On projectively equivalent metrics near points of bifurcation

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Abstract

Let Riemannian metrics g and $\bar{g}$ on a connected manifold M^n have the same geodesics (considered as unparameterized curves). Suppose the eigenvalues of one metric with respect to the other are all different at a point. Then, by the famous Levi-Civita's Theorem, the metrics have a certain standard form near the point.

Our main result is a generalization of Levi-Civita's Theorem for the points where the eigenvalues of one metric with respect to the other bifurcate.

1 Introduction

1.1 Metrics with the same geodesics

Definition 1. Two Riemannian metrics g and $\bar{g}$ on M^n are called *projectively equivalent*, if they have the same geodesics considered as unparameterized curves.

Trivial examples of projectively equivalent metrics can be obtained by considering proportional metrics g and $C \cdot g$, where C is a positive constant.

Definition 2. The Riemannian metrics g and $\bar{g}$ are said to be *strictly non-proportional* at $x \in M^n$, if the eigenvalues of g with respect to $\bar{g}$ are all different at x.

In the present paper we study projectively equivalent metrics which are strictly non-proportional at least at one point of the manifold. It is a very classical material. In 1865, Beltrami [1] found the first examples and formulated a problem of finding all pairs of projectively equivalent metrics. Locally the

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problem almost has been solved by Dini [6] for dimension two and Levi-Civita [16] for an arbitrary dimension: they obtained a local description of projectively equivalent metrics near the points where the eigenvalues of one metric with respect to the other do not bifurcate:

Theorem 1 (Levi-Civita [16]). *Consider two Riemannian metrics on an open subset $U^n \subset M^n$. Suppose the metrics are strictly-non-proportional at every point $x \in U^n$.*

Then the metrics are projectively equivalent on U^n, if and only if for every point $x \in U^n$ there exist coordinates $x_1, x_2, ..., x_n$ in some neighborhood of x such that in these coordinates the metrics have the following model form:

$$
\begin{aligned}
ds^2_{g_{model}} &= \Pi_1 dx_1^2 + \ \Pi_2 dx_2^2 \ + \cdots + \ \ \Pi_n dx_n^2, &(1)\\
ds^2_{\bar g_{model}} &= \rho_1 \Pi_1 dx_1^2 + \rho_2 \Pi_2 dx_2^2 + \cdots + \rho_n \Pi_n dx_n^2, &(2)
\end{aligned}
$$

where the functions Π_i and ρ_i are given by

$$
\begin{aligned}
\Pi_i &\stackrel{\text{def}}{=} (\lambda_i - \lambda_1)(\lambda_i - \lambda_2)\cdots(\lambda_i - \lambda_{i-1})(\lambda_{i+1} - \lambda_i)\cdots(\lambda_n - \lambda_i), \qquad (3)\\
\rho_i &\stackrel{\text{def}}{=} \frac{1}{\lambda_1 \lambda_2 ... \lambda_{n-1}} \frac{1}{\lambda_i}.
\end{aligned}
$$

where, for any i, the function λ_i is a smooth function of the variable x_i.

However, the global behavior of projectively equivalent metrics is not understood completely.

One of the main difficulties for global description of projectively equivalent metrics was the absence of local description of projectively equivalent metrics near the points where the eigenvalues of one metric with respect to the other bifurcate. As it has been proven in [23], if the manifold is not covered by the torus, it must have such points.

Our main result is a description of (strictly non-proportional at least at one point) projectively equivalent metrics near points where the eigenvalues of one metric with respect to the other bifurcate. The description needs some preliminary work and will be formulated at the very end of the paper, see Theorem 10 in Section 2.6. Here, we would like to note that the points of bifurcation behave quite regularly: the multiplicity of an eigenvalue is at most three; the points where the multiplicity of an eigenvalue is at least two are organized in totally-geodesic submanifold U^{n-2} of co-dimension two; the points where the multiplicity of the eigenvalue is three are organized in a closed totally-geodesic hypersurface (submanifold of co-dimension one) of U^{n-2}.

This local description is the main tool of the following

Theorem 2 (Topalov, Matveev). *Suppose M^n is a connected closed manifold. Let Riemannian metrics g and $\bar g$ on M^n be projectively equivalent. Suppose they are strictly non-proportional at least at one point. Then the manifold can be finitely covered by a product of spheres.*

We will neither prove nor explain this theorem in this paper. The proof will appear elsewhere, in a joint paper with P. Topalov.

As we will show in Section 2.3, see Corollary 5, a product of spheres always admits projective equivalent metrics strictly non-proportional at least at one point.

1.2 Geodesic rigidity conjecture

Conjecture 1. *Suppose M^n is closed connected. Let Riemannian metrics g and $\bar{g}$ on M^n be projectively equivalent and be nonproportional. Then the manifold can be covered by the sphere, or it admits a local product structure.*

Definition of the notion "local product structure" is in Section 2.2. Roughly speaking, local product structure is a metric and two orthogonal foliations of complimentary dimensions such that locally all three objects look as they came from the direct product of two Riemannian manifold.

Conjecture 1 is true for dimensions two [19, 20, 26] and three [25, 27]. Theorem 2 shows that conjecture is true, if we assume in addition that the metrics are strictly-non-proportional at least at one point. The following theorem shows that the conjecture is true on the level of fundamental groups:

Theorem 3 ([28]). *Let M^n be a closed connected manifold. Suppose two non-proportional Riemannian metrics g, $\bar{g}$ on M^n are projectively equivalent. If the fundamental group of M^n is infinite, then there exists a local product structure on M^n.*

If a manifold admits a local product structure, it admits two non-proportional projectively equivalent metrics. The direct product of manifolds admitting strictly-non-proportional projectively equivalent metrics also admit strictly-non-proportional projectively equivalent metrics (see Section 2.3 and Lemma 3 there). Every sphere admits strictly-non-proportional projectively equivalent metrics (essentially [1]; see also Corollary 5 in Section 2.3).

From the other side, not every manifold covered by the sphere can carry non-proportional projectively equivalent metrics. The lowest dimension where such phenomena is possible is three:

Theorem 4 ([25],[27]). *A closed connected 3-manifold admits non-proportional projectively equivalent metrics if and only if it is homeomorphic to a lens space or to a Seifert manifold with zero Euler number.*

Then, the Poincare homology sphere (which is certainly covered by S^3 [11]) does not admit non-proportional projectively equivalent metrics.

1.3 Integrable systems in the theory of projectively equivalent metrics

There exist two classical techniques for working with projectively equivalent metrics:

One is due to Dini [6], Lie [17], Levi-Civita [16], Weyl [35] and Eisenhart [7], and was actively developed by Russian and Japan geometry schools in the 50th-70th. Projective equivalence of two metrics is equivalent to a certain system of quasi-linear differential equation on the entries of these metrics. Combining this system with additional geometric assumptions (mostly written as a tensor formula) one can deduce topological restrictions on the manifold, see the surveys [29].

The second classical technique is due to E. Cartan [5] and was very actively developed by the French geometry school. The main observation is that, on the level of connections, projective equivalence is a very easy condition. In particular, given a connection, it is easy to describe all projectively equivalent connections. So the goal is to understand which of these connections are Levi-Civita connections of a Riemannian metric.

Unfortunately, both techniques are local, and all global (when the manifold is closed) result obtained with the help of these techniques require additional strong geometric assumptions (i.e., the metric is assumed to be Kähler or semisymmetric or generally semisymmetric or T-generalized semisymmetric or Einstein or of constant curvature or Ricci-flat or recurrent or admitting a concircular vector field or admitting a torse-forming vector field).

New methods for global investigation of projectively equivalent metrics have been suggested in [18, 34]. The main observation of [18, 34] is that the existence of $\bar{g}$ projectively equivalent to g allows one to construct commuting integrals for the geodesic flow of g, see Theorem 6 in Section 2.1.

When the metrics are strictly non-proportional at least at one point, the number of functionally independent integrals equals the dimension of the manifold so that the geodesic flow of g is Liouville-integrable and we can apply the well-developed machinery of integrable geodesic flows. In particular, in dimension two, Theorem 2 and Conjecture 1 follow directly from the description of metrics on surfaces with quadratically integrable geodesic flows from [2, 3, 12, 15].

For dimension three, the following theorem has been obtained in [21, 24] by combining the ideas applied in [32] for analytically-integrable geodesic flows with technique developed in [13] for quadratically-integrable geodesic flows:

Theorem 5 ([21, 24]). *Let M^3 be a connected closed manifold and g, $\bar{g}$ be projectively equivalent Riemannian metrics on M^n. Suppose there exists a point of the manifold where the number of different eigenvalues of g with respect to $\bar{g}$ equals 3. Then the following holds:*

1. *The first Betti number $b_1(M^3)$ is not greater than 3.*
2. *The fundamental group of the manifold is virtually Abelian.*
3. *If, in addition, there exists a point where the number of different eigenvalues of g with respect to $\bar{g}$ is less than 3, then $b_1(M^3) < 3$.*
4. *If $b_1(M^3) = 3$, then M^3 is homeomorphic to the torus T^n.*

For dimension three, Theorem 2 follows immediately from Theorem 5 modulo the Poincaré conjecture.

More precisely, by the second statement of Theorem 5, there exists a finite cover with the fundamental group isomorphic to $\underbrace{Z \times Z \times \ldots \times Z}_{k}$. By [8], the fundamental group of a compact 3-manifold has no subgroup isomorphic to $Z \times Z \times Z \times Z$, so k is as most 3.

By a result of Reidemeister (which has been formulated and proven in [9] and [33]), it follows that $Z \times Z$ can not be the fundamental group of a closed 3-manifold. Thus, k is either 3 or 1 or 0. If $k = 3$, then, by the fourth statement of Theorem 5, M^3 is covered by the torus $S^1 \times S^1 \times S^1$. If $k = 0$, the fundamental group of M^3 is finite so that (modulo the Poincaré conjecture) it is covered by S^3. If $k = 1$, then, modulo the Poincaré conjecture and in view of results of [36], M^3 is covered by the product $S^2 \times S^1$, see [24] for details.

