Finsler conformal Lichnerowicz conjecture

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Based on joint papers with
Troyanov (Lausanne), Rademacher (Leipzig), Zeghib (Lion)

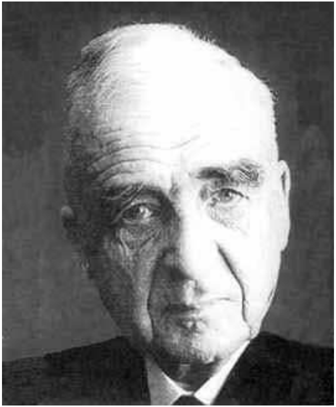
Slides of the presentation are on
www.minet.uni-jena.de/~matveev/

Paul Finsler

*11. April 1894 in Heilbronn; †29. April 1970 in Zürich;

worked mostly in set theory, foundation of mathematics and number theory (and never in differential or finsler geometry);

mostly known for introduction of finsler metrics in his thesis 1918.



Picture from
www.math.iupui.edu/~zshen/Finsler/people/Finsler html

Definition of finsler metrics

Euclidean norm:
$E : \mathbb{R}^n \to \mathbb{R}$ of the form
$E(v) = \sqrt{\sum_{i,j} a_{ij} v^i v^j}$,
where (a_{ij}) is a positively definite symmetric matrix

$\longrightarrow$

(Abstract) norm: $B : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$
such that $\forall v \in \mathbb{R}^n, \ \forall \lambda > 0$

(a) $B(\lambda \cdot v) = \lambda \cdot B(v)$,

(b) $B(u + v) \leq B(u) + B(v)$,

(c) $B(v) = 0 \iff v = 0$

$\downarrow$

(Local) Riemannian metric:
$g : \underbrace{\mathbb{R}^n}_{x} \times \underbrace{\mathbb{R}^n}_{v} \to \mathbb{R}_{\geq 0}$
of the form
$g(x, v) = \sqrt{\sum_{i,j} a_{ij}(x) v^i v^j}$,
where for every x
$(a_{ij}(x))$ is a positively definite symmetric matrix

$\longrightarrow$

$\downarrow$

(LOCAL) FINSLER METRIC:
$F : \underbrace{\mathbb{R}^n}_{x} \times \underbrace{\mathbb{R}^n}_{v} \to \mathbb{R}_{\geq 0}$ such that for every x
$F(x, \underbrace{\cdot}_{v}) : \mathbb{R}^n \to \mathbb{R}$ is a norm, i.e., satisfies
(a), (b), (c).

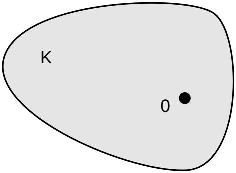
Formal Definition: Finsler metric ist a continuous function $F : TM \to \mathbb{R}$ such that for every $x \in M$ the restriction $F_{|T_xM}$ is a norm.
We say that the finsler metric is smooth if it is smooth on $TM - 0$.

How to visualize finsler metrics

It is known (Minkowski) that the unite ball determines the norm uniquely:

for a given convex body $K \in \mathbb{R}^n$ such that $0 \in int(K)$ there exists an unique norm B such that $K = \{x \in \mathbb{R}^n \mid B(x) \le 1\}$.

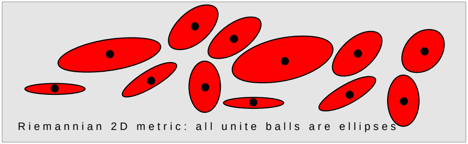
Thus, in order to describe a finsler metric it is sufficient to describe unite balls at every tangent space.



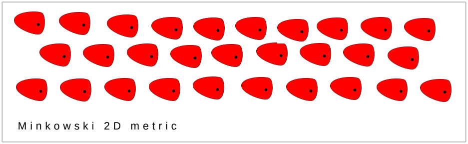
There exists a unique norm such that (the convex body) K is the unite ball in this norm

Examples:

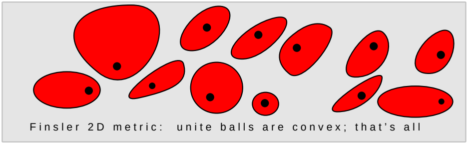
Riemannian metric: every unite ball is an ellipsoid symmetric w.r.t. 0.



Minkowski metric on $\mathbb{R}^n$: $F(x, v) = B(v)$ for a certain norm B, i.e., the metric is invariant w.r.t. the standard translations of $\mathbb{R}^n$.



