\lambda_i}=0,\ dI_{\bar\lambda_i}=0\},$$
$$T\,\mathrm{Sing}_i^2 = \{(x,\xi)\in TM \mid I_{\lambda_i}(x,\xi)=0,\ dI'_{\lambda_i}=0\},$$
$$T\,\mathrm{Sing}_i^3 = \{(x,\xi)\in TM \mid I_{\lambda_i}(x,\xi)=0,\ dI'_{\lambda_i}=0\}.$$

Note that for Case 1 the set of points x with $\lambda_i(x)=\bar\lambda_i\neq\lambda_{i+1}(x)$ or $\lambda_i(x)\neq\bar\lambda_i=\lambda_{i+1}(x)$, which is everywhere dense in Sing_i^1, is included in $\mathrm{Reg}(M)$. However, these points do not behave as other regular points and for certain purposes we will need to exclude them: by definition, let

$$\mathrm{Reg}^\natural(M)=\mathrm{Reg}(M)\Big\backslash\bigcup_i \mathrm{Sing}_i^1\,.$$

As we explained above, $M^n\setminus\mathrm{Reg}^\natural(M^n)$ is a finite union of closed totally geodesic submanifolds.

Let us note that for a fixed i only one of the submanifolds Sing_i^j, $j=1,2,3$, can be non-empty.

3. *Description of singular points*

Consider some mutually-different numbers $t_1,\dots,t_n\in\mathbb{R}$ and the respective integrals $I_{t_1},\dots,I_{t_n}$. Consider the Poisson action of the group $(\mathbb{R}^n,+)$ on TM^n: an element $(a_1,\dots,a_n)\in\mathbb{R}^n$ acts by time-one shift along the Hamiltonian vector field of the function $a_1I_{t_1}+\dots+a_nI_{t_n}$. As the functions are commuting integrals, the action is well-defined, smooth, symplectic, and preserves the integrals I_t and the Hamiltonian of the geodesic flow (see [**A**, §49] for details).

A point $(x,\xi)\in TM$ is said to be singular if the differentials $dI_{t_1},\dots,dI_{t_n}$ are linearly dependent at (x,ξ). An orbit of the action is said to be singular if it has a singular point. All points of a singular orbit are singular and have the same coefficients of the linear dependence.

Although the Poisson action depends on the choice of constants $t_1,\dots,t_n$, the property of (x,ξ) being singular does not depend on the choice of t_i as far as these numbers are all different.

3.1. *Singular points in Levi-Civita coordinates.* As we have remarked in §2.3, the submanifolds $T\,\mathrm{Sing}_i^j\subset TM$ consist of singularities.

The next theorem describes singular points that lie over a Levi-Civita chart $U^n\subset\mathrm{Reg}(M^n)$. Fix a point $x\in\mathrm{Reg}(M^n)$ and denote by $\bar\lambda_1,\dots,\bar\lambda_n$ the constants $\lambda_1(x),\dots,\lambda_n(x)$, respectively.

THEOREM 5. *Let the metrics g and $\bar g$ be given by formulas (5) and (6) in a neighborhood $U^n\subset M^n$. If the point $(y,\xi)=(x_1,\dots,x_m,\xi_1,\dots,\xi_m)\in T\,\mathrm{Reg}(M^n)$ is singular, then there exists $i\in\{1,\dots,n\}$ such that $dI_{\bar\lambda_i}=0$. Then* $I_{\bar\lambda_i}(x,\xi)=0$ *and at least one of the following statements holds:*

(1) *the derivative $\partial\lambda_i(x)/\partial x_i$ vanishes at x;*

(2) *The function $I'_{\bar\lambda_i}$ vanishes at (x,ξ).*

Moreover, if $M_{C(i)}(y)$ is compact, the whole geodesic passing through y with the velocity vector ξ is contained in $M_{C(i)}(y)$, where $C(i)$ is the same as in Lemma 2.

Actually, the assumption that $M_{C(i)}(y)$ is compact is not necessary: Theorem 5 remains true, if we replace this condition by the condition that $y \notin \mathrm{Sing}_i^1$. Our stronger assumption makes the proof shorter.