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2 Local description near bifurcations

Within this section, we assume that Riemannian metrics g and $\bar g$ are projectively equivalent, and that the manifold is connected. Our goal is to give a local description of (strictly-non-proportional at least at one point) projectively equivalent metrics g, $\bar g$ near the points where the eigenvalues of one metric with respect to the other bifurcate (i.e. are not all different). Sections 2.1,2.2,2.3 shows that it is sufficient to do these for two- and three-dimensional manifolds only; we do this in Sections 2.4,2.5 (Theorems 8,9) and combine all results in Section 2.6, see Theorem 10.

2.1 The multiplicity of an eigenvalue is not greater than three.

Let $g = (g_{ij})$ and $\bar g = (\bar g_{ij})$ be Riemannian metrics on a manifold M^n. Consider the (1,1)-tensor L given by the formula

$$L^i_j \stackrel{\mathrm{def}}{=} \left(\frac{\det(\bar g)}{\det(g)}\right)^{\frac{1}{n+1}} \bar g^{i\alpha} g_{\alpha j}. \qquad (4)$$

Then, L determines the family S_t, $t \in R$, of $(1,1)$-tensors

$$S_t \stackrel{\mathrm{def}}{=} \det(L - t\ \mathrm{Id})\,(L - t\ \mathrm{Id})^{-1}\,. \tag{5}$$

Remark 1. *Although $(L - t\ \mathrm{Id})^{-1}$ is not defined for t lying in the spectrum of L, the tensor S_t is well-defined for every t. Moreover, S_t is a polynomial in t of degree $n-1$ with coefficients being (1,1)-tensors.*

We will identify the tangent and cotangent bundles of M^n by g. This identification allows us to transfer the natural Poisson structure from T^*M^n to TM^n.

Theorem 6 ([18, 34]). *If g, $\bar g$ are projectively equivalent, then, for every $t_1, t_2 \in R$, the functions*

$$I_{t_i} : TM^n \to R, \quad I_{t_i}(v) \stackrel{\mathrm{def}}{=} g(S_{t_i}(v), v) \tag{6}$$

are commuting integrals for the geodesic flow of g.

Since L is self-adjoint with respect to $\bar g$, the eigenvalues of L are real. At every point $x \in M^n$, let us denote by $\lambda_1(x) \le ... \le \lambda_n(x)$ the eigenvalues of L at the point.

Remark 2. *The notation λ for the eigenvalues of L is compatible with the notations used inside Levi-Civita's Theorem: in Levi-Civita coordinates from Theorem 1, the tensor L is given by the diagonal matrix*

$$\mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n). \tag{7}$$

Definition 3. *A Riemannian manifold is called* **geodesic***, if every two points can be connected by a geodesic*

Corollary 1 ([26, 28]). *Let (M^n, g) be a geodesic Riemannian manifold. Let a Riemannian metric $\bar g$ on M^n be projectively equivalent to g. Then, for every $i \in \{1, ..., n-1\}$, for every $x, y \in M^n$, the following holds:*

1. *$\lambda_i(x) \le \lambda_{i+1}(y)$.*
2. *If $\lambda_i(x) < \lambda_{i+1}(x)$, then $\lambda_i(z) < \lambda_{i+1}(z)$ for almost every point $z \in M^n$.*

In order to prove Corollary 1, we need the following technical lemma. For every fixed $v = (\xi_1, \xi_2, ..., \xi_n) \in T_xM^n$, the function (6) is a polynomial in t. Consider the roots of this polynomial. From the proof of Lemma 1, it will be clear that they are real. We denote them by

$$t_1(x, v) \le t_2(x, v) \le ... \le t_{n-1}(x, v).$$

Lemma 1 ([26, 28]). *The following holds for every $i \in \{1, ..., n-1\}$:*

1. For every $v \in T_x M^n$,

$$\lambda_i(x) \le t_i(x,v) \le \lambda_{i+1}(x).$$

In particular, if $\lambda_i(x) = \lambda_{i+1}(x)$, then $t_i(x,v) = \lambda_i(x) = \lambda_{i+1}(x)$.

2. If $\lambda_i(x) < \lambda_{i+1}(x)$, then for every $\tau \in R$ the Lebesgue measure of the set

$$V_\tau \subset T_x M^n, \quad V_\tau \stackrel{\mathrm{def}}{=} \{v \in T_x M^n : t_i(x,v) = \tau\},$$

is zero.

Proof of Lemma 1: By definition, the tensor L is self-adjoint with respect to g. Then, for every $x \in M^n$, there exists "diagonal" coordinates in $T_x M^n$ where the metric g is given by the diagonal matrix $\mathrm{diag}(1,1,...,1)$ and the tensor L is given by the diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$. Then, the tensor (5) reads:

$$\begin{aligned} S_t &= \det(L - t\mathrm{Id})(L - t\mathrm{Id})^{(-1)} \\ &= \mathrm{diag}(\Pi_1(t), \Pi_2(t), ..., \Pi_n(t)), \end{aligned}$$

where the polynomials $\Pi_i(t)$ are given by the formula

$$\Pi_i(t) \stackrel{\mathrm{def}}{=} (\lambda_1 - t)(\lambda_2 - t)...(\lambda_{i-1} - t)(\lambda_{i+1} - t)...(\lambda_{n-1} - t)(\lambda_n - t).$$

Hence, for every $v = (\xi_1, ..., \xi_n) \in T_x M^n$, the polynomial $I_t(x,v)$ is given by

$$I_t = \xi_1^2 \Pi_1(t) + \xi_2^2 \Pi_2(t) + ... + \xi_n^2 \Pi_n(t). \tag{8}$$

Evidently, the coefficients of the polynomial I_t depend continuously on the eigenvalues λ_i and on the components ξ_i. Then, it is sufficient to prove the first statement of the lemma assuming that the eigenvalues λ_i are all different and that ξ_i are non-zero. For every $\alpha \ne i$, we evidently have $\Pi_\alpha(\lambda_i) \equiv 0$. Then,

$$I_{\lambda_i} = \sum_{\alpha=1}^{n} \Pi_\alpha(\lambda_i)\xi_\alpha^2 = \Pi_i(\lambda_i)\xi_i^2.$$

Hence $I_{\lambda_i}(x,v)$ and $I_{\lambda_{i+1}}(x,v)$ have different signs. Hence, the open interval $]\lambda_i, \lambda_{i+1}[$ contains a root of the polynomial $I_t(x,v)$. The degree of the polynomial I_t is equal $n-1$; we have $n-1$ disjoint intervals; each of these intervals contains at least one root so that all roots are real and the ith root lies between λ_i and λ_{i+1}. The first statement of the lemma is proved.

Let us prove the second statement of Lemma 1. Suppose $\lambda_i < \lambda_{i+1}$. Let first $\lambda_i < \tau < \lambda_{i+1}$. Then, the set

$$V_\tau \stackrel{\mathrm{def}}{=} \{v \in T_x M^n : t_i(x,v) = \tau\},$$

consists of the points v where the function $I_\tau(x,v) \stackrel{\mathrm{def}}{=} (I_t(x,v))_{|t=\tau}$ is zero; then it is a nontrivial quadric in $T_x M^n \equiv R^n$ and its measure is zero.

Let τ be one of the endpoints of the interval $[\lambda_i, \lambda_{i+1}]$. Without loss of generality, we can suppose $\tau = \lambda_i$. Let k be the multiplicity of the eigenvalue λ_i. Then, every coefficient $\Pi_\alpha(t)$ of the quadratic form (8) has the factor $(\lambda_i - t)^{k-1}$. Hence,

$$\hat{I}_t \stackrel{\text{def}}{=} \frac{I_t}{(\lambda_i - t)^{k-1}}$$

is a polynomial in t and $\hat{I}_\tau$ is a nontrivial quadratic form. Evidently, for every point $v \in V_\tau$, we have $\hat{I}_\tau(v) = 0$ so that the set V_τ is a subset of a nontrivial quadric in T_xM^n and its measure is zero. Lemma 1 is proved.

Proof of Corollary 1: The first statement of Corollary 1 follows immediately from the first statement of Lemma 1: Let us join the points $x, y \in M^n$ by a geodesic $\gamma : R \to M^n$, $\gamma(0) = x$, $\gamma(1) = y$. Consider the one-parametric family of integrals $I_t(x, v)$ and the roots

$$t_1(x, v) \leq t_2(x, v) \leq ... \leq t_{n-1}(x, v).$$

By Theorem 6, each root t_i is constant on every orbit $(\gamma, \dot{\gamma})$ of the geodesic flow of g so that

$$t_i(\gamma(0), \dot{\gamma}(0)) = t_i(\gamma(1), \dot{\gamma}(1)).$$

Using Lemma 1, we obtain

$$\lambda_i(\gamma(0)) \leq t_i(\gamma(0), \dot{\gamma}(0)), \quad \text{and} \quad t_i(\gamma(1), \dot{\gamma}(1)) \leq \lambda_{i+1}(\gamma(1)).$$

Thus $\lambda_i(\gamma(0)) \leq \lambda_{i+1}(\gamma(1))$ and the first statement of Corollary 1 is proved.

Let us prove the second statement of Corollary 1. Suppose $\lambda_i(y) = \lambda_{i+1}(y)$ for every point y of some subset $V \subset M^n$. Then, the value of λ_i is a constant (independent of $y \in V$). Indeed, by the first statement of Corollary 1,

$$\lambda_i(y_0) \leq \lambda_{i+1}(y_1) \ \text{ and } \ \lambda_i(y_1) \leq \lambda_{i+1}(y_0),$$

so that $\lambda_i(y_0) = \lambda_i(y_1) = \lambda_{i+1}(y_1) = \lambda_{i+1}(y_0)$ for every $y_0, y_1 \in V$.