Arbitrary finsler metric on $\mathbb{R}^4$:



Further, we will consider only smooth finsler metrics – that means that the balls are smooth and depend smoothly on the point.

Popular game: to take a notion from the Riemannian geometry and to try to generalize it to the finsler geometry

Good example (Finsler–Carathéodory 1918): *Length of a curve:* On a finsler manifold (M, F), the length of a smooth curve $c : [a, b] \to M$ is defined as

$$L(c) = \frac{1}{2} \int_a^b \big(F(c(t), c'(t)) + F(c(t), -c'(t))\big) dt.$$

The finsler length
(1) does not depend on the parameterization of the curve (because $F(x, \lambda v) = \lambda F(x, v)$ for $\lambda > 0$).
(2) coincides with the Riemannian (or Euclidean) length for Riemannian (resp. Euclidean) metrics.
(3) is an active object of investigation (there is a whole science about it)

Bad (!?) but still actively studied example

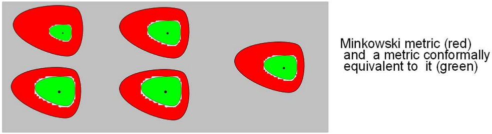
Riemannian curvature: there exist many generalizations of the Riemann curvature tensor for finsler manifolds; all of them are very complicated, none of them were used in other parts of mathematics or of sciences.

Remark. Cartan and Chern would probably disagree with me at this point: they worked a lot to understand what finsler curvature could be and gave two different definitions.

(Picture from http://www.msri.org/; photo of Chern by George Bergman)

Goal of today's lecture: conformal transformations

Def. Two finsler metrics F_1 and F_2 on one manifold M are *conformally equivalent*, if for a certain function $\lambda : M \to \mathbb{R}_{>0}$ we have:
$F_2(x, v) = F_1(x, v) \cdot \lambda(x)$.



Def. A diffeomorphism $\phi : M \to M$ is a *conformal transformation* (of F), if the pullback of F is conformally equivalent to F; i.e., if for a certain function $\lambda : M \to \mathbb{R}_{>0}$ we have $F(\phi(x), d_x\phi(v)) = F(x, v) \cdot \lambda(x)$.

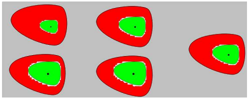
Special cases:

- Isometries: $\lambda \equiv 1$.
- Homotheties: $\lambda \equiv \mathrm{const}$.

Conformal transformations (isometries, homotheties) of finsler metrics is a popular (and "good" in the above sence) object to study; in 1986 it was a conference in Romania dedicated to conformal geometry of finsler metrics

Examples and main Theorem

(i) If $\phi : M \to M$ is an isometry for F, and $\lambda : M \to \mathbb{R}_{>0}$ is a function, then ϕ is a conformal transformation of $F_1 := \lambda \cdot F$.



(ii) Let F_m be a Minkowski metric on $\mathbb{R}^n$. Then, the mapping $x \mapsto \mathrm{const} \cdot x$ (for $\mathrm{const} \neq 0$) is a conformal transformation. Moreover, it is also a conformal transformation of $F := \lambda \cdot F_m$. Moreover, if ψ is an isometry of F_m, then $\psi \circ \phi$ is a conformal transformation of every $F := \lambda \cdot F_m$.

(iii) Let g be the standard (Riemannian) metric on the standard sphere S^n. Then, the standard Möbius transformations of S^n are conformal transformations of every metric $F := \lambda \cdot g$.

(In D2 Möbius transformations are fractional-linear transformations of $S^2 = \overline{\mathbb{C}}$.)

Main Theorem. That's all: Let ϕ be a conformal transformation of a connected (smooth) finsler manifold $(M^{n \geq 2}, F)$. Then (M, F) and ϕ are as in Examples (i, ii, iii) above.

Even in the Riemannian case, Main Theorem is nontrivial

Corollary (proved before by Alekseevsky 1971, Schoen 1995, (Lelong)-Ferrand 1996) Let ϕ be a conformal transformation of a connected RIEMANNIAN manifold $(M^{n\geq 2}, g)$. Then for a certain $\lambda : M \to \mathbb{R}$ one of the following conditions holds

(a) ϕ is an isometry of $\lambda \cdot g$, or

(b) $(M, \lambda \cdot g)$ is $(\mathbb{R}^n, g_{\mathrm{flat}})$,

(c) or $(S^n, g_{\mathrm{round}})$.

