Binet-Legendge ellipsoid in finsler geometry

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Based on the paper arXiv:1104.1647 joint with Marc Troyanov

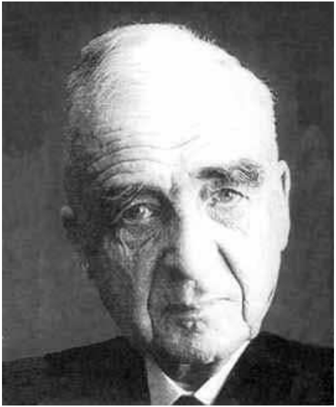
Abstract: I show a simple construction from convex geometry that solves many named problems in Finsler geometry

PAUL FINSLER

*11. April 1894 in Heilbronn; †29. April 1970 in Zürich;

worked mostly in set theory, foundation of mathematics and number theory (and never in differential or finsler geometry);

mostly known for introduction of finsler metrics in his thesis 1918.



Picture from
www.math.iupui.edu/~zshen/Finsler/people/Finsler html

Definition of finsler metrics: Finsler metric ist a continuous function $F : TM \to R$ such that for every $x \in M$ the restriction $F_{|T_xM}$ is a Minkowski norm, that is $\forall u, v \in T_xM$, $\forall \lambda > 0$

(a) $F(\lambda \cdot v) = \lambda \cdot F(v)$, (b) $F(u+v) \leq F(u) + F(v)$,

(c) $F(v) = 0 \iff v = 0$.

Euclidean norm:
$E : R^n \to R$ of the form
$E(v) = \sqrt{\sum_{i,j} a_{ij} v^i v^j}$,
where (a_{ij}) is a positively definite symmetric matrix

$\longrightarrow$

(Minkowski) norm:
$B : R^n \to R_{\geq 0}$ with

(a) $B(\lambda \cdot v) = \lambda \cdot B(v)$,

(b) $B(u+v) \leq B(u) + B(v)$,

(c) $B(v) = 0 \iff v = 0$

$\downarrow$ $\downarrow$

(Local) Riemannian metric:
$g : \underbrace{R^n}_{x} \times \underbrace{R^n}_{v} \to R_{\geq 0}$
of the form
$g_x(v,u) = \sum_{i,j} a_{ij}(x) v^i u^j$,
where for every x
$(a_{ij}(x))$ is a positively definite symmetric matrix

$\longrightarrow$

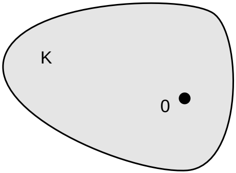
(LOCAL) FINSLER METRIC:
$F : \underbrace{R^n}_{x} \times \underbrace{R^n}_{v} \to R_{\geq 0}$ such that for every x
$F(x, \underbrace{\cdot}_{v}) : R^n \to R$ is a norm, i.e., satisfies
(a), (b), (c).

How to visualize finsler metrics

It is known (Minkowski) that the unit ball determines the norm uniquely:

for a given convex body $K \in R^n$ such that $0 \in int(K)$ there exists an unique norm B such that $K = \{x \in R^n \mid B(x) \leq 1\}$.

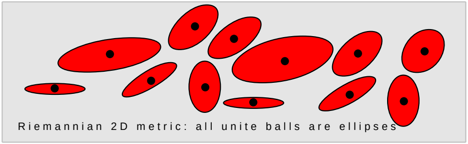
Thus, in order to describe a finsler metric it is sufficient to describe unit balls at every tangent space.



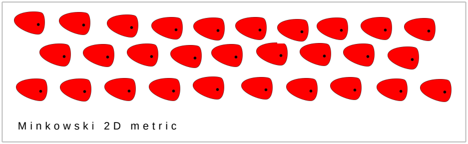
There exists a unique norm such that (the convex body) K is the unite ball in this norm

Examples:

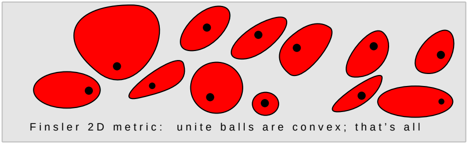
Riemannian metric: every unit ball is an ellipsoid symmetric w.r.t. 0.



Minkowski metric on R^n: $F(x, v) = B(v)$ for a certain norm B, i.e., the metric is invariant w.r.t. the standard translations of R^n.



Arbitrary finsler metric on R^n:



If the finsler metric is smooth, the balls are smooth and depend smoothly on the point.