We denote this constant by τ. Let us join the point x with every point of V by all possible geodesics. Consider the set $V_\tau \subset T_xM^n$ of the initial velocity vectors (at the point x) of these geodesics.

By the first statement of Lemma 1, for every geodesic γ passing through at least one point of V, the value $t_i(\gamma, \dot{\gamma})$ is equal to τ. By the second statement of Lemma 1, the measure of the set V_τ is zero. Since the set V lies in the image of the exponential mapping of the set V_τ, the measure of the set V is also zero. Corollary 1 is proved.

Corollary 2. *Suppose M^n connected. Let g and $\bar{g}$ be projectively equivalent on M^n. Suppose they are strictly non-proportional at least at one point. Then the following holds:*

1. *The metrics are strictly non-proportionally at almost every point of M^n.*
2. *At every point of M^n, the multiplicity of every eigenvalue of L is at most three.*

Proof: Suppose the metrics are simply non-proportional at x_1. Since the manifold is connected, for every point $x \in M^n$ there exists a sequence of open convex balls $B_i \subset M^n$ such that the first ball B_1 contains the point x_1, the last ball B_m contains the point X and the intersection $B_i \cap B_{i+1}$, $i < m$, is not empty. By the second statement of Corollary 1, the metrics are strictly non-proportional at almost every points of B_1. Then there are strictly non-proportional at almost every points of $B_1 \cap B_2$. Then, there exists a point of B_2 where the metrics are strictly non-proportional. Hence, by the second statement of Corollary 1, they are strictly non-proportional at almost every points of B_2. Iterating this argumentation $m-2$ times, we obtain that the metrics are strictly non-proportional at almost every points of B_m. Since the point x is arbitrary, the metrics are strictly non-proportional at almost every points of M^n.

Then, the multiplicity of an eigenvalue at the point x is not greater than 3: indeed, if we suppose that $\lambda_k(x) = \lambda_{k+3}(x)$, then, by the first statement of Corollary 1, $\lambda_{k+1} = \lambda_{k+2}$ at each point of B_m, which contradicts that the metrics are strictly non-proportional at almost every points of B_m. Corollary 2 is proved.

2.2 Splitting procedure

The goal of the next two sections is to show that for a local description of projectively equivalent metrics strictly-non-proportional at least at one point it is sufficient to describe them on two- and three-dimensional manifold only. We will need the following statement.

Corollary 3 ([4]). *Suppose the Riemannian metrics $g, \bar{g}$ on M^n are projectively equivalent. Then the Nijenhuis torsion of L vanishes.*

A self-contained proof of Corollary 3 can be found in [4]; here we prove the theorem assuming that the manifold is connected and the metrics are strictly-non-proportional at least at one point, which is sufficient for our goals.

Proof: Nijenhuis torsion is a tensor, so it is sufficient to check its vanishing at almost every point. By Corollary 1, almost every point of M^n is stable. In the Levi-Civita coordinates from Theorem 1, the tensor L is given by the diagonal matrix

$$\mathrm{diag}(\lambda_1, ..., , \lambda_n).$$

Since the eigenvalue λ_i depends on the variable x_i only, the Nijenhuis torsion of L is zero [10]. Corollary 3 is proved.

Definition 4. *A local-product structure on M^n is the triple (h, B_r, B_{n-r}), where h is a Riemannian metrics and B_r, B_{n-r} are transversal foliations of dimensions r and $n-r$, respectively (it is assumed that $1 \le r < n$), such that every point $p \in M^n$ has a neighborhood $U(p)$ with coordinates*

$$(\bar{x}, \bar{y}) = \big((x_1, x_2, ...x_r), (y_{r+1}, y_{r+2}, ..., y_n)\big)$$

such that the x-coordinates are constant on every leaf of the foliation $B_{n-r} \cap U(p)$, the y-coordinates are constant on every leaf of the foliation $B_r \cap U(p)$, and the metric h is block-diagonal such that the first $(r \times r)$ block depends on the x-coordinates and the last $((n-r) \times (n-r))$ block depends on the y-coordinates.

A model example of manifolds with local-product structure is the direct product of two Riemannian manifolds (M_1^r, g_1) and (M_2^{n-r}, g_2). In this case, the leaves of the foliation B_r are the products of M_1^r and the points of M_2^{n-r}, the leaves of the foliation B_{n-r} are the products of the points of M_1^r and M_2^{n-r}, and the metric h is the product metric $g_1 + g_2$.

Below we assume that

(a) The metrics g and $\bar{g}$ are projectively equivalent on a connected M^n.

(b) They are strictly-non-proportional at least at one point of M^n.

(c) There exists r, $1 \le r < n$, such that $\lambda_r < \lambda_{r+1}$ at every point of M^n.

We will show that (under the assumptions (a,b,c)) we can naturally define two local-product structures (h, B_r, B_{n-r}) and $(\bar{h}, B_r, B_{n-r})$ such that the restrictions of h and $\bar{h}$ to every leaf are projectively equivalent and strictly non-proportional at least at one point.

At every point $x \in M^n$, denote by V_x^r the subspaces of $T_x M^n$ spanned by the eigenvectors of L corresponding to the eigenvalues $\lambda_1, ..., \lambda_r$. Similarly, denote by V_x^{n-r} the subspaces of $T_x M^n$ spanned by the eigenvectors of L corresponding to the eigenvalues $\lambda_{r+1}, ..., \lambda_n$. By assumption, for every i, j such that $i \le r < j$, we have $\lambda_i \ne \lambda_j$ so that V_x^r and V_x^{n-r} are two smooth distributions on M^n. By Corollary 3, the distributions are integrable so that they define two transversal foliations B_r and B_{n-r} of dimensions r and $n-r$, respectively.

By construction, the distributions V_r and V_{n-r} are invariant with respect to L. Let us denote by L_r, L_{n-r} the restrictions of L to V_r and V_{n-r}, respectively. We will denote by χ_r, χ_{n-r} the characteristic polynomials of L_r, L_{n-r}, respectively.

Consider the (1,1)-tensor

$$C \stackrel{\text{def}}{=} ((-1)^r \chi_r(L) + \chi_{n-r}(L))$$

and the metric h given by the relation

$$h(u,v) \stackrel{\text{def}}{=} g(C^{-1}(u), v)$$

for every vectors u, v. (In the tensor notations, the metrics h and g are related by $g_{ij} = h_{i\alpha} C_j^\alpha$).

Consider the (1,1)-tensor

$$\bar{C} \stackrel{\text{def}}{=} \left(\frac{(-1)^r}{\det(L_{n-r})} \chi_r(L) + \frac{1}{\det(L_r)} \chi_{n-r}(L) \right)$$

and the metric $\bar{h}$ given by the relation

$$\bar{h}(u,v) \stackrel{\mathrm{def}}{=} \bar{g}(\bar{C}^{-1}(u),v)$$

for every vectors u, v.

Lemma 2. *The following statements hold:*

1. *The triples (h, B_r, B_{n-r}) and $(\bar{h}, B_r, B_{n-r})$ are local-product structures on M^n.*

2. *For every leaf of B_r, the restrictions of h and $\bar{h}$ to the leaf are projectively equivalent and strictly non-proportional at least at one point. For every leaf of B_{n-r}, the restrictions of h and $\bar{h}$ to the leaf are projectively equivalent and strictly non-proportional at least at one point.*

Proof: First of all, h and $\bar{h}$ are well-defined Riemannian metrics. Indeed, take an arbitrary point $x \in M^n$. At the tangent space to this point, we can find a coordinate system where the tensor L and the metric g are diagonal. In this coordinate system, the characteristic polynomials χ_r, χ_{n-r} are given by

$$\begin{array}{lcc} (-1)^r\chi_r & = & (t-\lambda_1)(t-\lambda_2)...(t-\lambda_r) \\ \chi_{n-r} & = & (\lambda_{r+1}-t)(\lambda_{r+2}-t)...(\lambda_n-t). \end{array} \tag{9}$$

Then, the (1,1)-tensors

$$\begin{array}{lcl} C & = & ((-1)^r\chi_r(L)+\chi_{n-r}(L)) \\ \bar{C} & = & ((-1)^r\det(L_{n-r})\chi_r(L)+\det(L_r)\chi_{n-r}(L)) \end{array}$$

are given by the diagonal matrices

$$\operatorname{diag}\left(\prod_{j=r+1}^{n}(\lambda_j-\lambda_1),...,\prod_{j=r+1}^{n}(\lambda_j-\lambda_r),\prod_{j=1}^{r}(\lambda_{r+1}-\lambda_j),...,\prod_{j=1}^{r}(\lambda_n-\lambda_j)\right), \tag{10}$$

$$\begin{aligned} \operatorname{diag}\Bigg(&\frac{1}{\lambda_{r+1}...\lambda_n}\prod_{j=r+1}^{n}(\lambda_j-\lambda_1),...,\frac{1}{\lambda_{r+1}...\lambda_n}\prod_{j=r+1}^{n}(\lambda_j-\lambda_r),\\ &\frac{1}{\lambda_1...\lambda_r}\prod_{j=1}^{r}(\lambda_{r+1}-\lambda_j),...,\frac{1}{\lambda_1...\lambda_r}\prod_{j=1}^{r}(\lambda_n-\lambda_j)\Bigg). \end{aligned} \tag{11}$$

We see that the tensors are diagonal and all its diagonal components are positive. Then, the tensors $C^{-1}, \bar{C}^{-1}$ are well-defined and $h, \bar{h}$ are Riemannian metrics.

By construction, B_r and B_{n-r} are well-defined transversal foliations of supplementary dimensions. In order to prove Lemma 2, we need to verify that, locally, the triples (h, B_r, B_{n-r}) and $(\bar{h}, B_r, B_{n-r})$ are as in Definition 4, that the restriction of h to a leaf is projectively equivalent to the restriction of $\bar{h}$, and that the restriction of h to a leaf is strictly non-proportional (at least at a point) to the restriction of $\bar{h}$.

