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**Andreas Cap, Karin Melnick, Galliano Valent and Vladimir
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the participants of this conference are warmly invited

Vladimir S. Matveev
Jena (Germany)

Symmetries of projective and
h-projective structures

www.minet.uni-jena.de/~matveev/

Def. Let $\Gamma = (\Gamma_{jk}^i)$ be a symmetric affine connection on M^n . A **geodesic** $c : I \rightarrow M$, $c : t \mapsto x(t)$ on (M, g) is given in terms of arbitrary parameter t as solution of

$$\frac{d^2 x^a}{dt^2} + \Gamma_{bc}^a \frac{dx^b}{dt} \frac{dx^c}{dt} = \alpha(t) \frac{dx^a}{dt}.$$

Better known version of this formula assumes that the parameter is affine (we denote it by s) and reads

$$\frac{d^2 x^a}{ds^2} + \Gamma_{bc}^a \frac{dx^b}{ds} \frac{dx^c}{ds} = 0.$$

Def. Two connections Γ and $\bar{\Gamma}$ on M are projectively equivalent, if every Γ -geodesic is a $\bar{\Gamma}$ geodesic.

Fact



(Levi-Civita 1896): The condition that Γ and $\bar{\Gamma}$ are projectively equivalent is equivalent to the existence of a one form ϕ on M such that $\bar{\Gamma}_{bc}^a = \Gamma_{bc}^a + \delta_b^a \phi_c + \delta_c^a \phi_b$

Assume M^{2n} carries a complex structure J .

Def. Let $\Gamma = (\Gamma_{jk}^i)$ be a symmetric affine connection on (M^{2n}, J) compatible w.r.t J (i.e., $\nabla J = 0$.)

An **h -planar curve** $c : I \rightarrow M$, $c : t \mapsto x(t)$ on (M, g) is given as solution of

$$\begin{aligned} \frac{d^2 x^a}{dt^2} + \Gamma_{bc}^a \frac{dx^b}{dt} \frac{dx^c}{dt} &= \alpha(t) \frac{dx^a}{dt} + \beta(t) \frac{dx^k}{dt} J_k^a \\ & (= (\alpha(t) + i \cdot \beta(t)) \cdot \frac{dx}{dt}.) \end{aligned}$$

In literature, h -planar curves are also called complex geodesics.

Def. Two connections Γ and $\bar{\Gamma}$ on M are h -projectively equivalent, if every Γ - h -planar curve is a $\bar{\Gamma}$ - h -planar curve.

Fact (T. Otsuki, Y. Tashiro 1954; the proof is a linear algebra:) The condition that Γ and $\bar{\Gamma}$ are h -projectively is equivalent to the existence of a one form ϕ on M such that

$$\bar{\Gamma}_{bc}^a = \Gamma_{bc}^a + \delta_b^a \phi_c + \delta_c^a \phi_b - J_b^a \phi_k J_c^k - J_c^a \phi_k J_b^k.$$

(The connection $\bar{\Gamma}$ is automatically compatible with J .)

Def. A **projective** structure is the equivalence class of symmetric affine connections w.r.t. **projective** equivalence.

Def. A **h -projective** structure on (M, J) is the equivalence class of symmetric affine connections w.r.t. **h -projective** equivalence.

Symmetries of the projective and h -projective structures.

Def. A vector field is a symmetry of a projective structure, if it sends geodesics to geodesics.

Def. A vector field is a symmetry of a h -projective structure, if it sends h -planar curves to h -planar curves.

Easy Theorem 1. A vector field $\frac{\partial}{\partial x^1}$ is a symmetry of a projective structure $[\Gamma]$, iff

$$\Gamma_{jk}^i(x^1, \dots, x^n) = \check{\Gamma}_{jk}^i(\underbrace{x^2, \dots, x^n}_{\text{no } x^1\text{-coord.}}) + \delta_j^i \phi_k(x^1, \dots, x^n) + \delta_k^i \phi_j(x^1, \dots, x^n).$$

Easy Theorem 2. A vector field $\frac{\partial}{\partial x^1}$ is a symmetry of a h -projective structure $[\Gamma]$, iff

$$\begin{aligned} \Gamma_{jk}^i(x^1, \dots, x^n) = & \check{\Gamma}_{jk}^i(\underbrace{x^2, \dots, x^n}_{\text{no } x^1\text{-coord.}}) + \delta_j^i \phi_k(x^1, \dots, x^n) + \delta_k^i \phi_j(x^1, \dots, x^n) \\ & - J_j^i \phi_a(x^1, \dots, x^n) J_k^a - J_k^i \phi_a(x^1, \dots, x^n) J_j^a. \end{aligned}$$

Thus, there is almost no sense to study a symmetry of projective or h -projective structures.

The questions that are still interesting but are not discussed in my talk:

Projective and h -projective structures admitting many symmetries:

For projective structures, the question was actively discussed:

- ▶ the two-dimensional case is classical (Lie, Tresse, Cartan)
- ▶ strong results of Yamaguchi 1979 in all dimensions
- ▶ interesting talk of Gianni Manno in two days.

Most of these results can be generalized for h -projective situation.

What I will discuss

I assume that the projective (or h -projective) structure contains the Levi-Civita connection of a (pseudo-)Riemannian metric.
Let us reformulate the condition of the existence of a metric in the projective and a projective class as a system of PDE.