Main Trick

$$\boldsymbol{F} \quad \xmapsto{\textit{canonically construct}} \quad \boldsymbol{g_F}$$

Given a (smooth) finsler metrics F we construct a (smooth) RIEMANNIAN metric on g_F such that

• The Riemannian metric g_F has the same (or better) regularity as the finsler metric F

• If F is Riemannian, i.e. if $F(x,\xi) = \sqrt{g_x(\xi,\xi)}$ for a some Riemannian metric g, then $g_F = g$

• If two finsler metrics F_1 and F_2 are conformally equivalent, i.e., if $F_1(x,\xi) = \lambda(x)F_2(x,\xi)$ for some function $\lambda : M \to R$, then the corresponding Riemannian metrics are also conformally equivalent with essentially the same conformal factor: $g_{F_1} = \lambda^2 g_{F_2}$

• If F_1 and F_2 are C^0-close, then so are g_{F_1} and g_{F_2}.

• If F_1 and F_2 are bilipschitzly equivalent, then so are g_{F_1} and g_{F_2}.

This allows to use the results and methods from (much better developed) Riemannian geometry to finsler geometry I will show many application

Construction of the (Binet-Legendre) Euclidean structure in every tangent space

For every convex body $K \subseteq V$ such that $0 \in \mathrm{int}(K)$, let us now construct an Euclidean structure in V.
Later, the role of V will play T_xM, and the role of K the unit ball in the norm $F_{|T_xM}$.
We take an arbitrary linear volume form Ω in V^* ($=$ dual vector space to V) and construct contravariant bilinear form $g^* : V^* \times V^* \to R$ by

$$g^*(\xi, \nu) := \frac{1}{Vol_\Omega(K)} \int_K \xi(k)\nu(k) d\Omega$$

(i.e., the function we integrate takes on $k \in K \subset V$ the value $\xi(k)\nu(k)$; ξ and ν are elements of V^*, i.e., are functions on V.)

Equivalent definition: $g^*(\xi, \nu) = \langle \xi_{|K}, \nu_{|K} \rangle_{L_2}$ where we fixed the linear volume form Ω on V by requiring $Vol_\Omega(K) = 1$.

g^* allows to identify canonically V and V^* and gives therefore an Euclidean structure on V, which we denote by g.

$$g^*(\xi,\nu) := \frac{1}{Vol_\Omega(K)}\int_K \xi(k)\nu(k)d\Omega$$

Evidently, g is a well-defined Euclidean structure

- it does not depend on Ω (because the only freedom is choosing Ω, multiplication by a constant, does not influence the result),
- It is bilinear and positive definite

  Moreover,
- g' constructed by $K' := \frac{1}{\lambda}\cdot K$ is given by $g' = \lambda^2 \cdot g$
- construction of g is canonical ($=$ does not depend on the coordinate system)

Remark 1. The construction is too easy to be new – our motivation came from classical mechanics, and our construction is close to one of the inertia ellipsoid (Poinsot, Binet, Legendre). In the convex geometry, Milman et al 1990 had a similar construction in an Euclidean space

Remark 2. There exist other constructions for example Vincze 2005 and M$\sim$, Rademacher, Troyanov, Zeghib 2009. The present construction has better properties

Thus, by a finsler metric F, we canonically constructed a Euclidean structure on every tangent space, i.e., a Riemannian metric g_F. If the finsler metric is smooth, then the Riemannian metric is also smooth.

This metric has the following property: $g_{\lambda \cdot F} = \lambda^2 \cdot g_F$.

In particular, if ϕ is isometry, similarity, or conformal transformation of F, it is an isometry, similarity, or conformal transformation of g_F.

First application: Wang's Theorem for all dimensions.