It is sufficient to verify the first two statements at almost every point of M^n. More precisely, it is known that the triple (h, B_r, B_{n-r}) is a local-product structure if and only if the foliations B_r and B_{n-r} are orthogonal and totally geodesic [30]. Clearly, if the foliations and the metric are globally given and smooth, if the foliations are orthogonal and totally-geodesic at almost every point then they are orthogonal and totally-geodesic at every point.

Similarly, if a foliation and two metrics are globally-given and smooth, if the restriction of the metrics to the leaves of the foliation is projectively equivalent almost everywhere then it is so at every point.

By Corollary 1, at almost every point of M^n the eigenvalues of L are different. Consider the Levi-Civita coordinates $x_1, ..., x_n$ from Theorem 1. In the Levi-Civita coordinates, L is given by (7). Then, by constructions of the foliations B_r and B_{n-r}, the coordinates $x_1, ..., x_r$ are constant on every leaf of the foliation B_{n-r}, the coordinates $x_{r+1}, ..., x_n$ are constant on every leaf of the foliation B_r.

Using (10,11), we see that, in the Levi-Civita coordinates, $h, \bar{h}$ are given by

$$\begin{aligned} h(\dot{x}, \dot{x}) &= \tilde{\Pi}_1 dx_1^2 + ... + \tilde{\Pi}_r dx_r^2 + \\ &+ \tilde{\Pi}_{r+1} dx_{r+1}^2 + ... + \tilde{\Pi}_n dx_n^2, \qquad (12) \\ \bar{h}(\dot{x}, \dot{x}) &= \tilde{\rho}_1 \tilde{\Pi}_1 dx_1^2 + \cdots + \tilde{\rho}_r \tilde{\Pi}_r dx_r^2 + \\ &+ \tilde{\rho}_n \tilde{\Pi}_{r+1} dx_{r+1}^2 \cdots \tilde{\rho}_n \tilde{\Pi}_n dx_n^2, \qquad (13) \end{aligned}$$

where the functions $\tilde{\Pi}_i, \tilde{\rho}_i$ are as follows: for $i \leq r$, they are given by

$$\begin{aligned} \tilde{\Pi}_i &\stackrel{\text{def}}{=} (\lambda_i - \lambda_1)...(\lambda_i - \lambda_{i-1})(\lambda_{i+1} - \lambda_i)...(\lambda_r - \lambda_i), \\ \tilde{\rho}_i &\stackrel{\text{def}}{=} \frac{1}{\lambda_i(\lambda_1 \lambda_2 ... \lambda_r)}. \end{aligned}$$

For $i > r$, the functions $\tilde{\Pi}_i, \tilde{\rho}_i$ are given by

$$\begin{aligned} \tilde{\Pi}_i &\stackrel{\text{def}}{=} (\lambda_i - \lambda_{r+1})...(\lambda_i - \lambda_{i-1})(\lambda_{i+1} - \lambda_i)...(\lambda_n - \lambda_i), \\ \tilde{\rho}_i &\stackrel{\text{def}}{=} \frac{1}{\lambda_i(\lambda_{r+1} \lambda_2 ... \lambda_n)}. \end{aligned}$$

We see that the restrictions of the metrics on the leaves of the foliations have the form from Levi-Civita's Theorem and, therefore, are projectively equivalent. We see that the metrics $h, \bar{h}$ are block-diagonal with the first $r \times r$ block depending on the variables $x_1, ..., x_r$ and the second $(n-r) \times (n-r)$ block depending on the remaining variables, so that (h, B_r, B_{n-r}) and $(\bar{h}, B_r, B_{n-r})$ are local-product structure.

The last thing to show is that the restrictions of the metrics to every leaf are strictly non-proportional at least at one point. Suppose it is not so; that is, there exists a leaf (say, of foliation B_r) and k, $0 < k < r$ such that $\lambda_k = \lambda_{k+1}$ at each point of the leaf. Then, by the first statement of Corollary 1, the eigenvalues λ_k, λ_{k+1} are constant on the leaf. Since the Nijenhuis torsion of L is zero, the eigenvalues λ_k, λ_{k+1} are constant along the leaves of the foliation B_{n-r} [10]. Then $\lambda_k = \lambda_{k+1}$ at each point of a neighborhood of the leaf, which contradicts Corollary 2. Thus the restrictions of the metrics to every leaf are strictly non-proportional at least at one point. Lemma 2 is proved.

2.3 Gluing procedure

Let B_r and B_{n-r}, $0 < r < n$, be transversal foliations of supplementary dimensions r and $n-r$ on a connected M^n. Suppose there exist Riemannian metrics h and $\bar h$ such that

(i) The triples (h, B_r, B_{n-r}) and $(\bar h, B_r, B_{n-r})$ are local-product structures.

(ii) For every fiber of B_r the restrictions of h, $\bar h$ to the fiber are projectively equivalent and strictly-non-proportional at least at one point. For every fiber of B_{n-r} the restrictions of h, $\bar h$ to the fiber are projectively equivalent and strictly-non-proportional at least at one point.

Let us denote by L_r (by L_{n-r}, respectively) the tensor (4) constructed for the restrictions of h, $\bar h$ to the tangent spaces of the leaves of B_r (of B_{n-r}, respectively). Assume in addition that

(iii) for every point of M^n, the eigenvalues of L_{n-r} are greater than that of L_r.

At every point of M^n, let us denote by χ_r, χ_{n-r} the characteristic polynomials of L_r, L_{n-r}, respectively. Let us denote by $P_r : T_xM^n \to T_xB_r$, $P_{n-r} : T_xM^n \to T_xB_{n-r}$ the orthogonal projections to the tangent spaces of the foliations B_r, B_{n-r}. (Since the foliations are orthogonal with respect to both metrics, it does not matter what metric we take here.) There exists a unique $(1,1)$-tensor L such that $L \circ P_r = L_r$ and $L \circ P_{n-r} = L_{n-r}$.

Consider the (1,1)-tensor

$$C \stackrel{\text{def}}{=} ((-1)^r \chi_r(L) + \chi_{n-r}(L)) \tag{14}$$

and the metric g given by the relation

$$g(u,v) \stackrel{\text{def}}{=} h(C(u),v) \tag{15}$$

for every vectors u, v. (In the tensor notations, the metric g is given by $h_{i\alpha}C^{\alpha}_{j}$.)

Consider the (1,1)-tensor

$$\bar C \stackrel{\text{def}}{=} \left(\frac{(-1)^r}{\det(L_{n-r})}\chi_r(L) + \frac{1}{\det(L_r)}\chi_{n-r}(L) \right) \tag{16}$$

and the metric $\bar h$ given by the relation

$$\bar g(u,v) \stackrel{\text{def}}{=} \bar h(\bar C(u),v) \tag{17}$$

for every vectors u, v.

Lemma 3. *The metrics g and $\bar{g}$ are projectively equivalent on M^n and are strictly-non-proportional at almost every point of M^n.*

Proof: First of all, g and $\bar{g}$ are well-defined Riemannian metrics. More precisely, at the tangent space to every point we can find a coordinate system where L is given by the diagonal matrix $\text{diag}(\lambda_1, ..., \lambda_n)$ assuming

$$\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_r < \lambda_{r+1} \leq ... \leq \lambda_n.$$

In this coordinate system, the characteristic polynomials χ_r and χ_{n-r} are given by (9). Hence, the tensors $C, \bar{C}$ are given by (10,11), and the condition (iii) guarantees that the metrics g and $\bar{g}$ are well-defined Riemannian metrics.

Let us show that the metrics g and $\bar{g}$ are projectively equivalent. It is sufficient to check it at almost every point. Clearly, at almost every point of M^n the eigenvalues of the tensor L are all different. Since by Levi-Civita's Theorem, the restriction of h, $\bar{h}$ to the leaves of B_r, B_{n-r} have the model form (1,2), we obtain that the metric $h, \bar{h}$ are given by (12,13). Using that $C, \bar{C}$ are given by (10,11), we obtain that the metrics g, $\bar{g}$ have precisely the form from Levi-Civita's Theorem, and, therefore, are projectively equivalent. Lemma 3 is proved.

Remark 3. *The notation C, $\bar{C}$, L, λ_i, g and $\bar{g}$ used in this section are not misleading and are compatible with the notations in Section 2.2. More precisely, if we take $g, \bar{g}$ satisfying assumptions (a,b,c), construct the metrics $h, \bar{h}$ and foliations B_r, B_{n-r}, then (by Lemma 2) the triples (h, B_r, B_{n-r}), $(\bar{h}, B_r, B_{n-r})$ satisfy conditions (i, ii, iii). Moreover, the tensor L given by (4) coincide with the tensor L constructed in this section. Therefore, the tensors (14,16) coincide with the tensors $C, \bar{C}$ from Section 2.2, and, therefore, the metrics constructed by (15,17) coincide with the initial metrics $g, \bar{g}$.*

Remark 4. *Levi-Civita's Theorem 1 follows from Lemmas 2,3.*

Proof: First of all, by direct calculation it is possible to verify that if the metrics are given by the Levi-Civita's model form (1,2), then they are projectively equivalent.

In order to prove that strictly non-proportional projectively equivalent metrics have (locally) the form (1,2), we use induction in dimension of the manifold.

For one-dimensional manifold it is nothing to prove; suppose Levi-Civita's Theorem is true for dimension $n-1$. Let us prove that, for dimension n, strictly non-proportional projectively equivalent metrics are locally given by (1,2).

If the metrics are strictly non-proportional at p, then $\lambda_1 < \lambda_2$ in a small neighborhood of p. Put $r = 1$ and construct the local-direct-product structures (h, B_1, B_{n-1}) and $(\bar{h}, B_1, B_{n-1})$. By definition 4, there exists a smaller neighborhood of p such that the foliations B_1 and B_{n-1} there look as they came from the direct product of the interval and the $(n-1)$-dimensional disk. Let us choose one leaf of the foliation B_1 and one leaf of the foliation B_{n-1}.