Proof of Theorem 5. Suppose the point (y, ξ) is singular. Then, there exist constants $(\mu_1, \dots, \mu_n) \neq (0, \dots, 0)$ such that at (y, ξ) the following holds:

$$\mu_1 dI_{\bar\lambda_1} + \dots + \mu_n dI_{\bar\lambda_n} = 0.$$

We will show that for every i such that $\mu_i \neq 0$ the differential $dI_{\bar\lambda_i}$ vanishes at (y, ξ). For every $j \in \{1, \dots, n\}$ consider the function $I_{\lambda_j(x)}(x, \eta) := (I_t(x, \eta))_{|t=\lambda_j(x)}$. In a small neighborhood of y, the function λ_j is smooth. Hence the function $I_{\lambda_j(x)}$ is smooth as well. At the point (y, ξ) we have

$$dI_{\lambda_j(y)} = dI_{\bar\lambda_j} + I'_{\bar\lambda_j} \cdot d\lambda_j.$$

We will work on the cotangent bundle to M^n. As we explained in §2.1, the function $I_{\lambda_j(x)}$ is equal to $(-1)^{j-1} p_j^2$ and its differential has coordinates

$$(\underbrace{0, \dots, 0}_{n+j-1}, 2 \cdot (-1)^{j-1} \cdot p_j, 0, \dots, 0).$$

As the function λ_j depends on x_j only, its differential is

$$\left(\underbrace{0, \dots, 0}_{j-1}, \frac{\partial \lambda_j}{\partial x_j}, 0, \dots, 0\right).$$

Thus, $dI_{\bar\lambda_j}$ at (y, ξ) is given by

$$\left(\underbrace{0, \dots, 0}_{j-1}, I'_{\bar\lambda_j} \cdot \frac{\partial \lambda_j}{\partial x_j}, \underbrace{0, \dots, 0}_{n-1}, 2 \cdot (-1)^{j-1} \cdot p_j, 0, \dots, 0\right).$$

We see that the differentials $dI_{\bar\lambda_j}$ do not combine: if $\mu_i \neq 0$, then $dI_{\bar\lambda_i} = 0$. Therefore, $p_i = 0$ (i.e. $\xi_i = 0$), which is equivalent to $I_{\bar\lambda_i}(x, \xi) = 0$, and at least one of the following holds: $\partial\lambda_i(x)/\partial x_i = 0$ or $I'_{\bar\lambda_i}(x, \xi) = 0$. The first part of the theorem is proven.

Now let us show that the geodesic γ such that $(\gamma(0), \dot\gamma(0)) = (y, \xi)$ is contained in $M_{C(i)}(y)$. As $M_{C(i)}(y)$ is compact, it is sufficient to prove that at almost every point of the geodesic the velocity vector of the geodesic is contained in $D_{C(i)}$. As Sing_k^j are totally geodesic submanifolds, the geodesic γ intersect them transversally, and it is sufficient to prove that the velocity vector of the geodesic lies in $D_{C(i)}$ in Levi-Civita's charts.

As $I_{\bar\lambda_i}$ is an integral and $dI_{\bar\lambda_i} = 0$ at (y, ξ), we obtain that $dI_{\bar\lambda_i}$ vanishes at every point $(\gamma(t), \dot\gamma(t))$. Then, as we explained above, in the Levi-Civita chart, the component ξ_i equals zero, so that the velocity vector of the geodesic lies in $D_{C(i)}$. Finally, the geodesic stays in $M_{C(i)}$ forever. □

Remark 3. Consider a point x from a Levi-Civita chart and a point $(x, \xi) \in TM$ over it with $\xi_i = 0$. If $n > 2$ we have enough freedom (two free coordinates ξ_j, $j \neq i$) to arrange $I'_{\bar\lambda_i} = 0$ (the coordinate representation of $I'_{\bar\lambda_i}$ is obtained from formula (4)), in which case (x, ξ) is a singularity. Thus the projection of singularities to M^n is dense and hence coincides with M^n for $n > 2$.

