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Integrable systems and geodesically equivalent metrics

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Video of a similar talk is at
http://www.msri.org/calendar/workshops/WorkshopInfo/446/show_workshop

Two separately developed theories,

- theory of geodesically equivalent metrics and
  Levi-Civita Painlevé Eisenhart Weyl Thomas Douglas Hall Rashevskii Solodovnikov Yamauchi Aminova Venzi Mikes Shandra
- theory of quadratically integrable Hamiltonian systems and separations of variables
  Benenti-systems, L-systems, cofactor systems, quasi-bi-hamiltonian systems, systems admitting special conformal Killing tensor
  Levi-Civita Painlevé Eisenhart Benenti Braden Ibort Magri Marmo Crampin Sarlet Tondo Saunders Cantrijn Kolokoltsov Rastelli Chanu Marciniak Ranada Santander Kiyohara Bolsinov Fomenko Kozlov Waalkens Dullin

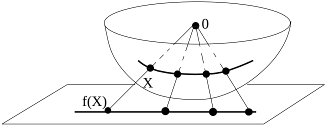
study essentially the same object.
We apply methods of one in the other

Definitions

1. Two metrics (on one manifold) are geodesically equivalent if they have the same unparametrized geodesics (notation: $g \sim \bar{g}$)

2. A vector field is projective w.r.t. a metric, if its flow takes (unparametrized) geodesics to geodesics.

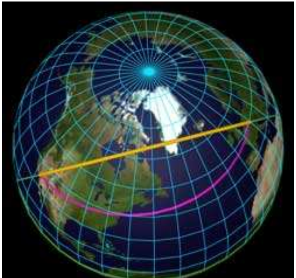
Example of Lagrange 1789



Radial projection $f : S^2 \to \mathbb{R}^2$ takes geodesics of the sphere to geodesics of the plane, because geodesics on sphere/plane are intersection of plains containing 0 with the sphere/plane.

Thus, for every Killing vector field on the plane ist pullback is a projective vector field on the sphere

Motivation of Lagrange:



Example of Beltrami

Beltrami (1865) modified Lagrange example to construct projective vector fields of the sphere:

For every $A \in SL(n+1) \xrightarrow{\text{we construct}}$
$a : S^n \to S^n, a(x) := \frac{A(x)}{|A(x)|}$

- a is a diffeomorphism
- a takes great circles (geodesics) to great circles (geodesics)
- a is an isometry iff $A \in O(n+1)$.

Thus, $Sl(n+1)$ acts by geodesic-preserving transformations on S^n, and its algebra $sl(n+1)$ can be viewed as the algebra of projective vector fields

Examples of Dini 1869 and S. Lie 1882

Theorem (Dini 1869) *The metric*
$(X(x) - Y(y))(dx^2 + dy^2)$
is geodesically equivalent to $\left(\frac{1}{Y(y)} - \frac{1}{X(x)}\right)\left(\frac{dx^2}{X(x)} + \frac{dy^2}{Y(y)}\right)$, *(if they have sense).*

Every two nonproportional geodesically equivalent metrics on the surface have this form in a neighbourhood of almost every point.

Lie 1882 The metric $(x^2 + y^2 + 1)(dx^2 + dy^2)$
admits three projective vector fields

$$\begin{aligned} v_1 &:= (y, -x) \\ v_2 &:= \left((x^2+1)y, y^2x)\right) \\ v_3 &:= \left((x(y^2 - x^2 - 1), y(y^2 + 1 - x^2)\right). \end{aligned}$$

(v_1 is the Killing vector field)

Relation with integrable systems

Given $g, \ \bar g$ on $M^n \xrightarrow{\text{we construct}} L := \bar g^{-1} g \cdot \left(\frac{\det \bar g}{\det g}\right)^{\frac{1}{n+1}}$

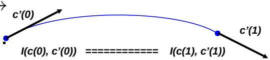
$\xrightarrow{\forall t \in \mathbb{R} \text{ define}} S_t := (L - t \cdot \mathbf{Id})^{-1} \cdot det(L - t \cdot \mathbf{Id})$

$\xrightarrow{\text{consider}} I_t : TM^n \to \mathbb{R}, I_t(\xi) := g(S_t(\xi), \xi).$

Theorem (Topalov, M∼ 1998):

If $g \sim \bar g$, then, $\forall t_1, t_2 \in \mathbb{R}$, the functions I_{t_i} are commuting integrals for the geodesic flow of g (i.e. for the Hamiltonian $H(\xi) := g(\xi, \xi)$)

A function $I : TM \to \mathbb{R}$ is an integral of the geodesic flow of g, if for every geodesic $\gamma : \mathbb{R} \to M$ the function $I(\gamma, \dot\gamma) : \mathbb{R} \to \mathbb{R}$ is constant in t.