Since the leaf of B_1 is one-dimensional, there exists a coordinate x_1 there such that the restriction of the metrics $h, \bar{h}$ are respectively given by dx_1^2 and

$\frac{dx_1^2}{\lambda_1^2}$. Since the restrictions of the metrics $h, \bar h$ to the leaf of B_{n-1} are strictly non-proportional and projectively equivalent, by Levi-Civita's Theorem, there exists a coordinate system $x_2, ..., x_n$ there such that the restrictions of the metrics $h, \bar h$ to the leaf of B_{n-1} are respectively given by

$$\begin{array}{ccc} \hat\Pi_2 dx_2^2 & +...+ & \hat\Pi_n dx_n^2 \\ \frac{1}{\hat\rho_2}\hat\Pi_2 dx_2^2 & +...+ & \frac{1}{\hat\rho_n}\hat\Pi_n dx_n^2, \end{array}$$

where the functions $\hat\rho_i$ and $\hat\Pi_i$ are related to the functions ρ_i and Π_i from Levi-Civita's Theorem by the formulae

$$\hat\rho_i = \frac{\rho_i}{\lambda_1}, \quad \hat\Pi_i = \frac{\Pi_i}{\lambda_i - \lambda_1}.$$

Because of the local-product structure, these coordinates of the leaf of B_1 and on the leaf of B_{n-1} give us a coordinate system in the neighborhood of p. By direct calculations, $-\chi_1 = (t - \lambda_1)$, $\chi_{n-1} = (\lambda_2 - t)(\lambda_3 - t)...(\lambda_n - t)$. Then,

$$-\chi_1(L) = \mathrm{diag}(0, \lambda_2 - \lambda_1, \lambda_3 - \lambda_1, ..., \lambda_n - \lambda_1),$$

$$\chi_{n-1}(L) = \mathrm{diag}(\Pi_1, 0, 0, ..., 0),$$

and the determinants $\det(L_1), \det(L_{n-1})$ are equal to λ_1, $\lambda_2\lambda_3...\lambda_n$, respectively.

Using that the metric h, $\bar h$ are the products of their restrictions to the leaf of B_1 and the leaf of B_{n-1}, and in view of Remark 3, we obtain that the metrics $g, \bar g$ are precisely in the model form (1,2).

2.4 Dimension 2

The goal of this section is to give a local description of projectively equivalent metrics on surfaces near the points where the eigenvalues of L bifurcate. In dimension two, the inverse of Theorem 6 also takes place:

Theorem 7 ([18, 22, 19]) *Let g, $\bar g$ be Riemannian metrics on M^2. They are projectively equivalent if and only if the function*

$$F : TM^2 \to R, \quad F(\xi) \stackrel{\mathrm{def}}{=} \left(\frac{det(g)}{det(\bar g)}\right)^{\frac{2}{3}} \bar g(\xi, \xi) \tag{18}$$

is an integral of the geodesic flow of g.

We see that the integral F is quadratic in velocities. Thus the existence of an integral quadratic in velocities (for the geodesic flow of g) allows one to construct a metric projectively equivalent to g (at least locally).

Now, in in the two-dimensional case, the local description of metrics with quadratically integrable geodesic flows has been obtained in [2], see also [3] (Basing on the technique developed in [15]). Combining this description with Theorem 7, we obtain the following

Theorem 8. *Let g and $\bar{g}$ be projectively equivalent on a (2-dimensional) connected surface M^2. Suppose they are non-proportional at least at one point. Assume they are proportional at $p \in M^2$. Then, precisely one of the following possibilities takes place:*

1. *There exist coordinates u, v in a neighborhood of p, and there exists function λ of one variable such that the metrics have the following model form*

$$
\begin{aligned}
ds_g^2 &= 2\frac{\lambda(\rho+u)-\lambda(u-\rho)}{\rho}(du^2+dv^2) &(19)\\
ds_{\bar{g}}^2 &= \left(\frac{\lambda(u+\rho)-\lambda(u-\rho)}{\rho\lambda(u-\rho)\lambda(u+\rho)}\right)^2\left[\left(\rho\frac{\lambda(u+\rho)+\lambda(u-\rho)}{\lambda(u+\rho)-\lambda(u-\rho)}\right)(du^2+dv^2)\right.\\
&- \left. udu^2-2vdudv+udv^2\right] &(20)
\end{aligned}
$$

   where $\rho \stackrel{\mathrm{def}}{=} \sqrt{u^2+v^2}$.

2. *There exist coordinates u, v in a neighborhood of p, there exists a functions f of one variable, and there exists a positive constant λ_1 such that the metrics have the following model form*

$$
\begin{aligned}
ds_g^2 &= f(\rho^2)(du^2+dv^2) &(21)\\
ds_{\bar{g}}^2 &= \frac{f(\rho^2)}{\lambda_1(\lambda_1+\lambda_1\rho^2 f(\rho^2))^2}\left(\left(1+f(\rho^2)v^2\right)du^2-2f(\rho^2)uvdudv\right.\\
&+ \left.\left(1+f(\rho^2)u^2\right)dv^2\right), &(22)
\end{aligned}
$$

   where $\rho \stackrel{\mathrm{def}}{=} \sqrt{u^2+v^2}$.

3. *There exist coordinates u, v in a neighborhood of p, there exists function f of one variable, and there exists a positive constant λ_2 such that the metric g has the form (21) and the metric $\bar{g}$ is given by*

$$
\begin{aligned}
ds_{\bar{g}}^2 &= \frac{f(\rho^2)}{\lambda_2(\lambda_2-\lambda_2 f(\rho^2))^2}\left(\left(1-f(\rho^2)v^2\right)du^2+2f(\rho^2)uvdudv\right.\\
&+ \left.\left(1-f(\rho^2)u^2\right)dv^2\right), &(23)
\end{aligned}
$$

   where $\rho \stackrel{\mathrm{def}}{=} \sqrt{u^2+v^2}$.

Theorem 8 is true also in the other direction: if Riemannian metrics are given by formulae (19,20) or (21,22) or (21,23), then they are projectively equivalent. Of course, in order the formulae to define Riemannian metrics (at least in a small neighborhood of $(0,0)$), the functions f and λ must be smooth, positive and satisfy the conditions $\lambda'(0) > 0$, $f'(0) > 0$.

Remark 5. *The most natural coordinate system for projectively equivalent metrics near a point of bifurcation has singularity at the point:*

1. *In the elliptic coordinate system $x_1^2 = \rho - u$, $x_2^2 = \rho + u$ (which has a singularity at $(0,0)$), the metrics (19,20) are given (up to multiplication by 4) by*
$$(\lambda(x_1) - \lambda(x_2))(dx_1^2 + dx_2^2),$$
$$\left(\frac{1}{\lambda(x_2)} - \frac{1}{\lambda(x_1)}\right)\left(\frac{dx_1^2}{\lambda(x_1)} + \frac{dx_2^2}{\lambda(x_2)}\right),$$
which is precisely the Levi-Civita form for dimension two. In particular, the eigenvalues $\lambda_1(u,v)$, $\lambda_2(u,v)$ of L are equal to $\lambda(u-\rho)$, $\lambda(u+\rho)$, respectively.

2. *In the polar coordinate system $u = e^r\cos(\phi), v = e^r\sin(\phi)$ the metrics (21,22) and (23) are given by*
$$(e^{2r}f(e^{2r}))(dr^2 + d\phi^2),$$
$$\frac{1}{\lambda_1}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_1 + \lambda_1 e^{2r}f(e^{2r})}\right)\left(\frac{d\phi^2}{\lambda_1} + \frac{dr^2}{\lambda_1 + \lambda_1 e^{2r}f(e^{2r})}\right),$$
$$\frac{1}{\lambda_2}\left(\frac{1}{\lambda_2 - \lambda_2 e^{2r}f(e^{2r})} - \frac{1}{\lambda_2}\right)\left(\frac{dr^2}{\lambda_2 - \lambda_2 e^{2r}f(e^{2r})} + \frac{d\phi^2}{\lambda_2}\right),$$
respectively. We see that they are in the Levi-Civita form (up to the factors λ_1 and λ_2), and that the eigenvalues of L for the pair of metrics (21,22) are $\lambda_1(u,v) = \lambda_1$, $\lambda_2(u,v) = \lambda_1 + \lambda_1\rho^2 f(\rho^2)$, and the eigenvalues of L for the pair of metrics (21,23) are $\lambda_1(u,v) = \lambda_2 - \lambda_2\rho^2 f(\rho^2)$, $\lambda_2(u,v) = \lambda_2$.

Wee see that the first possibility (from Theorem 8) for projectively equivalent metrics is characterized by the condition that λ_1, λ_2 of L for are non-constant, and the second possibility is characterized by the condition λ_1 is constant, λ_2 is not constant, and the third possibility is characterized by the condition λ_2 is constant, λ_1 is not constant.

2.5 Dimension 3

The goal of this section is to describe (strictly-non-proportional at least at a point) projectively equivalent metrics on a 3-manifold near the points where the metrics are proportional. This will be made in Theorem 9. In order to prove it, we need Corollary 4 and Lemma 4.