If (x, ξ) is restricted to lie in a singular Liouville leaf $\mathcal{L}_c = \{I_{\bar\lambda_j} = \mathrm{const}_j \mid 1 \le j \le n\}$, we do not have that much freedom to obtain subjectivity of the projection, but still the projection can be very complicated. For example, consider a Liouville leaf such that $I_{\bar\lambda_i}(x, \xi) = 0$ on it. Consider the function λ_i on its projection. Let $K_i = \{x_i \mid \lambda_i(x_i) = \bar\lambda_i, \partial\lambda_i/\partial x_i = 0\}$. Consider the $(n-1)$-dimensional disk D^{n-1} with coordinates $(x_j \mid j \neq i)$. Then, by Theorem 5, the set $D^{n-1} \times K_i$ is locally contained in the projection of the set of singularities (and for certain leaves locally coincides with the projection of the set of singularities). However, the closed set K_i can be quite complicated, for instance, a Cantor set.

3.2. *Removable singularities.* Our next goal is to show that certain singular points are artificially singular: if we use a finite cover and choose the integrals appropriately, they become regular.

Suppose the eigenvalue λ_i is constant. From the proof of Theorem 5 it follows that for every $x \in \mathrm{Reg}_{\{i\}}(M)$ and $\xi \in D_{\mathsf{C}(i)}(x) \subset T_x M^n$ the differential dI_{λ_i} vanishes at (x, ξ). We will show that this singularity is *removable*, in the sense that on an appropriate finite cover we can find a function J_i, that is linear in velocities and such that $J_i^2 = (-1)^{i-1} I_{\lambda_i}$. This relation immediately implies that J_i commutes with the functions I_t. Since I_{λ_i} is an integral, J_i is an integral as well. As it is linear in velocities, it corresponds to a Killing vector field. We will show that this Killing vector field is non-zero at x, which automatically implies that the differential of this integral does not vanish at (x, ξ).

In the Levi-Civita coordinates, $I_{\lambda_i} = (-1)^{i-1} p_i^2$ and we can put $J_i = \pm p_i$. Clearly, in the Levi-Civita coordinate system, $J_i(\eta) := g(v_i, \eta)$, where $v_i = \pm\partial/\partial x_i$.

Note that the vector field $\partial/\partial x_i$ satisfies conditions (10), and that near every regular point every vector field satisfying (10) is the vector field $\partial/\partial x_i$ of a certain Levi-Civita coordinate system.

Thus, in order to show that (at least on a finite cover) there exists a smooth function J_i such that it is linear in velocities and such that $J_i^2 = (-1)^{i-1} I_{\lambda_i}$, it is sufficient to prove the following theorem.

THEOREM 6. *Suppose λ_i is constant. Then at least on a double cover of M^n there exists a smooth vector field v_i satisfying (10) at every point $x \in M^n$.*

Remark 4. Conditions (10) imply that the zeros of v_i coincide with $\bigcup_{j=2,3} \mathrm{Sing}_i^j$. As v_i is a Killing vector field, Sing_i^3 is a totally-geodesic submanifold.

Proof of Theorem 6. First we show that at least on the double cover there exists a continuous vector field v_i with the required properties. In order to do this, it is sufficient to prove the following semi-local statement:

$$\text{Locally near every point } x \text{ there exist precisely two continuous vector fields } v_i \text{ satisfying (10).} \tag{S}$$

If $\lambda_{i-1}(x) \neq \lambda_i \neq \lambda_{i+1}(x)$, then $y \in \mathrm{Reg}_{\{i\}}(M)$. Then, $\Pi_i(\lambda_i) \neq 0$. Hence, $v_i \neq 0$ in a small neighborhood of x and the statement (S) is trivial.

Let us consider $x \in \mathrm{Sing}_i^j$, where $j = 2$ or 3, and prove the statement in a small disk neighborhood $U^n \ni x$.

First, if a vector field v_i satisfies (10), then the vector field $-v_i$ satisfies (10) as well. As Sing_i is nowhere dense, the fields do not coincide. Therefore, we obtain at least two different required vector fields.