The family contains n integrals which are functionally independent almost everywhere, if and only if there exists a point such that the minimal polynomial of $g^{-1}\bar g$ has degree n.

There is no problem to introduce potential energy in the picture:
Bolsinov, M∼ 2003/ Crampin, Scarlet 2003/ Benenti 2004/ Kruglikov, M∼ 2006

There is no problem to quantize the system (replace the integrals by commuting differential operators) (Topalov, M∼ 2001)

Plan:

- Geometric sense of the integrals
- One application of geodesic equivalence to integrable systems: Sinjukov-Topalov hierarchy as a way of constructing integrable systems
- One application of integrable systems to geodesic equivalence: What closed manifolds admit geodesically equivalent Riemannian metrics?
- Combining methods from integrable systems and differential geometry: solution of Lie Problem and of Lichnerowicz-Obata conjecture
- Geodesic equivalence in general relativity

Symplectic nature of these integrals

(Topalov 1997 $\overset{\text{independently}}{---}$ Tabachnikov 1998 , Foulon 1986 , Pollicott)
Consider Hamiltonian systems

(N^{2n}, ω, H, X_H) and $(\bar{N}^{2n}, \bar{\omega}, \bar{H}, X_{\bar{H}})$

and their energy surfaces
$Q^{2n-1} := \{H(x) = h\}$ and $\bar{Q}^{2n-1} := \{\bar{H}(x) = \bar{h}\}$

Suppose there exists $m : Q^{2n-1} \to \bar{Q}^{2n-1}$ such that $\mathrm{d}m(X_H) = \lambda(x) X_{\bar{H}}$

Then we can construct integrals for X_H:
indeed: consider $\sigma := \omega_{|Q}, \bar{\sigma} := \bar{\omega}_{|\bar{Q}}$ and the pull-back $m^*\bar{\sigma}$.

Lemma: *The flow of X_H preserves σ, $m^*\bar{\sigma}$.*

Proof: $L_{X_H} m^*\bar{\sigma} = \imath_{X_H} \mathrm{d}\left[m^*\bar{\sigma}\right] + \mathrm{d}\left[\imath_{X_H} m^*\bar{\sigma}\right] = 0 + 0 = 0.$

Since the forms σ, $m^*\bar{\sigma}$ are preserved by the flow, a function constructed invariantly by using these forms must automatically be an integral. So the coefficients of the characteristic polynomial of one form with respect to the second are integrals.

We can construct many new integrable systems

Given g, $\bar{g}$ let us construct L as above: $L := \bar{g}^{-1} g \cdot \left(\frac{\det \bar{g}}{\det g}\right)^{\frac{1}{n+1}}$

For every $(1,1)$-tensor B, define: $\begin{array}{l} g_B(\xi,\eta) := g(B(\xi),\eta) \\ \bar{g}_B(\xi,\eta) := \bar{g}(B(\xi),\eta) \end{array}$

Theorem (Topalov, Matveev 2001): Assume $g \sim \bar{g}$. Then, for every real-analytic function F, the metrics $g_{F(L)}$ and $\bar{g}_{F(L)}$ are geodesically equivalent (if they have sense.)

The example of Beltrami gives us many pairs of geodesically equivalent metrics on the sphere. If we apply the above Theorem to it for functions $F(x) = x$ and $F(x) = x^2$, we get the metrics of the ellipsoid and of the Poisson spheres.

Moreover, all the systems have the same foliations into Liouville tori. In particular, they have the same singularities of the Liouville foliation. Depending on the type of the eigenvalues of A, all nodegenerate singularities appears here. In particular, monodromy of focus-focus singularity (first observed by Duistermaat 1980, whose topology was described by M$\sim$ 1996, and popularized by R. Cushman and Vu Ngok) is closely related here with Hopf foliation.

Application to topology: motivation:

- Beltrami 1865: La seconda ... generalizzazione ... del nostro problema, vale a dire: riportare i punti di una superficie sopra un' altra superficie in modo che alle linee geodetiche della prima corrispondano linee geodetiche della seconda.

