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Lie, Beltrami and Schouten problems

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www.minet.uni-jena.de/∼matveev/

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2. A vector field is projective w.r.t. a metric, if its flow takes (unparametrized) geodesics to geodesics.

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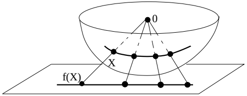

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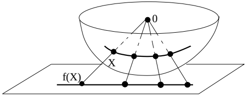

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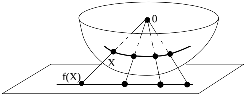

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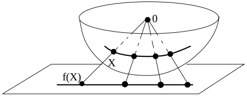

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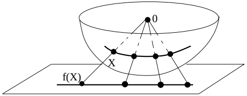

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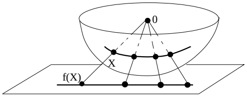

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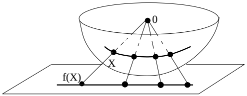

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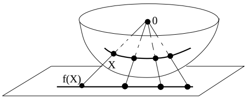

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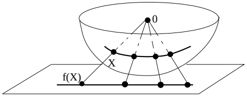

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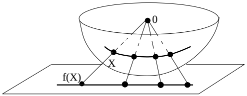

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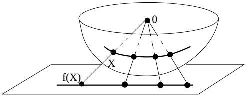

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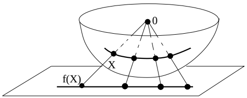

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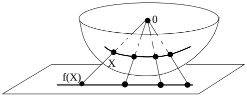

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Pseudo-Riemannian case: Two more series of normal forms: Bolsinov, Pucacco, M$\sim$ 2008

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English translation: Describe all 2 dim metrics admitting at least two projective vector fields

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Repeating Def: *A vector field is projective (w.r.t. a metric), if its flow takes unparameterized geodesics to geodesics.*

Theorem (Bryant, Manno, M$\sim$ 2007) *If a two-dimensional metric g of nonconstant curvature has at least 2 projective vector fields such that they are linear independent at the point p, then there exist coordinates x, y in a neighborhood of p such that the metrics are as follows.*

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1. $\epsilon_1 e^{(b+2)x} dx^2 + \epsilon_2 b e^{bx} dy^2$, *where* $b \in \mathbb{R} \setminus \{-2, 0, 1\}$ *and* $\epsilon_i \in \{-1, 1\}$
2. $a\left(\epsilon_1 \frac{e^{(b+2)x} dx^2}{(e^{bx}+\epsilon_2)^2} + \frac{e^{bx} dy^2}{e^{bx}+\epsilon_2}\right)$, *where* $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R} \setminus \{-2, 0, 1\}$ *and* $\epsilon_i \in \{-1, 1\}$
3. $a\left(\frac{e^{2x} dx^2}{x^2} + \epsilon \frac{dy^2}{x}\right)$, *where* $a \in \mathbb{R} \setminus \{0\}$, *and* $\epsilon \in \{-1, 1\}$
4. $\epsilon_1 e^{3x} dx^2 + \epsilon_2 e^x dy^2$, *where* $\epsilon_i \in \{-1, 1\}$,
5. $a\left(\frac{e^{3x} dx^2}{(e^x+\epsilon_2)^2} + \frac{\epsilon_1 e^x dy^2}{(e^x+\epsilon_2)}\right)$, *where* $a \in \mathbb{R} \setminus \{0\}$, $\epsilon_i \in \{-1, 1\}$,
6. $a\left(\frac{dx^2}{(cx+2x^2+\epsilon_2)^2 x} + \epsilon_1 \frac{x dy^2}{cx+2x^2+\epsilon_2}\right)$, *where* $a > 0$, $\epsilon_i \in \{-1, 1\}$, $c \in \mathbb{R}$.

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Pseudo-Riemannian case was considered: Venzi 85, Barnes 93, Hall 95.

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Lie Problem is equivalent to "Which of them come from a metric"

**Theorem (R. Liouville 1889 – Eastwood/ M∼ 2007–
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Theorem (R. Liouville 1889 – Eastwood/ M$\sim$ 2007– Bryant/Manno/M$\sim$ 2007) *A 2-dimensional metric g has a given projective connection*

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Standard algorithm to understand the dimension of the space of solutions

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Applying the above procedure to the projective connections admitting 2 point symmetries, we solve the Lie 2nd problem:

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Of cause, we also have a hand-written proof.

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where A is a 2×2 matrix.

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- One can try to apply Topalov-Sinjukov hierarhy
  - as a source of integrable systems
  - as test systems for different ways of quantizing
- one can try to combine geodesic equivalence with additional tensor assumptions
  - the metric is Einstein (Joint project with Kiosak)
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Thanks a lot!!!