Theorem. *Let (M^n, F) be a C^2-smooth connected Finlser manifold. If the dimension of the space of Killing vector fields of (M, F) is greater than $\frac{n(n-1)}{2} + 1$, then F is actually a Riemannian metric.*
History: For $n \neq 2, 4$ Theorem was proved 1947 by H.C. Wang. This theorem answers a question of S. Deng and Z. Hou (2007).
Proof. I will use: if ϕ is an isometry of F, then it is an isometry of g_F.
Let $r > \frac{n(n-1)}{2} + 1$ be the dimension of the space of Killing vector fields. Take a point x and choose $r - n$ linearly independent Killing vector fields $K_1, \ldots, K_{r-n}$ vanishing at x. The point x is then a fixed point of the corresponding local flows $\phi_t^{K_1}$, ... , $\phi_t^{K_{r-n}}$.
Then, for every t, the differentials of $\phi_t^{K_1}$, ... , $\phi_t^{K_{r-n}}$ at x are linear isometries of (T_xM, g_F).
Thus, the subgroup of $SO(T_xM, g_F)$ preserving the function $F_{|T_xM}$ is at least $r - n$ dimensional.
Now, it is well-known that every subgroup of $SO(T_xM, g_F)$ of dimension $r - n > \frac{n(n-1)}{2} + 1 - n = \frac{(n-2)(n-1)}{2}$ acts transitively on the g_F-sphere $S^{n1} \subset T_xM$. Then, the ratio $F(\xi)^2/g(\xi, \xi)$ is constant for all $\xi \in T_xM$ and the metric F is actually a Riemannian metric $\square$

The Liouville Theorem for Minkowski spaces and the solution to a problem by Matsumoto.

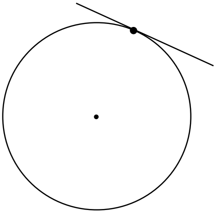
Theorem. *Let (V, F) be an non-euclidean Minkowski space. If $\phi : U_1 \to U_2$ is a conformal map between two domains $U_1 \subset V$ and $U_2 \subset V$, then ϕ is (the restriction of) a similarity, that is the composition of an isometry and a homothety $x \mapsto \mathrm{const} \cdot x$.*

Remark. Theorem generalizes classical result of Liouville for Minkowski metrics: Liouville has shown 1850 that every conformal transformation of the standard $(R^{n\geq 3}, g_{\mathrm{euclidean}})$ is a similarity or a Möbius transformation, i.e., a composition of a similarity and an inversion. We see that for noneuclidean finsler metrics only similarities are allowed.

Theorem answers the question of Matsumoto 2001.

Proof of: Every conformal mapping of a Minkowski space is a similarity

Proof for $dim(M) > 2$**.** I will use: if ϕ is a conformal transformation of F, then it is a conformal transformation of g_F. Moreover, if ϕ is a conformal transformation of F and similarity of g_F, then it is a similarity of F.
We consider the metric g_F. It is Euclidean; w.l.o.g. we think that $g_F = dx_1^2 + ... + dx_n^2$.
Then, by the classical Liouville Theorem 1850, ϕ is as we want or a Möbius transformation, i.e., a composition of of a similarity and an inversion. We thus only need to prove that *a composition of of a similarity an inversion cannot be a conformal map of some non euclidean Minkowski norm on* R^n, which is an easy exercise.



The differential of the inversion at every point of the sphere
is the reflection with respect to the tangent line to the sphere.
The only convex body invariant with respect to all such reflection
is the standard ball

Conformally flat compact Finsler Manifolds

Def. A metric F is conformally flat, if locally, in a neighborhood of every point, it is conformally Minkowski.

Corollary. *Any smooth connected compact conformally flat non Riemannian Finsler manifold is either a Bieberbach manifolds or a Hopf manifolds. In particular, it is finitely covered either by a torus T^n or by $S^{n-1} \times S^1$.*

Proof. Assuming M to be non Riemannian, it follows from Theorem from the previous slide that these changes of coordinates are euclidean similarities.

The manifold M carries therefore a similarity structure.

Compact manifolds with a similarity structure have been topologically classified by N. H. Kuiper (1950) and D. Fried (1980): they are either Bieberbach manifolds (i.e. R^n/Γ, where Γ is some crystallographic group of R^n), or they are Hopf-manifolds i.e. compact quotients of $R^n \setminus \{0\} = S^{n-1} \times R_+$ by a group G which is a semi-direct product of an infinite cyclic group with a finite subgroup of $O(n+1)$.

Finsler spaces with a non trivial self-similarity

A C^1-map $f : (M, F) \to (M', F')$ is a *similarity* if there exists a constant $a > 0$, $a \neq 1$ (called the *dilation constant*) such that $F(f(x), df_x(\xi)) = a \cdot F(x, \xi)$ for all $(x, \xi) \in TM$.

Theorem. Let (M, F) be a forward complete connected C^0-Finsler manifold (the manifold M is of class C^1, the metric F is C^0). If there exists a non isometric self-similarity $f : M \to M$ of class C^1, then (M, F) is a Minkowski space.