  English Translation: **DESCRIBE** all geodesically equivalent metrics

  - locally was done by Dini (1869) for dim 2, Levi-Civita (1896) for dim n (almost everywhere for Riem. case)
  - I will answer the topologicall question: what closed manifolds can admit nonproportional geodesically equivalent Riemannian metrics
  - I also know the description of all metrics – too long for this talk

What closed manifolds admit geodesically equivalent Riemannian metrics?

Theorem (Matveev 2008) *Suppose M is closed connected. Let Riemannian metrics g and $\bar{g}$ on M be geodesically equivalent and nonproportional. Then, one of the following statements holds:*

- *the manifold is diffeomorphic to a reducible space form:*

$$M \overset{diffeo}{\approx} S^n/_G, \text{ where } G \subset O(n+1) \quad \begin{array}{c} \textit{is discrete} \\ \textit{acts freely on } S^n \\ = G_1 + G_2 \end{array},$$

OR

- *it admits a metric with reducible holonomy group.*

Corollary 1 (Topalov, M$\sim$ 2001): A closed orientable surface admitting nonproportional geodesically equivalent metrics is S^2 or T^2.

Corollary 2 (M$\sim$ 2003): A closed 3-manifold admitting nonproportional geodesically equivalent metrics is $L_{p,q}$ or Seifert manifold with zero Euler number.

Proof of Corollary 1:

(Two geodesically equivalent metrics on the surface if genus ≥ 2 are proportional)

In dimension 2, the integral I_0 is

$$I_0(\xi) := \left(\frac{\det(g)}{\det(\bar{g})}\right)^{\frac{2}{3}} \bar{g}(\xi, \xi).$$

Assume the surface is neither torus nor the sphere. The goal is to show that g and $\bar{g}$ are proportional.

Because of topology, there exists x_0 such that $g_{|x_0} = \bar{g}_{|x_0}$ (after the appropriate scaling of one metric). We assume $g_{|x_1} \neq \bar{g}_{|x_1}$ and find a contradiction.

Explanation of Corollary 2: (*A closed 3-manifold admitting nonproportional geodesically equivalent metrics is $L_{p,q}$ or Seifert manifold with zero Euler number.*)

Assume $dim(M) = 3$

Case 1: There exists a point of the manifold such that that the polynomial $det(g - \lambda \bar{g})$ has 3 different roots. Then, the geodesic flow of g is Liouville-integrable.

Theorem (Kruglikov, Matveev 2006): *Then, the topological entropy of g vanishes.*

(And therefore modulo the Poincare conjecture the manifold can be covered by S^3, $S^2 \times S^1$ or by $S^1 \times S^1 \times S^1$.)

Case 2: At every point the number of roots of the polynomial is ≤ 2.

Then precisely the same trick as in dimension 2 works.

Two small goals: Problems of Lie

Lie 1882:

Problem I: *Es wird verlangt, die Form des Bogenelementes einer jeden Fläche zu bestimmen, deren geodätische Kurven eine infinitesimale Transformation gestatten.*

English translation:

Describe all 2 dim metrics admitting

- **Problem I: one projective vector field**
- **Problem II: many projective vector fields**

Lie 1882:

Problem II: *Man soll die Form des Bogenelementes einer jeden Fläche bestimmen, deren geodätische Kurven mehrere infinitesimale Transformationen gestatten.*

Both problems are local, in a neighborhood of a generic point, pseudoriemannian metrics are allowed

Solution of the 2nd Lie Problem

Theorem (Bryant, Manno, M$\sim$ 2007) *If a two-dimensional metric g of nonconstant curvature has at least 2 projective vector fields such that they are linear independent at the point p, then there exist coordinates x, y in a neighborhood of p such that the metrics are as follows.*