The story

This statement is known as *conformal Lichnerowicz conjecture* $\sim$ 1960
1970: Obata proved it under the assumption that M is closed.
1971: Alekseevsky proved it for all manifolds.
1974–1996: (Lelong)-Ferrand gave another proof using her theory of quasiconformal mappings
1995: Schoen: New proof using completely new ideas

Remark. *In the pseudo-Riemannian case, the analog of Main Theorem is a open actively studied conjecture (Baum, Leistner, Leitner, Kühnel, Melnick, Frances, ...)*

Main trick and the plan of the proof

I reduce Main Theorem to its (Riemannian) Corollary which was proved before:

1. Main Trick: given a (smooth) finsler metrics F we construct a RIEMANNIAN metric on g_F such that $g_{\lambda F} = \lambda g_F$.

$$F \quad \xmapsto{\text{canonically construct}} \quad g_F$$

Then, any conformal transformation of F (homothety, isometry resp.) is a conformal transformation (homothety, isometry, resp.) of g_F.

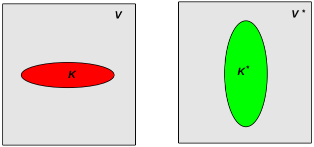
2. Plan. By Corollary, the exist two cases:

(a) conformal transformation ϕ is an isometry of a certain $\lambda \cdot g_F$; then, it is an isometry of $\lambda \cdot F$. (Case (i) of Main Theorem)

(b) for a certain λ the metric $\lambda \cdot g_F$ is the standard metric of S^n or $\mathbb{R}^n$. The conformal transformations of S^n or $\mathbb{R}^n$ are described, by Liouville 1850 in dim $n = 2$, and by S. Lie 1872 in dim $n \geq 3$. Playing with this description, we obtain the result (Cases (ii) and (iii) of Main Theorem).

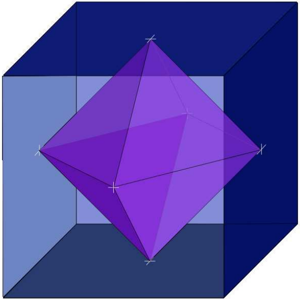
A standard object from convex geometry

Let V be a $\mathbb{R}-$vector space, and V^* ist dual space. For every convex body K such that $0 \in \operatorname{int}(K)$ we consider $K^* \subseteq V^*$ given by $K^* := \{\xi \in V^* \mid \ \xi(k) < 1 \text{ for all } k \in K\}$.



Well-known facts:

- K^* is a convex (compact) body,
- $(\lambda K)^* = \frac{1}{\lambda} K^*$,
- K^* does not depend on the coordinate system



If we have a fixed Euclidean metric in V, we can identify V and V^*, and dual body becomes the polar body

Construction of the Euclidean structure in every tangent space (avereging construction)

For every convex body $K \subseteq V$ such that $0 \in \mathrm{int}(K)$, let us now construct an Euclidean structure in V (later, the role of V will play T_xM, and the role of K the unite ball in the norm $F_{|T_xM}$). We take an arbitrary linear volume form Ω in V^* and put $g(v) := \sqrt{\frac{1}{Vol(K^*)} \int_{K^*} \xi(v)^2 d\Omega}$ (i.e., the function we integrate takes on $\xi \in K^* \subseteq V^*$ the value $\xi(v)^2$.)
Evidently,

- $g(v)$ is well-defined:
  - it does not depend on Ω (because the only freedom is choosing Ω, multiplication by a constant, does not influence the result),
- $g(\lambda \cdot v) = |\lambda| g(v)$ (i.e., g is homogeneous of degree 1)
- $g(v) = 0 \iff v = 0$
- g' constructed by $K' := \lambda \cdot K$ is given by $g' = \lambda \cdot g$ (because $(\lambda \cdot K)^* = \frac{1}{\lambda} K^*$.
- g does not depend on the coordinate system

g given by $g(v) := \sqrt{\frac{1}{Vol(K^*)}\int_{K^*}\xi(v)^2 d\Omega}$ is Euclidean