Remark. In the case of smooth Finsler manifolds, Theorem is known. A first proof was given by Heil and Laugwitz in 1974, however R. L. Lovas, and J. Szilasi found a gap in the argument and gave a new proof in 2009.

In the proof, I will use:

(Fact 1.) if f is similarity for F, then it is a similarity for g_F;

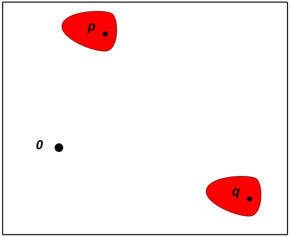
(Fact 2.) A similarity of a forward-complete manifold always has a fixed point, i.e. x such that $f(x) = x$ (since for every x the sequence $x, f(x), f(f(x)), f(f(f(x))), ...$ is forward Cauchy and its limit is a fixed point.)

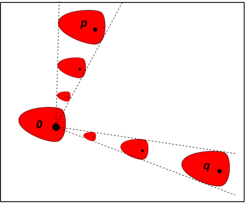
(Fact 3.) A Riemannian metric admitting similarity with a fixed point is flat. Indeed, for smooth metrics this statement reduces to a classical Riemannian argument, since the existence of a non trivial self-similarity in a C^2-Riemannian manifold easily implies that the sectional curvature of that manifold vanishes because otherwise it goes to infinity at the sequence of points $y, f(y), f(f(y)), f(f(f(y)))... \to x$. For nonsmooth metrics, the proof is slightly more tricky and is given in our paper; though it is known to experts in metric geometry.

Proof. By Fact 3, g_F is the standard Euclidean metric, and the similarity f is a similarity of R^n.

We consider two points $p, q \in R^n$. Our goal is to show that the unit ball in q is the parallel translation of the unit ball in p.

Let us first assume for simplicity that f is already a homothety $x \mapsto C \cdot x$ for a constant $1 > C > 0$ (we known that actually it is $\psi \circ \phi$, where ψ is an isometry and ϕ a homothety; I will explain on the next slide that w.l.o.g. $\psi = Id$)





We consider the points $p, f(p) = C \cdot p, f \circ f(p) = C^2 \cdot p$, ... , $\stackrel{\text{converge}}{\longrightarrow} 0$.

The unit ball of the push-forward $f_*^k(F)$ of the metric at the point $f^k(p)$ are as on the picture; therefore, the unit ball of $\frac{1}{C^k} f_*^k(F)$ at the point $f^k(p)$ is the parallel translation of the unit ball at the unit ball at the point p. But the unit ball of $\frac{1}{C^k} f_*^k(F)$ at $f^k(p)$ is the unit ball of F!

Thus, for every k the unit ball of F at $f_k(p)$ is the parallel translation of the unit ball of F at p.

Sending $k \to \infty$, we obtain that the unit ball at $0 = \lim_{k\to\infty} f^k(p)$ is the parallel translation of the unit ball at p. The same is true for q. Then, the unit ball at q is the parallel translation of the unit ball at p

Why we can think that the similarity f is a homothety, and not the composition $\psi \circ \phi$, where $\psi \in O(n)$ is an isometry and ϕ is a homothety

Because the group $O(n)$ is compact. Hence, any sequence of the form $\psi, \psi^2, \psi^3, ...$, has a subsequence converging to Id.
Thus, in the arguments on the previous slide we can take the subsequence $k \to \infty$ such that

$$(\psi \circ \phi)^k \overset{\phi\circ\psi=\psi\circ\phi}{=} \underbrace{\psi^k}_{\sim Id} \circ \phi^k$$

is "almost" ϕ^k, and the proof works.

Proof of Szabo's theorem 1982

Def. A Finsler metric is *Berwald*, if there exists a symmetric affine connection $\Gamma = (\Gamma^i_{jk})$ such that the parallel transport with respect to this connection preserves the function F. In this case, we call the connection Γ the *associated connection*.

Example 1. Riemannian metrics are always Berwald. For them, the associated connection coincides with the Levi-Civita connection.

Def. We say that the metric is *essentially Berwald*, if it Berwald but not Riemannian

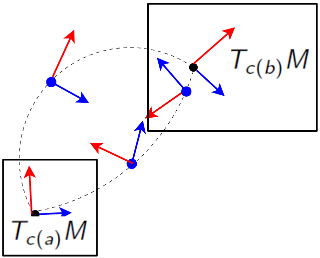
Example 2. Minkowski (nonriemannian) metric is essentially Berwald — with the flat associated connection

Theorem (Szabo 1982) The associated connection of an essentially Berwald finsler metric is Levi-Civita connection of a certain Riemannian metric. Moreover, the Riemannian metric is decomposable or symmetric of rank ≥ 2.