1. $\varepsilon_1 e^{(b+2)\,x} dx^2 + \varepsilon_2 b e^{b\,x} dy^2$, *where* $b \in \mathbb{R} \setminus \{-2, 0, 1\}$ *and* $\varepsilon_i \in \{-1, 1\}$
2. $a\left(\varepsilon_1 \frac{e^{(b+2)\,x} dx^2}{(e^{b\,x}+\varepsilon_2)^2} + \frac{e^{b\,x} dy^2}{e^{b\,x}+\varepsilon_2}\right)$, *where* $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R} \setminus \{-2, 0, 1\}$ *and* $\varepsilon_i \in \{-1, 1\}$
3. $a\left(\frac{e^{2\,x} dx^2}{x^2} + \varepsilon \frac{dy^2}{x}\right)$, *where* $a \in \mathbb{R} \setminus \{0\}$, *and* $\varepsilon \in \{-1, 1\}$
4. $\varepsilon_1 e^{3x} dx^2 + \varepsilon_2 e^{x} dy^2$, *where* $\varepsilon_i \in \{-1, 1\}$,
5. $a\left(\frac{e^{3x} dx^2}{(e^{x}+\varepsilon_2)^2} + \frac{\varepsilon_1 e^{x} dy^2}{(e^{x}+\varepsilon_2)}\right)$, *where* $a \in \mathbb{R} \setminus \{0\}$, $\varepsilon_i \in \{-1, 1\}$,
6. $a\left(\frac{dx^2}{(cx+2x^2+\varepsilon_2)^2 x} + \varepsilon_1 \frac{x dy^2}{cx+2x^2+\varepsilon_2}\right)$, *where* $a > 0$, $\varepsilon_i \in \{-1, 1\}$, $c \in \mathbb{R}$.

Example for 1st Problem of Lie: infinitesimal homotheties

Def. A vector field is an infinitesimal homothety, if its flow multiply the metric by a constant (depending on the time).

Example. $\frac{\partial}{\partial x} = (1,0)$ is an infinitesimal homothety for the metric $e^{\lambda x}\left(E(y)dx^2 + F(y)dxdy + G(y)dy^2\right)$.

Every infinitesimal homothety is a projective vector field.

Def: Two metrics g and $\bar g$ (on one manifold) are geodesically equivalent if they have the same unparametrized geodesics

Of cause, if v is projective w.r.t. g, then it is projective w.r.t. every geodesically equivalent $\bar g$

For explicitly given metric g, it is possible (and relatively easy with the help of Maple) to describe the space of all geodesically equivalent metric $\bar g$:
Shulikovsky 1954 – Kruglikov 2007 – Bryant-Dunajski-Eastwood 2008

Theorem (M∼ 2008): *Let v be a projective vector field on $(M^2, \bar{g})$. Assume the restriction of $\bar{g}$ to no neighborhood has an infinitesimal homothety. Then, there exists a coordinate system in a neighborhood of almost every point such that certain metric g geodesically equivalent to $\bar{g}$ is given by*

1. $ds_g^2 = (X(x) - Y(y))(X_1(x)dx^2 + Y_1(y)dy^2), v = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, where
   - 1.1 $X(x) = \frac{1}{x}, \;\; Y(y) = \frac{1}{y}, X_1(x) = C_1 \cdot \frac{e^{-3x}}{x}, \; Y_1(y) = \frac{e^{-3y}}{y}$.
   - 1.2 $X(x) = \tan(x), \; Y(y) = \tan(y), X_1(x) = C_1 \cdot \frac{e^{-3\lambda x}}{\cos(x)}$, $Y_1(y) = \frac{e^{-3\lambda y}}{\cos(y)}$.
   - 1.3 $X(x) = C_1 \cdot e^{\nu x}, \; Y(y) = e^{\nu y}, X_1(x) = e^{2x}, \; Y_1(y) = \pm e^{2y}$.
2. $ds_g^2 = (Y(y) + x)dxdy, \; v = v_1(x,y)\frac{\partial}{\partial x} + v_2(y)\frac{\partial}{\partial y}$, where
   - 2.1 $Y = e^{\frac{3}{2y}} \cdot \frac{\sqrt{y}}{y-3} + \int_{y_0}^{y} e^{\frac{3}{2\xi}} \cdot \frac{\sqrt{\xi}}{(\xi-3)^2} d\xi$, $v_1 = \frac{y-3}{2}\left(x + \int_{y_0}^{y} e^{\frac{3}{2\xi}} \cdot \frac{\sqrt{\xi}}{(\xi-3)^2} d\xi\right), \; v_2 = y^2$.
   - 2.2 $Y = e^{-\frac{3}{2}\lambda \arctan(y)} \cdot \frac{\sqrt[4]{y^2+1}}{y-3\lambda} + \int_{y_0}^{y} e^{-\frac{3}{2}\lambda \arctan(\xi)} \cdot \frac{\sqrt[4]{\xi^2+1}}{(\xi-3\lambda)^2} d\xi$, $v_1 = \frac{y-3\lambda}{2}\left(x + \int_{y_0}^{y} e^{-\frac{3}{2}\lambda \arctan(\xi)} \cdot \frac{\sqrt[4]{\xi^2+1}}{(\xi-3\lambda)^2} d\xi\right), \; v_2 = y^2 + 1$.
   - 2.3 $Y(y) = y^{\nu}, \; v_1(x,y) = \nu x, \; v_2 = y$.