Next, there exist no more than two such vector fields. Indeed, such a vector field v_i must vanish along Sing_i^j, as $\Pi_i(\lambda_i)$ equals zero there, and it is non-zero in the complement. This complement is connected, because Sing_i^j has codimension two (by the proven part of Lemma 3 and as we explained in §2.3), and the claim follows.

At last, let us prove that such a continuous field v_i exists in the small disk neighborhood $U^n \ni x$. As $U^n \setminus \mathrm{Sing}_i^j$ is connected, we can define v_i in one of two possible ways at some point x_0 and extend by continuity along paths in $U^n \setminus \mathrm{Sing}_i^j$. We need to show that the result is well-defined.

In order to do this we connect two paths ϕ_0, ϕ_1 from x_0 to x_1 in $U^n \setminus \mathrm{Sing}_i^j$ by a homotopy ϕ_τ in U^n. The paths and the homotopy can be assumed smooth. As Sing_i^j has codimension two, we can perturb the homotopy and make it to be transversal to Sing_i^j. Thus, the intersection of $\mathrm{Image}_{\phi_\tau}$ with Sing_i^j is a finite set $\{(t_k, \tau_k)\} \in [0,1] \times [0,1]$ and it suffices to consider only one point of intersection $y_0 = \phi_{\tau_0}(t_0) = \phi(t_0, \tau_0) \in \mathrm{Sing}_i^j$. If we can find the required field v_i on a transversal two-dimensional disk at y_0, we are done.

As we explained in §2.3, at almost every point $y \in \mathrm{Sing}_i^j$ we have $\lambda_{i-1}(y) \neq \lambda_{i+1}(y)$. (Actually, for $j = 3$ this is true at every point.) Thus, without loss of generality, we can assume that $\lambda_{i-1}(y_0) \neq \lambda_{i+1}(y_0)$.

Assume $\lambda_{i-1}(y_0) \neq \lambda_i = \lambda_{i+1}(y_0)$. The case $\lambda_{i-1}(y_0) = \lambda_i \neq \lambda_{i+1}(y_0)$ is completely analogous.

Let $A = \{i, i+1\}$. Then $y_0 \in \mathrm{Reg}_A(M)$. Consider the leaf $M_A(y_0)$. This is a two-dimensional manifold transverse to Sing_i^j at y_0. The homotopy can be perturbed to have the image locally coinciding with $M_A(y_0)$. As $v_i \in D_A$, the problem, thanks to Lemma 1, is reduced to a local two-dimensional question on $M_A(y_0)$.

Consider the restriction of the metrics to $M_A(y_0)$. Denote by L_A the tensor (1) constructed for the restrictions of the metrics. We denote by $\lambda_A \leq \lambda'_A$ its eigenvalues. By Lemma 1, λ_A is constant, λ'_A is not. If there exists a (continuous) vector field v_A on M_A such that it vanishes precisely at y_0, such that it is eigenvector of L_A with eigenvalue λ_A and such that its length is $(\lambda'_A - \lambda_A)^{1/2}$, we are done. Indeed, by Lemma 1 the vector field v_i given by

$$\left(C^{-1/3}\left|\prod_{\alpha \neq i, i+1} (\lambda_i - \lambda_\alpha)\right|\right)^{1/2} v_A,$$

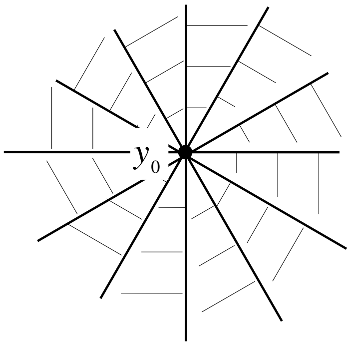

FIGURE 1. In dimension two, there exists a vector field orthogonal to all geodesics containing y_0.

where C is given by formula (8), satisfies the conditions (10). As

$$\left(C^{-1/3}\Big|\prod_{\alpha\neq i,i+1}(\lambda_i-\lambda_\alpha)\Big|\right)^{1/2}$$

is a smooth positive function, the existence of v_A implies the existence of v_i.