By Jordan – von Neumann 1935, a nonnegative function $g : V \to \mathbb{R}$ such that $g(\lambda \cdot v) = |\lambda| g(v)$ and $(g(v) = 0 \iff v = 0)$ is an Euclidean metric, if and only if the following "parallelogram condition" is fulfilled: $g(v+u)^2 + g(v-u)^2 = 2(g(v)^2 + g(u)^2)$. Let us check:

$$\frac{1}{Vol(K^*)}\int_{K^*}\xi(v+u)^2 d\Omega + \frac{1}{Vol(K^*)}\int_{K^*}\xi(v-u)^2 d\Omega$$
$$= \frac{1}{Vol(K^*)}\int_{K^*}\left(\xi(v+u)^2 + \xi(v-u)^2\right) d\Omega \quad =[\text{since } \xi \text{ is linear}]=$$
$$= \frac{1}{Vol(K^*)}\int_{K^*}\left(\xi(v)^2 + 2\cdot\xi(v)\cdot\xi(u) + \xi(u)^2 + \xi(v)^2 - 2\cdot\xi(v)\cdot\xi(u) + \xi(u)^2\right) d\Omega$$
$$= \frac{1}{Vol(K^*)}\int_{K^*} 2\cdot\xi(v)^2 d\Omega + \frac{1}{Vol(K^*)}\int_{K^*} 2\cdot\xi(u)^2 d\Omega$$
$$= 2(g(v)^2 + g(u)^2) \qquad \square$$

Remark 1. The construction is too easy to be new – our motivation came from classical mechanics, and our construction is close to one of the inertia ellipsoid (Poinsot, Binet, Legendre). In the convex geometry, Milman et al 1990 had a similar construction in an Euclidean space

Remark 2. There exist other constructions with the necessary properties – for example in the initial paper M∼, Rademacher, Troyanov, Zeghib we used another construction. The present construction is due to M∼ – Troyanov, and even does not require the unite ball to be smooth and even convex – we will see in applications why it is good

Thus, by a finsler metric F, we canonically constructed a Euclidean structure on every tangent space that smoothly depends on the point, i.e., a Riemannian metric g_F. This metric has the following property: $g_{\lambda \cdot F} = \lambda \cdot g_F$.
In particular,

- if ϕ is conformal transformation of F, it is a conformal transformation of g_F
- if ϕ is a conformal transformation of F, and is an isometry or homothety of g_F, it is an isometry resp. homothety of transformation of F.

Proof of Main Theorem

Let ϕ is a conformal transformation of F. Then, it is a conformal transformation of g_F. By the Riemannian version of Main Theorem, the following cases are possible:

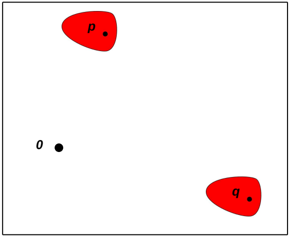
(Trivial case): ϕ is an isometry of a certain $\lambda \cdot g_F$. Then, it is an isometry of $\lambda \cdot F$.

(Case $\mathbb{R}^n$): After the multiplication F be an appropriate function, g_F is the standard Euclidean metric, and ϕ is a homothety of g_F.

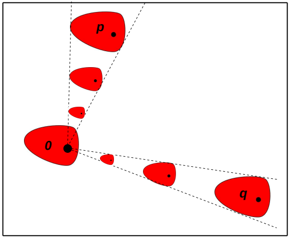
(Case S^n): After the multiplication F be an appropriate function, g_F is the standard metric on the sphere, and ϕ is a möbius transformation of the sphere.

(Case $\mathbb{R}^n$): After the multiplication F be an appropriate function, g_F is the standard Euclidean metric, and the conformal transformation ϕ is a homothety of g_F

We consider two points $p, q \in \mathbb{R}^n$. Our goal is to show that the unite ball in q is the parallel translation of the unite ball in q.



Let us first assume for simplicity that the conformal transformation ϕ is already a homothety $x \mapsto C \cdot x$ for a constant $1 > C > 0$ (we known that actually it is $\psi \circ \phi$, where ψ is an isometry; I will explain on the next slide that w.l.o.g. $\psi = Id$)



We consider the points $p, \phi(p) = C \cdot p, \phi \circ \phi(p) = \phi^2(p) = C^2 \cdot p$, ... , $\overset{\text{converge}}{\longrightarrow} 0$.

The unite ball of the push-forward $\phi_*^k(F)$ of the metric at the point $\phi^k(p)$ are as on the picture; therefore, the unite ball of $\frac{1}{C^k}\phi_*^k(F)$ at the point $\phi^k(p)$ is the parallel translation of the unite ball at the unite ball at the point p. But the unite ball of $\frac{1}{C^k}\phi_*^k(F)$ at $\phi^k(p)$ is the unite ball of F!