The initial proof of Szabo is complicated. With the help of the Binet-Legendre metric the proof is trivial — see the next slide

Proof of Szabo's theorem

Let F be an essential Berwald finsler metric on M. We consider the averaged Riemannian metric g_F. Let $c : [a, b] \to M$ be a smooth curve and $\tau_c : T_{c(a)}M \to T_{c(b)}M$ be the corresponding parallel transport with respect to the associated connection Γ of the Berwald metric. It is a linear map preserving the finsler unit ball.



Then, it preserves the Binet-Legendre metric g_F implying Γ is the Levi-Civita connection of the metric g_F.

Now consider all possible curves $c : [a, b] \to M$ such that $c(a) = c(b) := p$. For every such curve we have the endomorphism $\tau_c : T_pM \to T_pM$.

The set of such endomorphisms is *holonomy group of* Γ *at the point* p*:*

$H_p := \{\tau_c \mid c : [a, b] \to M \text{ is a smooth curve such that } c(a) = c(b) := p\} \subseteq O(g_{(p)})$.

H_p preserves $g_{(p)}$ and $F(p, \cdot)$ and therefore preserves the function $\tilde{m}(\xi) := \frac{F^2(p,\xi)}{g_{(p)}(\xi,\xi)}$. If the function $\tilde{m}$ is constant (on T_pM), the metric F at the point p is $\mathrm{const} \cdot \sqrt{g}$, i.e., is Riemannian. If the function $\tilde{m}$ is not constant, the holonomy group does not act transitively. Then, by the classical result of Berger(1955)–Simons(1962) the metric g is decomposable, or symmetric of rank ≥ 2, $\square$

One can describe all partially smooth Berwald spaces by the following construction. Choose an arbitrary smooth Riemannian metric g on M and choose an arbitrary Minkowski norm in the tangent space at some fixed point q that is invariant with respect to the holonomy group of g. Now extend this norm to all other tangent spaces by parallel translation with respect to the Levi-Civita connection of g. Since the norm is invariant with respect to the holonomy group, the extension does not depend on the choice of the curve connecting an arbitrary point to q, and is a partially smooth Berwaldian Finsler metric.

In the case the holonomy group of the Riemannian metric acts transitively on T_xM, the resulting Berwald metric is actually Riemannian; but if the Riemannian metric is decomposable or is symmetric of rank ≥ 2, then we obtain nontrivial examples of Berwald metrics.

In particular, if the Riemannian metric is globally symmetric and of rank ≥ 2, then the obtained Berwald metric is also globally symmetric.

Solution of Deng-Hou conjecture symmetric Finsler spaces

The Finsler manifold (M, F) is called *locally symmetric*, if for every point $x \in M$ there exists $r = r(x) > 0$ and an isometry $\tilde{I}_x : B_r(x) \to B_r(x)$ (called the *reflexion* at x) such that $\tilde{I}_x(x) = x$ and $d_x(\tilde{I}_x) = -\mathrm{id} : T_xM \to T_xM$. The largest $r(x)$ satisfying this condition is called the *symmetry radius* at x. The manifold (M, F) is called *globally symmetric* if the reflection $\tilde{I}_x$ can be extended to a global isometry: $\tilde{I}_x : M \to M$.

Theorem. Let (M, F) be a C^2-smooth Finsler manifold. If (M, F) is locally symmetric, then F is Berwald.

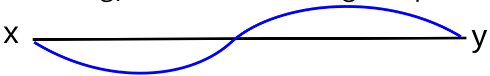
Remark. This theorem answers positively a conjecture of Deng-Hou 2009, where it has been proved for globally symmetric spaces.

Corollary. Every locally symmetric C^2-smooth Finsler manifold is locally isometric to a globally symmetric Finsler space.

Proof under the additional assumtion that the symmetry radius is locally bounded from zero.

The Binet-Legendge metric is a locally symmetric metric. Let us now show that the metrics g_F and F are affinely equivalent, that is, for every arclength parameterised F-geodesic $\tilde{\gamma}$ there exists a nonzero constant c such that $\tilde{\gamma}(c \cdot t)$ is an arclength parameterised g_F-geodesic.