Let us prove the existence of such a vector field v_A. At every $y \in M_A(y_0)$, $y \neq y_0$, denote by l_A the eigenspace of L_A corresponding to λ_A. Let us show that for every geodesic γ on $M_A(y_0)$ passing through y_0 the velocity vector $\dot\gamma(t)$ is orthogonal (in the restriction of g) to l_A at every $\gamma(t) \neq y_0$. Indeed, let I_t^A be the one-parametric family of the integrals from Theorem 2 constructed for the restrictions of g and $\bar g$ to $M_A(y_0)$. Consider the integral $I^A_{\lambda_A}$. At the tangent plane to every point z consider the coordinates such that the restriction of g to $M_A(y_0)$ is given by diag(1, 1) and L_A is diag(λ_A, λ'_A). In these coordinates, the integral I_t^A equals $(\lambda'_A - t)\xi_1^2 + (\lambda_A - t)\xi_2^2$, so that $I^A_{\lambda_A}$ is equal to $(\lambda'_A - \lambda_A)\xi_1^2$. We see that the integral vanishes on every geodesic γ passing through y_0. Because $\lambda'_A(z) \neq \lambda_A(z)$ for $z \neq y_0$, we obtain that the component ξ_1 of the velocity vector of γ at z vanishes, which means that the eigenvalue of L_A corresponding to λ_A is orthogonal to γ.

Clearly, in $M_A(y_0) \setminus y_0$ there exists a vector field of length one such that it is orthogonal to the geodesics passing through y_0, see Figure 1.

Multiplying this vector field by $(\lambda'_A - \lambda_A)^{1/2}$, we obtain a required vector field v_A on $M_A(y_0) \setminus y_0$. We put $v_A = 0$ at point y_0. As $(\lambda'_A - \lambda_A)^{1/2}$ converges to zero when x tends to y_0, the result is a required continuous vector field v_A on $M_A(y_0)$. Therefore, there exists a vector field v_i along $M_A(y_0)$ (satisfying (10)). Thus, the vector v_i at x_1 does not depend on the choice of path connecting x_0 and x_1. Finally, v_i is well-defined at the whole $U^n \setminus \mathrm{Sing}_i^j$, and is at least continuous on it.

At the points of $U^n \cap \mathrm{Sing}_i^j$ let us put v_i equal to zero. As $\Pi_i(\lambda_i)$ tends to zero when x approaches Sing_i^j, the vector field is continuous on U^n. Statement (S) is proven.

Then, at least on the double cover of M^n, there exists a continuous vector field v_i satisfying (10). Without loss of generality, we can assume that the vector field v_i is defined already on M^n.

Now let us prove that the vector field v_i is actually smooth. Clearly, it is smooth on the complement to Sing_i^j, because it coincides with the appropriate field $\partial/\partial x_i$ there. Denote by F_t the flow of the vector field v_i on $M^n \setminus (\mathrm{Sing}_i^2 \cup \mathrm{Sing}_i^3)$. This flow is globally (i.e. for every value of t) defined. Indeed, if $x \notin \mathrm{Sing}_i^2 \cup \mathrm{Sing}_i^3$, then $\lambda_{i-1}(x) < \lambda_i < \lambda_{i+1}(x)$. As v_i is an eigenvector of L with eigenvalue λ_i and the Nijenhuis tensor N_L vanishes (Corollary 3), for every t we have $\lambda_{i-1}(F_t(x)) = \lambda_{i-1}(x)$, $\lambda_{i+1}(F_t(x)) = \lambda_{i+1}(x)$. Therefore, the trajectory of the flow passing through x never approaches the set $\mathrm{Sing}_i^2 \cup \mathrm{Sing}_i^3$.

The function $J(\eta) := g(v_i, \eta)$ is an integral of the geodesic flow, depending linearly on the velocity. This implies that F_t acts by isometries on $M^n \setminus (\mathrm{Sing}_i^2 \cup \mathrm{Sing}_i^3)$. As $M^n \setminus (\mathrm{Sing}_i^2 \cup \mathrm{Sing}_i^3)$ is everywhere dense in M^n, the map F_t can be extended by completeness to act by isometries on the whole M^n. Thus, there exists a Killing vector field on M^n coinciding with v_i almost everywhere. As every Killing vector field is smooth, the vector field v_i is smooth. □

4. *Proof of Theorem 1*

We use induction by the dimension. If the dimension of the manifold is $n < 2$, Theorem 1 is trivial. Assume that for every dimension less than n Theorem 1 is true and consider $\dim M = n$.