Corollary 4. *Let g and $\bar g$ be projectively equivalent on connected M^3 and be strictly non-proportional at least at one point of M^3. Let $p \in M^3$. Suppose in a neighborhood of $p \in M^3$ the eigenvalue λ_2 is constant, and suppose $\lambda_1(p) <$*

$\lambda_2 = \lambda_3(p)$. *Then, there exists a neighborhood of p with coordinates x_1, x_2, x_3 where the metrics have the following model form:*

$$\begin{aligned} ds_g^2 &= (\lambda_1(x_1)-1)(\lambda_1(x_1)-1-\rho^2 f(\rho^2))dx_1^2 + f(\rho^2)\left(1+x_2^2 f(\rho^2)-\lambda_1(x_1)\right)dx_2^2 \\ &- 2f(\rho^2)x_2x_3dx_2^2dx_3^2 + \left(1+x_3^2 f(\rho^2)-\lambda_1(x_1)\right)dx_3^2, \end{aligned} \tag{24}$$

$$\begin{aligned} ds_{\bar g}^2 &= \frac{(\lambda_1(x_1)-1)(\lambda_1(x_1)-1-\rho^2 f(\rho^2))}{\lambda_1^2(x_1)(1+\rho^2 f(\rho^2))}dx_1^2 \\ &+ \frac{f(\rho^2)}{\lambda_1(x_1)(1+\rho^2 f(\rho^2))}\left[\left(1-\lambda_1(x_1)\frac{1+x_3^2 f(\rho^2)}{1+\rho^2 f(\rho^2)}\right)dx_2^2\right. \\ &+ \frac{2\lambda_1(x_1)x_2x_3}{1+\rho^2 f(\rho^2)}dx_2dx_3 \\ &+ \left.\left(1-\lambda_1(x_1)\frac{1+x_2^2 f(\rho^2)}{1+\rho^2 f(\rho^2)}\right)dx_3^2\right], \end{aligned} \tag{25}$$

where $\rho \stackrel{\text{def}}{=} \sqrt{x_2^2+x_3^2}$; f and λ_1 are functions of one variable and λ_2 is a positive constant.

Remark 6. *In the cylindrical coordinates $x_1 = u_1, x_2 = u_3\cos(u_2), v = u_3\sin(u_2)$ the metrics g, $\bar g$ almost have Levi-Civita form (1,2).*

Proof of Corollary 4: Since $\lambda_1 < \lambda_2$ at the point p, there exists a neighborhood of p where $\lambda_1 < \lambda_2$. Put $r = 1$, apply the splitting procedure from Section 2.2 and construct the metrics h and $\bar h$ and the foliations B_1, B_2. By Lemma 2, there exists a (possible, smaller) neighborhood of p isomorphic to a direct product of an interval and a 2-disc such that the metric h is the product metric $g_1 + g_2$ (where the metric g_1 is a metric on the interval and g_2 is a metric on the disc) and the metric $\bar h$ is the product metric $\bar g_1 + \bar g_2$ (where the metric $\bar g_1$ is a metric on the interval and $\bar g_2$ is a metric on the disc projectively equivalent to the metric g_2 and strictly-non-proportional to g_2 at least at one point). Since λ_2 is constant, the smallest eigenvalue of the tensor L constructed for the metrics g_2, $\bar g_2$ is constant. Since $\lambda_2 = \lambda_3$ at p, the metrics g_2, $\bar g_2$ are proportional at one point and, therefore, are given by (21,22) in an appropriate coordinate system x_2, x_3. There evidently exists a coordinate x_1 on the interval such that the metrics g_1, $\bar g_1$ are given by dx_1^2, $\frac{dx_1^2}{\lambda_1^2(x_1)}$ for an appropriate function λ. Applying the gluing procedure from Section 2.3, we obtain precisely the form (24,25). Corollary 4 is proved.

Lemma 4. *Consider the Riemannian metrics given by the formulae (24,25) in a neighborhood of the point $(0,0,0)$. Then,*

- *the plane $P := \{(x_1, x_2, x_3) :\ x_2 = 0\}$ is a totally geodesic submanifold;*
- *the eigenvalue λ_3 equals λ_2 precisely at the points where $x_2 = x_3 = 0$;*
- *the action of S^1 by the rotations*

$$(x_1, x_2, x_3) \mapsto (x_1, x_2\cos(\phi) - x_3\sin(\phi), x_2\sin(\phi) + x_3\cos(\phi))$$

preserves both metrics, and at every point $x \notin \{(x_1, x_2, x_3):\ x_2 = x_3 = 0\}$ *its orbits are tangent to the eigenspace of* L *corresponding to* λ_2,

- *at every point of the plane* P, *the vector* $\left(\frac{\partial}{\partial x_2}\right)$ *is the eigenvector of* L *with the eigenvalue* λ_2.

The lemma could be proved by direct computations. Actually, the first statement follows from the fact that the symmetry $(x_1, x_2, x_3) \mapsto (x_1, -x_2, x_3)$ is evidently an isometry; the second, third and the fourth statements follow from the observation that splitting-gluing procedure using in construction of metrics (24,25) is invariantly given in terms of metrics and therefore inherits all symmetries of the metrics g_1 and g_2.

Theorem 9. *Let* g *and* $\bar g$ *be projectively equivalent on a (3-dimensional) connected manifold* M^3. *Suppose they are strictly non-proportional at least at one point. Assume they are proportional at* $p \in M^3$. *Then, there exist coordinates* u_1, u_2, u_3 *in a neighborhood of* p, *a function* λ *of one variable and a positive constant* C, *such that the metrics have the following model form*

$$\begin{aligned} ds_g^2 &= 2\frac{\lambda(\rho+u_1)-\lambda(u_1-\rho)}{\rho}\left(du_1^2+\left(d\sqrt{u_2^2+u_3^2}\right)^2\right)\\ &+ C(\lambda(0)-\lambda(u_1-\rho))(\lambda(u_1+\rho)-\lambda(0))\,(u_3du_2-u_2du_3)^2 \qquad (26)\\ ds_{\bar g}^2 &= \left(\frac{\lambda(u_1+\rho)-\lambda(u_1-\rho)}{\rho\lambda(u_1-\rho)\lambda(u_1+\rho)}\right)^2\left[\left(\rho\frac{\lambda(u_1+\rho)+\lambda(u_1-\rho)}{\lambda(u_1+\rho)-\lambda(u_1-\rho)}\right)\left(du_1^2+\left(d\sqrt{u_2^2+u_3^2}\right)^2\right)\right.\\ &- \left. u_1du_1^2-2\sqrt{u_2^2+u_3^2}du_1d\sqrt{u_2^2+u_3^2}+u_1\left(d\sqrt{u_2^2+u_3^2}\right)^2\right]\\ &+ C\frac{(\lambda(0)-\lambda(u_1-\rho))(\lambda(u_1+\rho)-\lambda(0))}{\lambda(0)^2\lambda(u_1-\rho)\lambda(u_1+\rho)}\,(u_3du_2-u_2du_3)^2 \qquad (27)\end{aligned}$$

where $\rho \stackrel{\text{def}}{=} \sqrt{u_1^2+u_2^2+u_3^2}$.

Remark 7. *The most natural coordinate system here are cylindrical-elliptic:*

$$x_1 = \rho - u_1;\ x_2 = \sqrt{C}\arccos\left(\frac{u_2}{\sqrt{u_2^2+u_3^2}}\right);\ x_3 = \rho + u_1,$$

where the metrics have the Levi-Civita form (1,2) (with $\lambda_1 = \lambda(\rho - u_1)$, $\lambda_2 = \lambda(0)$, *and* $\lambda_3 = \lambda(\rho+u_1)$.*) In particular, if two Riemannian metrics are given in the form (26,27), they are projectively equivalent. If* λ *is smooth and positive with* $\lambda'(0) > 0$, *then the formulae (26,27) define Riemannian metrics.*

Proof: Let the metrics be proportional at x_0. Take a small $\epsilon-$ball B_ϵ (in the metric g) with the center in x_0. If ϵ is small enough, x_0 is the only point of the ball where the metrics are proportional. Indeed, suppose they are proportional at three points y_1, y_2, y_3 of the ball. Then, for almost every point x of the ball, there exist three geodesics $\gamma_1, \gamma_2, \gamma_3$ such that $\gamma_i(0) = y_i$, $\gamma_i(1) = x$ and the velocity vectors $\dot\gamma_1(1), \dot\gamma_2(1), \dot\gamma_3(1)$ are mutually transversal. Let us show that $\lambda_1 = \lambda_2$ or $\lambda_2 = \lambda_3$ at x. Indeed, by Lemma 1, for every $i = 1, 2, 3$ we have that λ_2 is a double root of $I_t(\gamma_i(1), \dot\gamma_i(1))$. At T_xB_ϵ, consider the coordinate system where g and L are given by diagonal matrices diag$(1,1,1)$ and diag$(\lambda_1, \lambda_2, \lambda_3)$, respectively. In this coordinate system, the polynomial I_t is given by

$$(\lambda_2 - t)(\lambda_3 - t)\xi_1^2 + (\lambda_1 - t)(\lambda_3 - t)\xi_2^2 + (\lambda_1 - t)(\lambda_2 - t)\xi_2^2. \tag{28}$$

Then, the components (ξ_1, ξ_2, ξ_3) of the velocity vectors $\dot\gamma_i(1)$ satisfy

$$\left\{ \begin{array}{ccc} (\lambda_1(x) - \lambda_2)(\lambda_3(x) - \lambda_2)\xi_2^2 & = & 0 \\ (\lambda_1(x) - \lambda_2)(\xi_2^2 + \xi_3^2) + (\lambda_3(x) - \lambda_2)(\xi_2^2 + \xi_1^2) & = & 0. \end{array} \right. \tag{29}$$

If $\lambda_1(x) < \lambda_2 < \lambda_3(x)$, the solutions of (29) are organized into two intersected straight lines so that the velocity vectors $\dot\gamma_i(1)$ are not mutually transversal. Then, at almost every point of B_ϵ we have $\lambda_1 = \lambda_2$ or $\lambda_2 = \lambda_3$, which contradicts Corollary 1.