It is sufficient to show that for every sufficiently close points $x, y \in M$ the midpoints of the geodesic segments γ and $\tilde{\gamma}$ in the metrics g_F and F connecting the points x and y coincide.



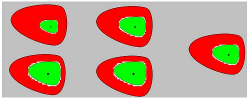
Indeed, if it is true, then the geodesics γ and $\tilde{\gamma}$ coincide on its dense subset implying they coincide.

Take a short F-geodesic $\tilde{\gamma} : [-\tilde{\varepsilon}, \tilde{\varepsilon}] \to M$. Let $\gamma : [-\varepsilon, \varepsilon] \to W$ be the unique shortest g_F-geodesic such that $\gamma(-\varepsilon) = \tilde{\gamma}(-\tilde{\varepsilon})$ and $\gamma(\varepsilon) = \tilde{\gamma}(\tilde{\varepsilon})$. Let $x = \tilde{\gamma}(0)$ be the midpoint of $\tilde{\gamma}$ and let I_x be the g_F reflexion centered at x. Then, $I_x(\gamma(-\varepsilon)) = I_x(\tilde{\gamma}(-\tilde{\varepsilon})) = \tilde{\gamma}(\tilde{\varepsilon}) = \gamma(\varepsilon)$ implying $I_x(\gamma(0)) = \gamma(0)$. By uniqueness of the fixed point of I_x, it follows that $\gamma(0) = x = \tilde{\gamma}(0)$.

Thus, all geodesic segments γ and $\tilde{\gamma}$ coincide after the affine reparameterization By the classical result of Chern-Shen, the metric F is Berwald.

Examples of conformal transformations and Theorem

(i) If $\phi : M \to M$ is an isometry for F, and $\lambda : M \to R_{>0}$ is a function, then ϕ is a conformal transformation of $F_1 := \lambda \cdot F$.



(ii) Let F_m be a Minkowski metric on R^n. Then, the mapping $x \mapsto \mathrm{const} \cdot x$ (for $\mathrm{const} \neq 0$) is a conformal transformation. Moreover, it is also a conformal transformation of $F := \lambda \cdot F_m$. Moreover, if ψ is an isometry of F_m, then $\psi \circ \phi$ is a conformal transformation of every $F := \lambda \cdot F_m$.

(iii) Let g be the standard (Riemannian) metric on the standard sphere S^n. Then, the standard Möbius transformations of S^n are conformal transformations of every metric $F := \lambda \cdot g$.

Theorem (finsler verion of conformal Lichnerowicz conjecture).

That's all: Let ϕ be a conformal transformation of a connected (smooth) finsler manifold $(M^{n\geq 2}, F)$. Then (M, F) and ϕ are as in Examples (i, ii, iii) above.

Even in the Riemannian case, Main Theorem is nontrivial

Corollary (proved before by Alekseevsky 1971, Schoen 1995, (Lelong)-Ferrand 1996) Let ϕ be a conformal transformation of a connected RIEMANNIAN manifold $(M^{n\geq 2}, g)$. Then for a certain $\lambda : M \to R$ one of the following conditions holds

(a) ϕ is an isometry of $\lambda \cdot g$, or

(b) $(M, \lambda \cdot g)$ is (R^n, g_{flat}),

(c) or (S^n, g_{round}).

The story: This statement is known as *conformal Lichnerowicz conjecture* ~ 1960
1970: Obata proved it under the assumption that M is closed.
1971: Alekseevsky proved it for all manifolds; later many mathematicians (for example Yoshimatsu 1976 amd Gutschera 1995) claimed the existence of flaws in the proof
1974–1996: (Lelong)-Ferrand gave another proof using her theory of quasiconformal mappings
1995: Schoen: New proof using completely new ideas
Remark. *In the pseudo-Riemannian case, the analog of Theorem is still an open actively studied conjecture, a counterexample recently announced by Derdzinski 2011 appeared to be wrong.*

Proof

Let ϕ is a conformal transformation of F. Then, it is a conformal transformation of g_F. By the Riemannian version of Main Theorem, the following cases are possible:

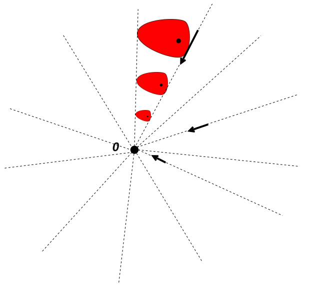
(Trivial case): ϕ is an isometry of a certain $\lambda \cdot g_F$.
Then, it is an isometry of $\lambda^2 \cdot F$.