Vanishing of the topological entropy for the lift of a dynamical system to a finite cover (of a closed manifold) implies vanishing of the topological entropy of the original system. Thus, we assume that already on M^n for every constant eigenvalue λ_i we can associate a global vector field v_i from Theorem 6. Therefore, for every constant λ_i, we globally define the integral J_i such that its differential does not vanish over the points of $\mathrm{Reg}(M^n)$, it commutes with all integrals I_t and it is functionally dependent with the integral I_{λ_i}.

By geodesic flow we will understand the restriction of the Hamiltonian system on TM^n with the Hamiltonian $H(\xi) := g(\xi, \xi)$ to $T_1M^n = \{\xi \in TM^n \mid H(\xi) = 1\}$. The symplectic form on TM^n came from T^*M^n via standard identification by g.

As T_1M^n is compact, the variational principle (see, for example, [**KH**, Theorem 4.5.3]) holds, and we obtain

$$h_{\mathrm{top}}(g) = \sup_{\mu \in \mathfrak{B}} h_\mu(g).$$

Here $\mathfrak{B}$ is the set of all invariant ergodic probability measures on T_1M^n and h_μ is the entropy of an invariant measure μ. Recall that a measure is said to be *ergodic*, if $\mu(B)(1 - \mu(B)) = 0$ for all μ-measurable invariant Borel sets B.

Therefore, in order to prove Theorem 1, it is sufficient to prove that $h_\mu(g) = 0$ for all $\mu \in \mathfrak{B}$. Fix one such measure and let $\mathrm{Supp}(\mu)$ be its support (the set of $x \in M^n$ such that every neighborhood $U_\epsilon(x)$ has positive measure).

As the measure is ergodic, its support lies on a level surface of every invariant continuous function. Then, $\mathrm{Supp}(\mu)$ is included into a Liouville leaf Υ. (Recall that *a Liouville leaf* is a connected component of the set $\{I_{t_1} = c_1, \dots, I_{t_n} = c_n\}$, where $c_1, \dots, c_n$ are constants.)

Suppose a point $\xi \in \mathrm{Supp}(\mu)$ is non-singular or is a removable singular point (in the sense that every I_{λ_i} such that $dI_{\lambda_i} = 0$ can be replaced by a linear integral J_i

such that $dJ_i \neq 0$). Then, a small neighborhood $U(\xi)$ of ξ in $\mathrm{Supp}(\mu)$:

- has positive measure in μ;
- contains only points that are non-singular or removable-singular.

We will show that these two conditions imply that the entropy of μ is zero.

By the Implicit Function Theorem, Υ is n-dimensional near ξ. Denote by $O(\xi)$ the orbit of the Poisson action of $(\mathbb{R}^n, +)$ containing ξ. As it is also n-dimensional, in a small neighborhood of ξ it coincides with Υ. Thus, $U(\xi) \subset O(\xi)$.

The orbits of the Poisson action and the dynamic on them are well-studied (see, for example, [**A**, §49]). There exists a diffeomorphism to

$$T^k \times \mathbb{R}^{n-k} = \underbrace{S^1 \times \cdots \times S^1}_{k} \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n-k}$$

with the standard coordinates $\phi_1, \ldots, \phi_k \in (\mathbb{R} \bmod 2\pi)$, $t_{k+1}, \ldots, t_n \in \mathbb{R}$ such that in these coordinates (the push-forward of) every trajectory of the geodesic flow is given by the formula

$$\begin{aligned}&(\phi_1(\tau), \ldots, \phi_k(\tau), t_{k+1}(\tau), \ldots, t_n(\tau))\\ &\qquad = (\phi_1(0) + \omega_1\tau, \ldots, \phi_k(0) + \omega_k\tau, t_{k+1}(0) + \omega_{k+1}\tau, \ldots, t_n(0) + \omega_n\tau),\end{aligned}$$

where the constants $\omega_1, \ldots, \omega_n$ are universal on $T^k \times \mathbb{R}^{n-k}$.