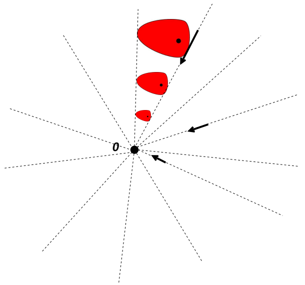
Sending $k \to \infty$, we obtain that the unite ball at $0 = \lim_{k\to\infty} \phi^k(p)$ is the parallel translation of the unite ball at p. The same is true for q. Then, the unite ball at q is the parallel translation of the unite ball at p

Why we can think that the conformal transformation is a homothety ϕ, and not the composition $\psi \circ \phi$, where $\psi \in O(n)$ is an isometry

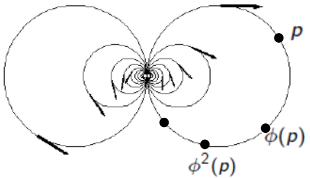
Because the group $O(n)$ is compact. Hence, any sequence of the form $\psi, \psi^2, \psi^3, ...$, has a subsequence converging to Id. Thus, in the arguments on the previous slide we can take the subsequence $k \to \infty$ such that $(\psi \circ \phi)^k \stackrel{\phi \circ \psi = \psi \circ \phi}{=} \psi^k \circ \phi^k$ is "almost" ϕ^k, and the proof works.

(Case S^n): After the multiplication F be an appropriate function, g_F is the standard “round” (Riemannian) metric on the sphere

Conformal transformation of S^n were described by J. Liouville 1850 in dim n $= 2$, and by S. Lie 1872. For the sphere, the analog of the picture (a) for the conformal transformation (which are homotheties) of $\mathbb{R}^n$ is the picture (b).



Picture (a)



Picture (b)

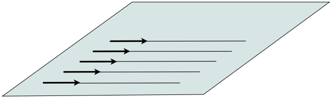
One can generalize of the proof for $\mathbb{R}^n$ to the case S^n (the principal observation that sequence of the points $p, \phi(p), \phi^2(p), \ldots$ converges to a fixed point is also true on the sphere; the analysis is slightly more complicated).

Facts: J. Liouville 1850, S. Lie 1872

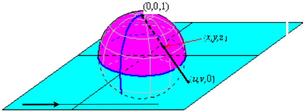
Fact 1. Let ϕ be a conformal nonisometric orientation-preserving transformation of the round sphere $(S, g_{\rm round})$. Then, there exists a one parameter subgroup $(\mathbb{R}, +) \subset \mathrm{Conf}(S, g_{\rm round})$ containing ϕ.

Fakt 2. Any one-parametric subgroup of $(\mathbb{R}, +) \subset \mathrm{Conf}(S, g_{\mathrm{round}})$ which is not a subgroup of $Iso(S, g_{\mathrm{round}})$ can be constructed by one of the following ways:

- Way 1. (General case)

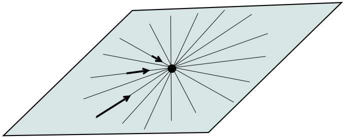


  (i) One takes the sliding rotation $\Phi_t : x \to \exp(tA) + tv$, where A is a skew-symmetric matrix such that the vector v is its eigenvector



  (ii) and then pullback this transformation to the sphere with the help of stereographic projection

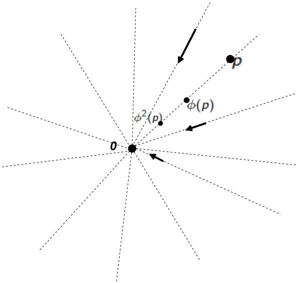
- Way 2. (Special case)



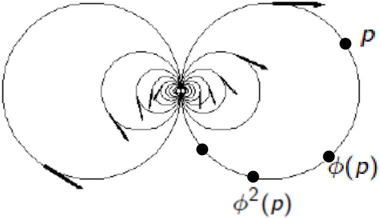
  (i) One takes $\Psi \circ \Phi$, where Φ is a homothety on the plane and Ψ is a rotation on the plane

  (ii) and then pullback this transformation to the sphere with the help of stereographic projection