Consider the set

$$U \stackrel{\rm def}{=} \{x \in B_\epsilon : \ (\lambda_1(x) - \lambda_2)(\lambda_3(x) - \lambda_2) = 0\}.$$

Lemma 5. *The set U is a totally geodesic connected submanifold of B_ϵ of dimension 1. (In other words, U is a geodesic segment).*

Proof: First of all, there exists $x_1 \in B_\epsilon$ where precisely two eigenvalues of L coincide. Indeed, take the $\frac{\epsilon}{2}-$ sphere $S_{\frac{\epsilon}{2}}$ with the center in x_0 and consider the exponential mapping $\exp : T_{x_0}B_\epsilon \to B_\epsilon$. Suppose there exists no point of $S_{\frac{\epsilon}{2}}$ where precisely two eigenvalues of L coincide. Then, at every point of the sphere $S_{\frac{\epsilon}{2}}$ the eigenspace of L corresponding to λ_2 has dimension one. Let us show that it is tangent to the sphere. It is sufficient to show that it is orthogonal to the geodesic connecting the point of the sphere with the point x_0. Denote the initial velocity vector of the geodesic by ξ. At the tangent space at this point, consider the coordinate system where g and L are given by diagonal matrices diag$(1,1,1)$ and diag$(\lambda_1, \lambda_2, \lambda_3)$, respectively. In this coordinate system, the polynomial I_t is given by (28). Since the geodesic goes through the point where $\lambda_1 = \lambda_2 = \lambda_3$, by Lemma 1, λ_2 is the double-root $I_t(\xi)$. Then the components ξ_1, ξ_2, ξ_3 of ξ satisfy the system (29). From the first equation of the system, in view that $\lambda_1 < \lambda_2 < \lambda_3$, we obtain that the component ξ_2 is zero so that ξ is orthogonal to the eigenspace of L corresponding to λ_2. Thus, the eigenspaces of L corresponding to λ_2 give us a smooth one-dimensional distribution on the 2-sphere which is impossible because the Euler characteristic of the sphere. Finally, there exists $x_1 \in S_{\frac{\epsilon}{2}}$ where precisely two eigenvalues of L coincide.

Denote by γ the geodesic going through x_1 and x_0. Let us show that at every point of this geodesic at least two eigenvalues of L coincide.

We assume that $\gamma(1) = x_1$ and $\gamma(0) = x_0$.

At each tangent space, we can find coordinates such that g and L are given by the diagonal matrixes $\mathrm{diag}(1,1,1)$ and $\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$, respectively. Consider the function

$$I'_t \stackrel{\mathrm{def}}{=} \left(\frac{d}{dt}I_t\right).$$

For every fixed t, the function I'_t is an integral of the geodesic flow. By (28),

$$\begin{aligned} I'_t &= (\lambda_2 - t + \lambda_3(x) - t)\xi_1^2 + (\lambda_1(x) - t + \lambda_3(x) - t)\xi_2^2 + (\lambda_2 - t + \lambda_3(x) - t)\xi_3^2, \\ I'_{\lambda_2} &= (\lambda_1(x) - \lambda_2)(\xi_2^2 + \xi_3^2) + (\lambda_3(x) - \lambda_2)(\xi_2^2 + \xi_1^2). \end{aligned} \tag{30}$$

By Lemma 1, λ_2 is a double-root of the polynomial $I_t(\gamma,\dot\gamma)$. It follows from Lemma 1 that the leading coefficient of the polynomial $I'_t(\gamma,\dot\gamma)$ is not zero.

Let us prove that the differential dI'_{λ_2} vanishes at each point of the geodesic orbit $(\gamma,\dot\gamma)$. Since I'_{λ_2} is an integral, it is sufficient to show this at the point $(\gamma(1),\dot\gamma(1))$ only. We have,

$$I'_{\lambda_1(x)} = (\lambda_3(x) - \lambda_1(x))(\xi_1^2 + \xi_2^2) + (\lambda_2 - \lambda_1(x))(\xi_1^2 + \xi_3^2).$$

We see that the function $I'_{\lambda_1(x)}$ is non-negative. At the point $(\gamma(1),\dot\gamma(1))$, it coincides with I'_{λ_2} and, therefore, is zero. Then, it has a minimum at the point $(\gamma(1),\dot\gamma(1))$, and its differential vanishes.

Let us show that the differential of the function $I'_{\lambda_1(x)} - I'_{\lambda_2}$ also vanishes at the point $(\gamma(1),\dot\gamma(1))$. Indeed, the function I'_t is a linear polynomial in t with non-zero leading coefficient at the point $(\gamma(1),\dot\gamma(1))$. Since $\lambda_1(x) < \lambda_2$, then the function $I'_{\lambda_1(x)} - I'_{\lambda_2}$ is either everywhere positive or everywhere negative. Since it vanishes at $(\gamma(1),\dot\gamma(1))$, the differential of $I'_{\lambda_1(x)} - I'_{\lambda_2}$ vanishes at the point $(\gamma(1),\dot\gamma(1))$.

Thus the differential dI'_{λ_2} is zero at each point of the geodesic orbit $(\gamma,\dot\gamma)$. At each point of γ, the components $\frac{\partial I'_{\lambda_2}}{\partial \xi_i}$ of dI'_{λ_2} are

$$2(\lambda_3 - \lambda_2)\xi_1,\ 2(\lambda_3 - \lambda_2 - \lambda_2 + \lambda_1)\xi_2,\ 2(\lambda_1 - \lambda_2)\xi_3.$$

Since the differential vanishes, all its components are equal to zero. Then, $\lambda_2 = \lambda_3$ or $\lambda_1 = \lambda_2$ or $\xi_2 \neq 0$.

On the other hand, by (28), using that $I_{\lambda_2} = 0$, we see that $(\lambda_2 - \lambda_1(x))(\lambda_2 - \lambda_3(x)) = 0$ or $\xi_2 = 0$.

Finally, every point of γ lies in U.

Now let us prove that γ actually coincides with U. Assume γ does not coincide with U; that is, there exist a point x_2 where $\lambda_1 = \lambda_2$ or $\lambda_2 = \lambda_3$ not lying on the geodesic γ. Then, for almost every $x \in B_\epsilon$, there exist three geodesics $\gamma_1,\gamma_2,\gamma_3$ such that $\gamma_i(1) = x$ and $\gamma_i(0) \in U$ and the vectors $\dot\gamma_i(1) = x$ are linearly independent. From other side, the components ξ_1,ξ_2,ξ_3 of each of these vectors (in the coordinate system where g and L are given by diagonal matrices $\mathrm{diag}(1,1,1)$ and $\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$) satisfy the first equation of the system (29). Then, $x \in U$ which contradicts Lemma 1. Finally, U coincides with γ.

Now let us show that there exists a smooth vector field v_2 on B_ϵ such that:

1. v_2 is a Killing vector field with respect to both metrics,

2. $g(v_2, v_2) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)$ (in particular, v_2 vanishes at U),

3. $Lv_2 = \lambda v_2$,

4. the integral curves of v_2 are homeomorphic to the circle.

Let us prove that there exists v_2 on $B_\epsilon \setminus U$ satisfying (2),(3). Since at every point of $B_\epsilon \setminus U$ the eigenspace of L corresponding to λ_2 is one-dimensional, at every point there exist precisely two vectors satisfying conditions (2),(3).

Take a point $x_2 \in B_\epsilon \setminus U$ and choose a vector satisfying conditions (2),(3) at x_2. Then we can smoothly choose the vector field satisfying (2),(3) locally, and, therefore, along every curve. Thus it is sufficient to show that the result does not depend on the curve. Take two curves c_1, c_2 connecting x_2 and an arbitrary point $x \in B_\epsilon \setminus U$.

Take a point $x_1 \in U$, $x_1 \neq x_0$. Without loss of generality we can assume that $\lambda_1(x_1) < \lambda_2(x_1) = \lambda_3(x_1)$. By Corollary 4 in a small neighborhood W of x_1 the metrics are given by the model form (24,25). Clearly, there exists curves $\tilde c_1$, $\tilde c_2$ such that c_1 is homotop to $\tilde c_1$, c_2 is homotop to $\tilde c_2$, and $\tilde c_1$ coincides with $\tilde c_2$ outside of W. As it follows from Corollary 4, we can choose v_2 satisfying (2),(3) in W. Since the curves coincide outside of W, there exists v_2 on $B_\epsilon \setminus U$ satisfying (2),(3).

In Levi-Civita coordinates from Theorem 1, the vector field v_2 equals $\pm\frac{\partial}{\partial x_2}$ and therefore is Killing with respect to the metrics g and $\bar g$. Put the vector field v_2 equal to zero at every point of U. Since every isometry of g is a diffeomorphism, the vector field is smooth everywhere. Consider the exponential mapping from the point x_0. As we have proven before, the v_2 is tangent to the images of the spheres on $T_{x_0}B_\epsilon$, and, therefore, generates a Killing vector field on every such sphere. Then its integral curves are closed [2].

Consider the point x_1 where $\lambda_1(x_1) < \lambda_2(x_1) = \lambda_3(x_1)$. In the neighborhood of the point there exists coordinates x_1, x_2, x_3 where the metrics are given by the form (24,25). Consider the exponential mapping $exp : T_{x_1}B_\epsilon \to B_\epsilon$, the 2-plane $P \subset T_{x_1}B_\epsilon$ spanned by $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_3}$, and the image $exp(P)$ of this plane under the exponential mapping.

Since ϵ is small, $exp(P)$ is a two-dimensional submanifold of B_ϵ. Let us show that $exp(P)$ is totally geodesic, (so that the restrictions of the metrics to $exp(P)$ are projectively equivalent), that the eigenvalues of the tensor L constructed for the restriction of the metrics to $exp(P)$ are λ_1, λ_3, and that at every point $exp(P)$ is orthogonal to the vector v_2.

It is sufficient to prove both facts at almost every point of $exp(P)$. Take a point $p_2 \in exp(P)$ where $\lambda_1 < \lambda_2 < \lambda_3$. Denote by γ the geodesic connecting this point with p_1, $\gamma(1) = p_1$, $\gamma(2) = p_2$. In a neighborhood of p_1, the listed above statements are true because of Corollary 4. In a neighborhood of another points, the listed above statements are true because of Levi-Civita's Theorem.