We see that if at least one of the constants $\omega_{k+1}, \ldots, \omega_n$ is not zero, every point of $U(\xi)$ is wandering in $\mathrm{Supp}(\mu)$ (see [**KH**, §3, Ch. 3] for a definition), which contradicts the invariance of the measure. Then, the entropy of μ is zero.

If all constants $\omega_{k+1}, \ldots, \omega_n$ are zero, the coordinates $t_{k+1}, \ldots, t_n$ are constants on the trajectories of the geodesic flow. As μ is ergodic, they are constant on the points of $\mathrm{Supp}(\mu)$. Then, $\mathrm{Supp}(\mu)$ is (diffeomorphic to) the torus $T^{\bar{k}}$ of dimension $\bar{k} \leq k$, and the dynamics on $\mathrm{Supp}(\mu)$ is (conjugate to) the linear flow on $T^{\bar{k}}$. Then, the entropy of μ is zero (see, for example, [**KH**, Proposition 3.2.1]).

Now suppose that $\mathrm{Supp}(\mu)$ contains only singular points which are not removable. If all of them belong to $\bigcup_{i,j} T\,\mathrm{Sing}_i^j$, then (because the measure is ergodic) $\mathrm{Supp}(\mu)$ is a subset of a certain $T\,\mathrm{Sing}_i^j$. As Sing_i^j is totally geodesic, and by induction hypothesis the topological entropy on Sing_i^j is zero, the entropy of μ is also zero.

The last case is when $\mathrm{Supp}(\mu)$ contains a singular point which is not removable and which does not belong to $\bigcup_{i,j} T\,\mathrm{Sing}_i^j$. Then, as all Sing_i^j are totally geodesic, and there are finitely many of them, $\mathrm{Supp}(\mu)$ contains a singular point ξ which is not removable and such that its projection does not belong to $\bigcup_{i,j} \mathrm{Sing}_i^j$. Then, the projection of a small neighborhood $U(\xi) \subset \mathrm{Supp}(\mu)$ of ξ does not contain points of $\bigcup_{i,j} \mathrm{Sing}_i^j$.

From Theorems 5 and 6 it follows that for certain $\bar{\lambda}_i$ such that λ_i is not constant the differentials of $I_{\bar{\lambda}_i}$ vanish at ξ. As the number of such $\bar{\lambda}_i$ is finite, and the measure is ergodic, we obtain that there exists i such that:

- $dI_{\bar{\lambda}_i} = 0$ at every point of $\mathrm{Supp}(\mu)$;
- the eigenvalue λ_i satisfies the assumptions of Lemma 2 (otherwise the singularity is removable or ξ lies in $\bigcup_{i,j} T\,\mathrm{Sing}_i^j$).

Hence, by Lemma 2, for every point y of the projection of $U(\xi)$ we have that $M_{\mathsf{C}(i)}(y)$ is compact. Then, by Theorem 5, for every $\eta \in U(\xi)$, the projection of the trajectory of the geodesic flow passing through η stays on the corresponding $M_{\mathsf{C}(i)}$. As all $M_{\mathsf{C}(i)}$ passing through the projection of $U(\xi)$ are compact and do not intersect one another, a trajectory staying in one $T_1M_{\mathsf{C}(i)}$ never approaches another $T_1M_{\mathsf{C}(i)}$. Thus, as μ is ergodic, all points of $\mathrm{Supp}(\mu)$ belong to a certain $T_1M_{\mathsf{C}(i)}(y)$. Then, the dynamics on $\mathrm{Supp}(\mu)$ is a subsystem of the geodesic flow for the restriction of g to $M_{\mathsf{C}(i)}(y)$. (Indeed, if a geodesic of a metric lies on a submanifold, then it is a geodesic in the restriction of the metric to the submanifold.) Finally, by induction assumptions, the entropy of μ is zero.