Repeating: geodesically equivalent 2-dim metrics and the quadratic integrals of the geodesic flow

Def. A function $F : T^*M^2 \to \mathbb{R}, \quad a(x,y)p_x^2 + b(x,y)p_x p_y + c(x,y)p_y^2.$ is called **a quadratic integral** of the geodesic flow of g, if $\{H, F\} = 0$, where $H := \frac{1}{2} g^{ij} p_i p_j : T^*M \to \mathbb{R}$ is the kinetic energy corresponding to the metric.

- Quadratic intergrals are classical (Jacobi 1839, Liouville, Darboux, Eisenhart, ...),
- are useful in physics (Wittaker, Birkhoff,...) (conservative quantities, separation of variables)
- are useful for description of metrics with the same geodesics: as explained today (Dini,Darboux $< - - - - - -- >$ Topalov, M$\sim$ 1998),

  Theorem: $g \sim \bar g$ iff $I : TM^2 \to \mathbb{R}, \quad I(\xi) := \bar g(\xi, \xi) \left(\frac{\det(g)}{\det(\bar g)} \right)^{2/3}$ is an integral of the geodesic flow of g. We identify TM and T^*M by g and obtain a quadratic integral in the above sense.

Why to study quadratic integrals ?

- Because they are classical (Jacobi 1839, Liouville, Darboux, Eisenhart, ...)
- Because they are useful in physics (Wittaker, Birkhoff,...) (conservative quantities, separation of variables)

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- Because they are useful for description of metrics with the same geodesics:

Theorem (Dini,Darboux $< - - - - - -- >$ Topalov, M$\sim$ 1998)
$g \sim \bar g$ iff the function

$$I : TM^2 \to \mathbb{R}, \quad I(\xi) := \bar g(\xi, \xi) \left(\frac{\det(g)}{\det(\bar g)} \right)^{2/3}$$

is an integral of the geodesic flow of g. We identify TM and T^*M by g.

What people studied about quadratic integrals?

- Local description/classification.

  Theorem In a neighborhood of almost every point there exist coordinates x, y such that the metric and the integral are

Liouville case	Complex-Liouville case	Jordan-block case
$(X(x) - Y(y))(dx^2 \pm dy^2)$	$\Im(h)dxdy$	$(1 + xY'(y))\,dxdy$
$\frac{X(x)p_y^2 \pm Y(y)p_x^2}{X(x)-Y(y)}$	$p_x^2 - p_y^2 + 2\frac{\Re(h)}{\Im(h)}p_x p_y$	$p_x^2 - 2\frac{Y(y)}{1+xY'(y)}p_x p_y$
Dini, Liouville 1869	Its complexification	Bolsinov, Pucacco, M$\sim$08

  where $\Re(h)$ and $\Im(h)$ are the real and imaginary parts of a holomorphic function h of the variable $z := x + i \cdot y$.

- Superintegrability (when there exist many quadratic integrals) (Koenigs 1896, Winternitz 1969 , Miller, Kalnins, Kress (nowdays)).

Theorem (Koenigs 1896, Lie 1882 $< - - - >$ **Bryant, Manno, M$\sim$07)**

The space of quadratics integrals is $\geq$ 4-dimensional	$\iff$	the space of projective vector fields is $\geq$ 3-dimensional

We will use it for Lie Problems

The PDE for the function $a(x,y)p_x^2 + b(x,y)p_x p_y + c(x,y)p_y^2$ to be a quadratic integral for the metric $ds_g^2 = e^{f(x,y)}dxdy$ is

$$\begin{cases} a_y &= 0\,, \\ a_x + b_y + 2f_x a + f_y b &= 0\,, \\ b_x + c_y + f_x b + 2f_y c &= 0\,, \\ c_x &= 0\,. \end{cases} \tag{1}$$

We see that the system is overdetermined and of finite type.