(Case R^n): After the multiplication F be an appropriate function, g_F is the standard Euclidean metric, and ϕ is a homothety of g_F; we closed this case when we worked with similarities.

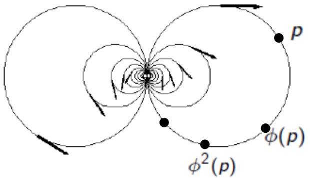
(Case S^n): After the multiplication F by an appropriate function, g_F is the standard metric on the sphere, and ϕ is a möbius transformation of the sphere.

(Case S^n): After the multiplication F be an appropriate function, g_F is the standard "round" (Riemannian) metric on the sphere

Conformal transformation of S^n were described by J. Liouville 1850 in dim n = 2, and by S. Lie 1872. For the sphere, the analog of the picture (a) for the conformal transformation (which are homotheties) of R^n is the picture (b).



Picture (a)



Picture (b)

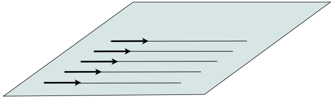
One can generalize of the proof for R^n to the case S^n (the principal observation that sequence of the points $p, \phi(p), \phi^2(p), ...$ converges to a fixed point is also true on the sphere; the analysis is slightly more complicated).

Facts: J. Liouville 1850, S. Lie 1872

Fact 1. Let ϕ be a conformal nonisometric orientation-preserving transformation of the round sphere (S, g_{round}). Then, there exists a one parameter subgroup $(R, +) \subset \mathrm{Conf}(S, g_{\mathrm{round}})$ containing ϕ.

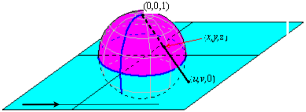
Fakt 2. Any one-parametric subgroup of $(R,+) \subset \mathrm{Conf}(S, g_{\mathrm{round}})$ which is not a subgroup of $Iso(S, g_{\mathrm{round}})$ can be constructed by one of the following ways:

- Way 1. (General case)



  (i) One takes the sliding rotation $\Phi_t : x \to \exp(tA) + tv$, where A is a skew-symmetric matrix such that the vector v is its eigenvector

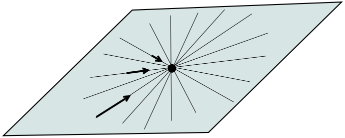
  (ii) and then pullback this transformation to the sphere with the help of stereographic projection



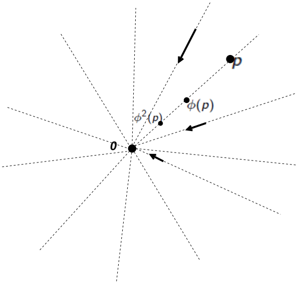
- Way 2. (Special case)

  (i) One takes $\Psi \circ \Phi$, where Φ is a homothety on the plane and Ψ is a rotation on the plane

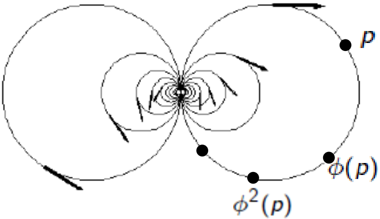
  (ii) and then pullback this transformation to the sphere with the help of stereographic projection



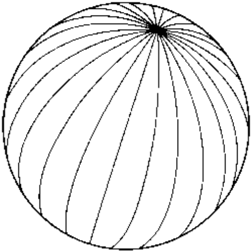
A neighborhood of the pole on the sphere is as on the picture:



Special case



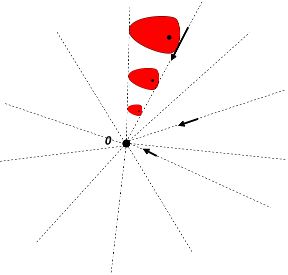
General case



in the special case two points of the sphere have such neighborhood (south and north pols), in the general case only one

The proof for the special case

In the special case, we can repeat the repeat the proof for the case (R^n, g_{flat}):



We obtain that the metric on $S^n - \{\text{south pole}\}$ is conformally diffeomorph to $(R^n, g_{\text{minkowski}})$; the conformal diffeomorphism is the stereographic projection S_{SP}.