Thus, for every ergodic probabilistic invariant measure μ its entropy is zero. Finally, the topological entropy is zero. □

Note added in proof: we received from B. Hasselblatt the following alternative argument of the end of the proof. He confirmed our suggestion that the support of an invariant ergodic Borel probability measure is the closure of some orbit. Here are the arguments. Every non-empty (relatively) open subset of the support has positive measure (by definition of support) and therefore almost every orbit visits it infinitely often (by the Birkhoff ergodic theorem). A compact manifold is second countable, so the intersection of the sets of full measure, obtained from the elements of a countable base, consists of orbits that are dense in the support, and this set has full measure, hence is non-empty. Thus we can consider one orbit, classify it according to singularity and then study the closure by already established properties.

5. *Topological restrictions for manifolds with infinite fundamental group: announcement*

THEOREM 7. *Suppose the Riemannian metrics g and $\bar g$ on a closed connected manifold M^n are geodesically equivalent and strictly non-proportional at least at one point. Then some finite cover of M^n is diffeomorphic to the product $Q^k \times T^{n-k}$ of a rational-elliptic manifold and the torus.*

The proof of this theorem is lengthy and will appear elsewhere. Here we sketch the proof only. It uses Corollary 1, methods developed in [**M1, M4**] and classical results of [**CG**].

In [**M1**], it was shown that if a manifold with non-proportional geodesically equivalent metrics has an infinite fundamental group, it admits a local product structure (equal to a new Riemannian metric and two orthogonal foliations of complementary dimensions B_k and B_{n-k} such that in a small neighborhood of almost every point all three objects look as if they come from the Riemannian product of two Riemannian manifolds). In [**M4**, Lemma 2], it was shown that (assuming that the initial metrics g and $\bar g$ are strictly non-proportional at least at one point), the restriction of the local-product metric to the leaves of the foliations admits a metric that is geodesically equivalent to it and strictly non-proportional to it at almost every point. By applying the same construction to the leaves, we obtain that M^n admits a Riemannian metric h and m orthogonal foliations $B_{k_1}, B_{k_2}, \dots, B_{k_m}$ of complementary dimension $k_1 + k_2 + \cdots + k_m = n$ such that:

- the restriction of the metric h to B_{k_1} is flat;
- the leaves of $B_{k_2}, B_{k_3}, \dots, B_{k_m}$ are compact and have finite fundamental group (this is actually the lengthy part of the proof; its proof it similar to the proof of [**M1**, Theorem 2], but one can not apply [**M1**, Theorem 2] directly and should essentially repeat all steps of its proof in a slightly different setting);
- the restriction of h to each of $B_{k_2}, B_{k_3}, \dots, B_{k_m}$ admits a metric that is geodesically equivalent to it and is strictly non-proportional to it at least at one point;
- locally, in a neighborhood of every point, the metric h and the foliations B_{k_i} look as if they (simultaneously) came from the direct product of m Riemannian manifolds.

Then, by Corollary 1, the universal cover of $B_{k_2} \times B_{k_3} \times \cdots \times B_{k_m}$ is rational elliptic, and Theorem 7 follows from [**CG**, Theorem 9.2].

Theorem 7 allows us to generalize homotopic and homologic properties listed before Corollary 1. Let $\operatorname{rank} \pi_1(M)$ be the minimal number of generators of the fundamental group, which span a subgroup of finite index, and $\pi_{*+1}(M) = \bigoplus_{i>1} \pi_i(M)$ be the sum of abelian higher homotopy groups. Denote $\dim_{\mathbb{Q}} \pi_*(M) = \operatorname{rank} \pi_1(M) + \dim \pi_{*+1}(M) \otimes \mathbb{Q}$.

COROLLARY 4. *Suppose the Riemannian metrics g and $\bar{g}$ on a closed connected manifold M^n are geodesically equivalent and strictly non-proportional at least at one point. Then,*

$$\dim_{\mathbb{Q}} \pi_*(M^n) \le n, \quad \dim H_*(M^n; \mathbb{Q}) \le 2^n \quad \textit{and} \quad \chi(M^n) \ge 0.$$

For small dimensions, in view of Theorem 1, Corollary 4 follows from [**PP2**].

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