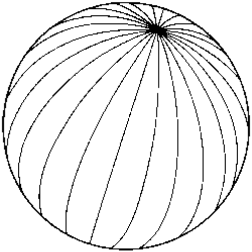
A neighborhood of the pole on the sphere is as on the picture:



Special case



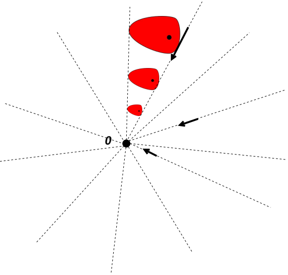
General case



in the special case two points of the sphere have such neighborhood (south and north pols), in the general case only one

The proof for the special case

In the special case, we can repeat the repeat the proof for the case $(\mathbb{R}^n, g_{\text{flat}})$:



We obtain that the metric on $S^n - \{\text{south pole}\}$ is conformally diffeomorph to $(\mathbb{R}^n, g_{\text{minkowski}})$; the conformal diffeomorphism is the stereographic projection S_{SP}.

Similarly, (because there is no essential difference between south and north pole) we obtain that the metric on $S^n - \{\text{north pole}\}$ is conformally equivalent to $(\mathbb{R}^n, g_{\text{minkowski}})$, the conformal diffeomorphism is the stereographic projection S_{NP}.

Then the superposition $S_{SP}^{-1} \circ S_{NP}$ is a conformal diffeomorphism of the metric F restricted to $S^n - \{\text{south pole, north pole}\}$.

From the school geometry we know that the superposition $S_{SP}^{-1} \circ S_{NP}$ is the inversion

$I : \mathbb{R}^n \to \mathbb{R}^n, \quad I(x_1, ..., x_n) = \left(\frac{x_1}{x_1^2 + ... + x_n^2}, ..., \frac{x_n}{x_1^2 + ... + x_n^2} \right).$

Thus, in order to show that the finsler metric F is Riemannian, it is sufficient to show, that the push-forward $I_* F$ of a Minkowski metric F is conformally equivalent to (another) Minkowski metric if and only if they are Riemannian, which is an easy exercise.

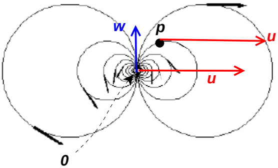
The proof for the general case

We have: the finsler metric F is invariant with respect to ϕ.
The goal: to prove that the metric is conformally equivalent to $g_{\text{averaged}} = g_{\text{round}}$. We consider the following two functions:

$M(q) := \max_{\eta \in T_q S^n,\ \eta \neq 0} \frac{F(q,\eta)}{g_{(q)}(\eta)} - \min_{\eta \in T_q S^n,\ \eta \neq 0} \frac{F(q,\eta)}{g_{(q)}(\eta)}$.

$M(q) = 0 \iff F(q,\cdot)$ **is proportional to** $g_{(q)}(\cdot)$**.**

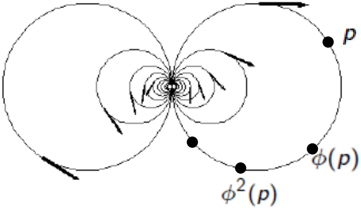
$m(q) := \frac{F(q,v(q))}{g_{(q)}(v(q))}$, where v is the generator of the 1-parameter group of the conformal transformations containing ϕ. **Both functions are invariant with respect to** ϕ**.** Let us first show that the function M is zero at the point 0.



We will show that for every vector u at 0 we have $\frac{F(0,u)}{g_{(0)}(u)} = \frac{F(0,w)}{g_{(0)}(w)}$, where w is as on the picture.
We take a point p very close to 0 such that at this point u is proportional to v with a positive coefficient. Such points exist in arbitrary small neighborhood of 0. We have:

$$\frac{F(p,u)}{g_{(p)}(u)} = \frac{F(p,v)}{g_{(p)}(v)} := m(p) \overset{m(p) \text{ is invariant w.r.t. } \phi}{=} m(0) = \frac{F(0,w)}{g_{(0)}(w)}.$$

Replacing p by a sequence of the points converging to 0 (such that at these points u is proportional to v) we obtain that $\frac{F(0,u)}{g_{(0)}(u)} = \frac{F(0,w)}{g_{(0)}(w)}$ implying $M(q) = 0$ implying $F(0,\cdot) = \lambda \cdot g_{(0)}(\cdot)$.