Similarly, (because there is no essential difference between south and north pole) we obtain that the metric on $S^n - \{\text{north pole}\}$ is conformally equivalent to $(R^n, g_{\text{minkowski}})$, the conformal diffeomorphism is the stereographic projection S_{NP}.

Then the superposition $S_{SP}^{-1} \circ S_{NP}$ is a conformal diffeomorphism of the metric F restricted to $S^n - \{\text{south pole, north pole}\}$.

From the school geometry we know that the superposition $S_{SP}^{-1} \circ S_{NP}$ is the inversion

$I : R^n \to R^n, \quad I(x_1, ..., x_n) = \left(\frac{x_1}{x_1^2 + ... + x_n^2}, ..., \frac{x_n}{x_1^2 + ... + x_n^2} \right).$

Thus, in order to show that the finsler metric F is Riemannian, it is sufficient to show, that the push-forward $I_* F$ of a Minkowski metric F is conformally equivalent to (another) Minkowski metric if and only if they are Riemannian, which is an easy exercise.

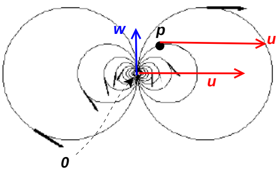
The proof for the general case

We have: the finsler metric F is invariant with respect to ϕ.
The goal: to prove that the metric is conformally equivalent to $g_F = g_{\mathrm{round}}$. We consider the following two functions:

$M(q) := \max_{\eta\in T_qS^n,\ \eta\neq 0} \frac{F(q,\eta)}{g_{(q)}(\eta)} - \min_{\eta\in T_qS^n,\ \eta\neq 0} \frac{F(q,\eta)}{g_{(q)}(\eta)}$.

$M(q) = 0 \iff F(q,\cdot)$ **is proportional to** $g_{(q)}(\cdot)$**.**

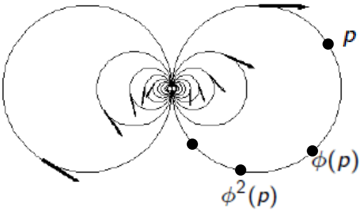
$m(q) := \frac{F(q,v(q))}{g_{(q)}(v(q))}$, where v is the generator of the 1-parameter group of the conformal transformations containing ϕ. **Both functions are invariant with respect to** ϕ**.** Let us first show that the function M is zero at the point 0.



We will show that for every vector u at 0 we have $\frac{F(0,u)}{g_{(0)}(u)} = \frac{F(0,w)}{g_{(0)}(w)}$, where w is as on the picture.
We take a point p very close to 0 such that at this point u is proportional to v with a positive coefficient. Such points exist in arbitrary small neighborhood of 0. We have:

$$\frac{F(p,u)}{g_{(p)}(u)} = \frac{F(p,v)}{g_{(p)}(v)} := m(p) \overset{m(p)\text{ is invariant w.r.t. }\phi}{=} m(0) = \frac{F(0,w)}{g_{(0)}(w)}.$$

Replacing p by a sequence of the points converging to 0 (such that at these points u is proportional to v) we obtain that $\frac{F(0,u)}{g_{(0)}(u)} = \frac{F(0,w)}{g_{(0)}(w)}$ implying $M(q) = 0$ implying $F(0,\cdot) = \lambda \cdot g_{(0)}(\cdot)$.



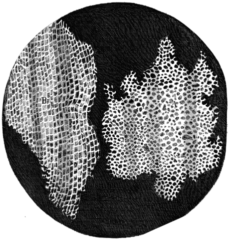
We have:

- $M(0) = 0$,
- M is invariant w.r.t. ϕ and continuous,
- For every point p the sequence $p, \phi(p), \phi^2(p), \ldots$ converges to 0.

Then, $M \equiv 0$ implying the metric F is actually a Riemannian metric, $\square$

What to do next: possible applications in sciences

Finsler geometers always emphasis possible applications of finsler metrics in geometry – certain phenomena in sciences (for examples light prolongation in cristalls or certain processes in organic cells) can be described with the help of finsler metrics.



Picture from Wikipedia common

Picture from Wikipedia common

Picture from chemistry.about.com

Unfortunately, the "standard" finsler methods appeared to be too complicated to be used.

We suggest to replace t´he Finsler metric F by a Riemannian metric g_F, and then to analyze it. Of cause, we loose a lot of information, but get an object which is easier to investigate.

Note that we even do not require that the "unit ball" is smooth and convex.

Thank you for your attention!!!