Finally, since $exp(P)$ is totally geodesic, the restrictions of the metrics g and $\bar g$ to $exp(P)$ are projectively equivalent. They are proportional at x_0, they are non-proportional at every other point, and the eigenvalues of L constructed for the restriction are non-constant. By Theorem 8 the metrics are given by (21,22). Finally, the metrics g and $\bar g$ are as in Theorem 9.

2.6 General case and realization

Let $M_1^{n_1}$, $M_2^{n_2}$ be connected manifolds. Suppose Riemannian metrics $g_1, \bar g_1$ on $M_1^{n_1}$ are projectively equivalent and are strictly-non-proportional at least at one point. Suppose Riemannian metrics $g_2, \bar g_2$ on $M_2^{n_2}$ are projectively equivalent and are strictly-non-proportional at least at one point. Assume in addition that, for every $i < j$, the eigenvalues of L_i are less than the eigenvalues of L_j. Consider the direct product $M_1^{n_1} \times M_2^{n_2}$ with the canonical transversal foliations B_{n_1} and B_{n_2}. (The leaves of the foliation B_{n_1} are the products of $M_1^{n_1}$ and a point of $M_2^{n^2}$, the leaves of the foliation B_{n_2} are the products of the point of $M_1^{n^1}$ and $M_2^{n_2}$.) Consider the Riemannian metric $h \stackrel{\mathrm{def}}{=} g_1 + g_2$ and $\bar h \stackrel{\mathrm{def}}{=} \bar g_1 + \bar g_2$ on the product $M_1^{n_1} \times M_2^{n_2}$. It is easy to see that the foliations and the metrics satisfy the assumptions (i,ii,iii) of Section 2.3. Then, by Lemma 3, the metrics g and $\bar g$ given by (15,17) are projectively equivalent and are strictly-non-proportional at least at one point of the manifold. Thus, given two triples $(M_1^{n_1}, g_1, \bar g_1)$, $(M_2^{n_2}, g_2, \bar g_2)$, we constructed the triple $(M_1^{n_1} \times M_2^{n_2}, g, \bar g)$. We will denote this operation by "$\oplus$":

$$(M_1^{n_1}, g_1, \bar g_1) \oplus (M_2^{n_2}, g_2, \bar g_2) = (M_1^{n_1} \times M_2^{n_2}, g, \bar g).$$

It is easy to check that the operation is associative:

$$((M_1^{n_1}, g_1, \bar g_1) \oplus (M_2^{n_2}, g_2, \bar g_2)) \oplus (M_3^{n_3}, g_3, \bar g_3) = (M_1^{n_1}, g_1, \bar g_1) \oplus ((M_2^{n_2}, g_2, \bar g_2) \oplus (M_3^{n_3}, g_3, \bar g_3)).$$

Consider connected manifolds $M_1^{k_1}, M_2^{k_2}, ..., M_m^{k_m}$ with projectively equivalent metric $g_1, \bar g_1; g_2, \bar g_2; ...; g_m, \bar g_m$, respectively. Assume the metrics are strictly-non-proportional at least at one point. We denote by L_i the tensor (4) constructed for $g_i, \bar g_i$, $i = 1, ..., m$. Assume in addition that, for every $i < j$, the eigenvalues of L_i are less than the eigenvalues of L_j. Then, we can canonically construct projectively equivalent metric on the product of the manifolds, and these metrics are strictly-non-proportional at least at one point.

Corollary 5. *A product of spheres admits projectively equivalent metrics which are strictly-non-proportional at least at one point.*

Proof: Basically we will show that if connected manifolds $M_1^{k_1}, M_2^{k_2}, ..., M_m^{k_m}$ admit projectively equivalent strictly-non-proportional at least at one point metric $g_1, \bar g_1; g_2, \bar g_2; ...; g_m, \bar g_m$, then the product of the $M_1^{k_1} \times M_2^{k_2} \times ... \times M_m^{k_m}$ also admit projectively equivalent metrics strictly-non-proportional at least at one point.

Since the manifolds $M_1^{k_1}, M_2^{k_2}$ are closed, the eigenvalues of the tensors (4) constructed for the metrics $g_i, \bar g_i$ are bounded. By definition, the tensor (4) constructed for the metrics $g_i, \bar{g}_i$ and the tensor (4) constructed for the metrics $C \cdot g_i, \bar{g}_i$, where C is a positive constant, are related by

$$L_{new} = C^{\frac{1}{n_2+1}} L_{old}.$$

Thus without loss of generality we can assume that for $i < j$ the eigenvalues of the tensor (4) constructed for the metrics $g_j, \bar{g}_j$ are greater than the eigenvalues of the tensor (4) constructed for the metrics $g_i, \bar{g}_i$. Then, by Lemma 3,

$$(M_1^{k_1}, g_1, \bar{g}_1) \oplus (M_2^{k_2}, g_2, \bar{g}_2)) \oplus ... \oplus (M_m^{k_m}, g_m, \bar{g}_m)$$

gives us projectively equivalent metrics on $M_1^{k_1} \times M_2^{k_2} \times ... \times M_m^{k_m}$ strictly-non-proportional at least at one point.

Finally, in order to prove Corollary 5, we need to show that the sphere S^n admits two projectively equivalent metrics which are strictly-non-proportional at least at one point. Essentially, it was done in [1]: the metric g is the restriction of the Euclidean metrics $dx_1^2 + ... + dx_{n+1}^2$ to the sphere

$$S^n \stackrel{\text{def}}{=} \{(x_1, x_2, ..., x_{n+1}) \in R^{n+1} : \ x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1\}.$$

The metric $\bar{g}$ is the pull-back l^*g, where the mapping $l : S^n \to S^n$ is given by $l : v \mapsto \frac{A(v)}{\|A(v)\|}$, where A is an arbitrary linear non-degenerate transformation of R^{n+1}.

The metrics g and $\bar{g}$ are projectively equivalent. Indeed, the geodesics of g are great circles (the intersections of planes that go through the origin with the sphere). The mapping A is linear and, hence, takes planes to planes. Since the normalization $w \mapsto \frac{w}{\|w\|}$ takes planes to their intersections with the sphere, the mapping l takes great circles to great circles. Thus the metrics g and $\bar{g}$ are projectively equivalent. It is easy to verify that for almost all linear transformations A, (and in particular for $A = \text{diag}(a_1, ..., a_{n+1})$, where $a_1 < a_2 < .... < a_{n+1}$), the metrics g and $\bar{g}$ are strictly-non-proportional at almost every point. Corollary 5 is proved.

Definition 5. By *prime standard triples* we will mean:

- In the one-dimensional case, $(I, dx^2, \frac{1}{\lambda(x)} dx^2)$, where I is an interval with the coordinate x and λ is a smooth positive function.

- In the two-dimensional case, $(D^2, g, \bar{g})$, where D^2 is a 2-disc with coordinates u, v and the metrics $g, \bar{g}$ are given either by the formulae (19,20), (21,22) or (21,23), respectively.

- In the three-dimensional case, $(D^3, g, \bar{g})$, where D^3 is a 3-disc with coordinates u_1, u_2, u_3 and the metrics $g, \bar{g}$ are given by the formulae (24,25).

A disk D^n with two projectively equivalent metrics g, $\bar{g}$ will be called *standard* is there exists the prime standard triples $(D_i, g_i, \bar{g}_i)$, $i = 1, ..., m$, such that

$$(D_1, g_1, \bar{g}_1) \oplus (D_2, g_2, \bar{g}_2)) \oplus ... \oplus (D_3, g_m, \bar{g}_m) = (D^n, g, \bar{g}).$$

The sign "=" here means the existence of a diffeomorphism between $D_1\times\ldots\times D_m$ and D^n which is an isometry with respect to both metrics.

Theorem 10. *Suppose projectively equivalent Riemannian metrics g and $\bar g$ on a connected manifold M^n and are strictly non-proportional at least at one point. Then, every point p of the manifold has a neighborhood U such that the triple $(U, g, \bar g)$ is standard.*

Proof: We use induction in dimension. For dimension one the statement is trivial; for dimension two, the statement follows from Theorem 8. Assume $n \ge 3$, and suppose the theorem is true for dimensions less than n. Let us prove that it is true for dimension n. By Corollary 1, there exists $1 \le r \le 3$ such that $\lambda_1(p) = \ldots = \lambda_r(p) < \lambda_{r+1}(p) \le \ldots \le \lambda_n(p)$. If $r = n$, then $n = 3$; then, the theorem follows from Theorem 9.

Suppose $n > r$. Then, $\lambda_r < \lambda_{r+1}$ at every point of a small neighborhood of p, and we can apply the splitting procedure from Section 2.2. We obtain the metrics $h, \bar h$ and the foliations B_r and B_{n-r} such that the local product structures (h, B_r, B_{n-r}) and $(\bar h, B_r, B_{n-r})$ satisfy the conditions (i,ii,iii) of Section 2.3. By definition of a local product structure, a (possibly, smaller) neighborhood of p, is a direct product of two discs D^r and D^{n-r}, the metrics h and $\bar h$ are the product metrics $g_1 + g_2$, $\bar g_2 + \bar g_2$, where g_i is projectively equivalent to $\bar g_i$. By the induction assumption, the triples $(D^r, g_1, \bar g_1)$ and $(D^{n-r}, g_2, \bar g_2)$ are standard; therefore, $(D^r, g_1, \bar g_1) \oplus (D^{n-r}, g_2, \bar g_2)$ is standard as well. In view of Remark 3, $(D^r, g_1, \bar g_1) \oplus (D^{n-r}, g_2, \bar g_2)$ is precisely $(D^r \times D^{n-r}, g, \bar g)$. Theorem 10 is proved.

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