PDE for "$\bar{g}$ is geodesically equivalent to a given g"

Is nonlinear . One can linearize it with the help of quadratic integrals.

- First observation:
  If $g \sim \bar{g}$ then $I(\xi) := \bar{g}(\xi,\xi)\left(\frac{\det(g)}{\det(\bar{g})}\right)^{2/3}$ is an integral
- Second observation: I is integral $\iff \{I, H\}_g = 0 \iff$
  $\left\{\left(\frac{\det(g)}{\det(\bar{g})}\right)^{2/3}\bar{g}(\xi,\xi), H(\xi)\right\}_g = 0$
  The last equation is linear in $\bar{g}/\det(\bar{g})^{2/3}$.

The linear equation on $a := \bar{g}/\det(\bar{g})^{2/3}$ (R. Liouville 1889)

$$\begin{cases} \frac{\partial a_{11}}{\partial x} + 2\,K_0 a_{12} - 2/3\,K_1 a_{11} = 0 \\ 2\,\frac{\partial a_{12}}{\partial x} + \frac{\partial a_{11}}{\partial y} + 2\,K_0 a_{22} + 2/3\,K_1 a_{12} - 4/3\,K_2 a_{11} = 0 \\ \frac{\partial a_{22}}{\partial x} + 2\,\frac{\partial a_{12}}{\partial y} + 4/3\,K_1 a_{22} - 2/3\,K_2 a_{12} - 2\,K_3 a_{11} = 0 \\ \frac{\partial a_{22}}{\partial y} + 2/3\,K_2 a_{22} - 2\,K_3 a_{12} = 0 \end{cases}$$

where $K_0 = -\Gamma^2_{11}$, $K_1 := \Gamma^1_{11} - 2\Gamma^2_{12}$, $K_2 = -\Gamma^2_{22} + 2\Gamma^1_{12}$, $K_3 = \Gamma^1_{22}$.

Important observation: The system of PDE is projectively invariant: it does not depend on the connection within connections with the same geodesics.

PDE-background of the observation: K_i determine unparameterized geodesics.

Geometric background of observation: The integrals for g allow to construct integrals for $\bar{g}$, if $g \sim \bar{g}$, because g and $\bar{g}$ have the same geodesics.

How we solved Lie Problems

Let $\mathcal{A}$ be the space of all solutions of the above system (for a given metric g). It is a linear vector space . If $dim(\mathcal{A}) \geq 4$, then the metric admits 3 projective vector fields. If $dim(A) = 1$, all projective vector fields are infinitesimal homotheties.
We assume $dim(\mathcal{A}) = 2$ or $dim(\mathcal{A}) = 3$.

Let v be a projective vector field .
Since the system is projectively invariant , for every $a \in \mathcal{A}$, its Lie derivative $L_v a \in \mathcal{A}$. Thus, $L_v : \mathcal{A} \to \mathcal{A}$ is a linear map . Since $dim(\mathcal{A}) = 2$ or 3 , there exists a two-dimensional subspace $\hat{\mathcal{A}}$ invariant w.r.t. L_v .
In a basis $\sigma_1, \sigma_2 \in \hat{\mathcal{A}}$,
$L_v \binom{\sigma_1}{\sigma_2} = \mathrm{B} \binom{\sigma_1}{\sigma_2}$,
where B is a 2×2 matrix .
By choice of basis we can make
$\mathrm{B} = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$, or $\begin{pmatrix} \alpha & 1 \\ & \alpha \end{pmatrix}$, or $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$.

All together

- There are 3 cases for the matrix B of $L_v : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$.
- For a fixed matrix B, the condition $L_v \binom{\sigma_1}{\sigma_2} = \mathrm{B} \binom{\sigma_1}{\sigma_2}$ is a system of 6 PDE.
- There are 3 cases for the normal form of the pair $(\underbrace{g}_{\text{metric}}, \underbrace{F}_{\text{quadratic integral}})$ (which is essentially the same as (σ_1, σ_2)): Liouville, complex-liouville und Jordan-block cases.
- all together we have $9 = 3 \times 3$ cases to consider.
- In every case the data in the normal form of (σ_1, σ_2), i.e., the functions $X(x), Y(y)$ for Liouville case, h for complex-liouville case, $Y(y)$ for Jordan-block case, have at most two first derivatives.
- Thus, the equation $L_v \binom{\sigma_1}{\sigma_2} = \mathrm{B} \binom{\sigma_1}{\sigma_2}$ contains 6 PDE and 6 highest derivatives of the unknown functions. i.e., is a Frobenius system.
- Such systems can be solved by hands. We did it and solved the Lie Problems.