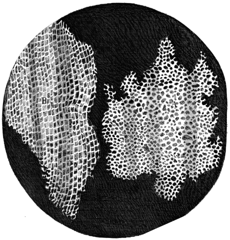
We have:

- $M(0) = 0$,
- M is invariant w.r.t. ϕ and continuous,
- For every point p the sequence $p, \phi(p), \phi^2(p), \ldots$ converges to 0.

Then, $M \equiv 0$ implying the metric F is actually a Riemannian metric, $\square$

What to do next: possible applications in sciences

Finsler geometers always emphasis possible applications of finsler metrics in geometry – certain phenomena in sciences (for examples light prolongation in cristalls or certain processes in organic cells) can be described with the help of finsler metrics.



Picture from Wikipedia common

Picture from Wikipedia common

Picture from chemistry.about.com

Unfortunately, the "standard" finsler methods appeared to be too complicated to be used.

Our proof suggests to use the averaging construction to replace the finsler metric by a Riemannian, and then to analyse it. Of cause, we loose a lot of information, but get an object which is easier to investigate.

Note that we even do not require that the "unite ball" is convex.

One more application in mathematics: Proof of Szabo's theorem 1982

Def. A Finsler metric is *Berwald*, if there exists a symmetric affine connection $\Gamma = (\Gamma^i_{jk})$ such that the parallel transport with respect to this connection preserves the function F. In this case, we call the connection Γ the *associated connection*.

Example 1. Riemannian metrics are always Berwald. For them, the associated connection coincides with the Levi-Civita connection.

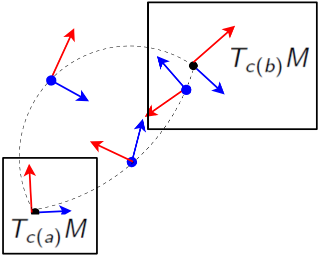
Def. We say that the metric is essentially Berwald, if it Berwald but not Riemannian

Example 2. Minkowski (nonriemannian) metric is essentially Berwald — with the flat associated connection

Theorem (Szabo 1982) The associated connection of an essentially Berwald finsler metric is Levi-Civita connection of a certain Riemannian metric. Moreover, the Riemannian metric is decomposable or symmetric of rank ≥ 2.

The initial proof of Szabo is complicated. With the help of the averaging metric the proof is trivial — see the next slide

Proof of Szabo's theorem



Let F be an essential Berwald finsler metric on M. We consider the averaged Riemannian metric g_F. Let $c : [a, b] \to M$ be a smooth curve and $\tau_c : T_{c(a)}M \to T_{c(b)}M$ be the corresponding parallel transport with respect to the associated connection Γ of the Berwald metric. It is a linear map preserving the finsler unite ball. Then, it preserves the averaged metric g_F implying it is the Levi-Civita connection of the metric g_F.

Now consider all possible curves $c : [a, b] \to M$ such that $c(a) = c(b) := p$. For every such curve we have the endomorphism $\tau_c : T_pM \to T_pM$.

The set of such endomorphisms is **holonomy group of Γ at the point p:** $H_p := \{\tau_c \mid c : [a, b] \to M \text{ is a smooth curve such that } c(a) = c(b) := p\} \subseteq O(g_{(p)})$.

H_p preserves $g_{(p)}$ and $F(p, \cdot)$ and therefore preserves the function $\tilde{m}(\xi) := \frac{F(p,\xi)}{g_{(p)}(\xi)}$. If the function $\tilde{m}$ is constant (on T_pM), the metric F at the point p is $\mathrm{const} \cdot g$, i.e., is Riemannian. If the function f is not constant, the holonomy group does not act transitively. Then, by the classical result of Berger(1955)–Simons(1962) the metric g is decomposable, or symmetric of rank ≥ 2, $\square$

What to do next: possible applications in mathematics

We already have additional applications, now it is your turn to try

Homework for you

Prove (without using Main Theorem) the following statement:

Let (M, F) be a complete finsler manifold admitting a homothety $\phi : M \to M$. Then, the homothety is an isometry, or $M = \mathbb{R}^n$ and F is the Minkowski metric.

The story behind:

- Riemannian case: well known at least since textbooks of Lichnerowicz and Kobayashi–Nomizu ($\sim$50th).
- Finsler case: claimed by Heil and Laugwitz in 1974/75;
- Lovas und Szilasi (2009) claimed a flaw in the proof of Heil and Laugwitz and suggested a correct proof; the proof uses very advanced finsler geometry and is complicated

With the help of averaging construction the proof is trivial — try!!!