If somebody did not manage to follow

Solution of Lie problem is based on

1. restriction on the dimension of the space of solutions of Liouville system due to connection with quadratic integrals
2. projective invariance of the Liouville system of equations due to connection with quadratic integrals
3. Jordan normal forms for matrices

   These allowed to simplify the equations: instead of 4 equations of the second order which immediately come out, we construct 6 equations of the first order. The price we paid: there are 9 different systems

4. solving 9 Frobenius systems.

The multidimensional version of Liouville equations

Sinjukov 1962/ Mikes, Berezovski 1989 / Bolsinov, M∼ 2003.

Theorem (Eastwood, M∼ 2007) *g is geodesically equivalent to a connection Γ^i_{jk} iff $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(n+1)}$ is a solution of*

$$\mathrm{Tracefreepartof}\left(\nabla_a \sigma^{bc}\right) = 0.$$

Here $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(n+1)}$ should be understood as an element of $S^2 M \otimes (\Lambda_n)^{2/(n+1)} M$. In particular,

$$\nabla_a \sigma^{bc} = \underbrace{\frac{\partial}{\partial x^a}\sigma^{bc} + \Gamma^b_{ad}\sigma^{dc} + \Gamma^c_{da}\sigma^{bd}}_{\text{Usual covariant derivative}} - \underbrace{\frac{2}{n+1}\Gamma^d_{da}\sigma^{bc}}_{\text{addition coming from volume form}}$$

THE SAME ADVANTAGES AS IN THE 2-DIM CASE:

1. It is a LINEAR PDE-system of the first order.
2. The system does not depend on the choice of Γ within a projective class. (short tensor calculations)
3. For dim2 it is the Liouville system we used to solve the Lie problems.

Application: questions of Schouten 1924: List all complete metrics admitting complete projective vector field

I proved **Lichnerowicz-Obata-Solodovnikov Conjecture (50th):** *Let a complete Riemannian manifold (of dim ≥ 2) admit a complete projective vector field. Then, the manifold is covered by the round sphere, or the vector field is affine.*

It is hard to relax the assumptions

History of L-O-S conjecture:

France (Lichnerowicz)	Japan (Yano, Obata, Tanno)	Soviet Union (Raschewskii)
Couty (1961) proved the conjecture assuming that g is Einstein or Kähler	Yamauchi (1974) proved the conjecture assuming that the scalar curvature is constant	Solodovnikov (1956) proved the conjecture assuming that all objects are real analytic and that $n \geq 3$.

Geodesically equivalent metrics in general relativity

Question (Weyl 1924) Can two Einstein metrics be geodesically equivalent? (but not affine equivalent)?

Partial answers: Couty 1961, Petrov 1964, Barnes 1992, Hall et al 1995–2007, Chen 1997–2002, Kim 2005

Theorem (Kiosak , M$\sim$ 2008): Complete Einstein manifolds are geodesically rigid: *Let $(M^{n\geq 3}, g)$ be a complete pseudoriemannian Einstein manifold.*
Then, every complete $\bar{g}$ geodesically equivalent to g has the same Levi-Civita connection with g.

The assumption that both metrics are complete is important: the 5-dimensional counterexamples are due to Formella 1982 and Mikes 1983.

In the most interesting dimension 4 the result is true also locally:

Theorem (Kiosak , M$\sim$ 2008): $4-$dim Einstein manifolds are geodesically rigid: *Let (M^4, g) be a pseudoriemannian Einstein manifold. Then, every $\bar{g}$ geodesically equivalent to g has the same Levi-Civita connection with g.*

Open questions and possible collaborations

- I did not solve the Lichnerowich conjecture in the pseudo-Riemannian case; joint project with Bolsinov and Kiosak; an interesting and unexpecting application of integrable systems on the semi-simple Lie algebras appeared there
- One can try to apply Topalov-Sinjukov hierarhy
  - as a source of integrable systems
  - as test systems for different ways of quantizing
- one can try to combine geodesic equivalence with additional tensor assumptions

Thanks a lot!!!