Theorem (Eastwood-Matveev 2006) The Levi-Civita connection of g lies in a projective class of a connection Γ_{jk}^i if and only if $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(n+1)}$ is a solution of

$$(\nabla_a \sigma^{bc}) - \frac{1}{n+1} (\nabla_i \sigma^{ib} \delta_a^c + \nabla_i \sigma^{ic} \delta_a^b) = 0. \quad (*)$$

Here $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(n+1)}$ should be understood as an element of $S^2 M \otimes (\Lambda_n)^{2/(n+1)} M$. In particular,

$$\nabla_a \sigma^{bc} = \underbrace{\frac{\partial}{\partial x^a} \sigma^{bc} + \Gamma_{ad}^b \sigma^{dc} + \Gamma_{da}^c \sigma^{bd}}_{\text{Usual covariant derivative}} - \underbrace{\frac{2}{n+1} \Gamma_{da}^d \sigma^{bc}}_{\text{addition coming from volume form}}$$

The equations (*) is a system of $\left(\frac{n^2(n+1)}{2} - n \right)$ linear PDEs of the first order on $\frac{n(n+1)}{2}$ unknown components of σ .

Theorem (Matveev-Rosemann 2011/Calderbank 2011) The Levi-Civita connection of g on (M^{2n}, J) lies in a h -projective class of a connection Γ_{jk}^i if and only if $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(2n+2)}$ is a solution of

$$\nabla_a \sigma^{bc} - \frac{1}{2n} (\delta_k^b \nabla_\ell \sigma^{\ell c} + \delta_a^c \nabla_\ell \sigma^{\ell b} + J_a^b J_m^c \nabla_\ell \sigma^{\ell m} + J_a^c J_m^b \nabla_\ell \sigma^{\ell m}) = 0. \quad (**)$$

Here $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(2n+2)}$ should be understood as an element of $S^2 M \otimes (\Lambda_n)^{1/(n+1)} M$. In particular,

$$\nabla_a \sigma^{bc} = \underbrace{\frac{\partial}{\partial x^a} \sigma^{bc} + \Gamma_{ad}^b \sigma^{dc} + \Gamma_{da}^c \sigma^{bd}}_{\text{Usual covariant derivative}} - \underbrace{\frac{1}{n+1} \Gamma_{da}^d \sigma^{bc}}_{\text{addition coming from volume form}}$$

Properties and advantages of the equations $(*)$ (resp. $(**)$)

1. They are linear PDE systems of finite type (close after two prolongations). In the projective case, there exists at most $\frac{(n+1)(n+2)}{2}$ -dimensional space of solutions. In the h -projective case, there exists at most $(n+1)^2$ -dimensional space of solutions.
2. THEY ARE PROJECTIVE (RESP. h -PROJECTIVE) INVARIANT:
THEY DO NOT DEPEND ON THE CHOICE OF A CONNECTION
WITHIN THE PROJECTIVE (RESP. h -PROJECTIVE) CLASS.

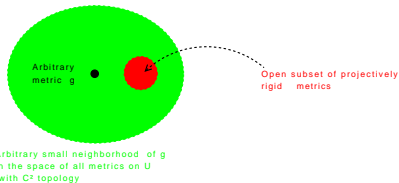
Since the equations are of finite type, it is expected that most projective and h -projective structure does not admit a metric in the projective (resp. h -projective) class: the expectation is true:

Theorem (Matveev arXiv:1101.2069) Almost every 4D metric is
will be explained

projectively rigid (i.e., every metric projectively equivalent to g is proportional to g) .

What we understand under almost every? We consider the standard uniform C^2 –topology: the metric g is ε –close to the metric \bar{g} in this topology, if the components of g and their first and second derivatives are ε –close to that of \bar{g} . ‘Almost every’ in the statement of Theorem above should be understood as

the set of geodesically rigid 4D metrics contains an open everywhere dense (in C^2 –topology) subset.



The result survives for all $n \geq 4$. The result survives in D3, if we replace the uniform C^2 – topology by the uniform C^3 –topology (based on Sinjukov 1954). In D2, the result is again true, if we replace the uniform C^2 – topology by the uniform C^6 –topology (based on nontrivial calculations of Kruglikov 2008 and Bryant–Dunajski–Eastwood 2009).

A similar result is true for h –projective structures (in this case, C^2 –topology is enough in all dimensions).

Main Theorems

Thus, “most” metrics do not admit projective or h -projective symmetry; but still locally there exist tons of examples of metrics admitting projective or h -projective symmetry.

Theorem (“Lichnerowicz-Obata conjecture”, Matveev 2007). Let (M, g) be a compact, connected Riemannian manifold of real dimension $n \geq 2$. If (M, g) cannot be covered by $(S^n, c \cdot g_{\text{round}})$ for some $c > 0$, then $\text{Iso}_0 = \text{Pro}_0$.

Theorem (“Yano-Obata conjecture”, Matveev–Rosemann, arXiv:1103.5613, 2011). Let (M, g, J) be a compact, connected Riemannian Kähler manifold of real dimension $2n \geq 4$. If (M, g, J) is not $(\mathbb{C}P(n), c \cdot g_{\text{FS}}, J_{\text{standard}})$ for some $c > 0$, then $\text{Iso}_0 = \text{HPro}_0$. (Here g_{FS} is the Fubini-Study metric).

Special cases were proved before by French, Japanese and Soviet geometry schools.

Lichnerowicz-Obata conjecture

France (Lichnerowicz)	Japan (Yano, Obata, Tanno)	Soviet Union (Raschewskii)
Couty (1961) proved the conjecture assuming that g is Einstein or Kähler	Yamauchi (1974) proved the conjecture assuming that the scalar curvature is constant	Solodovnikov (1956) proved the conjecture assuming that all objects are real analytic and that $n > 3$.

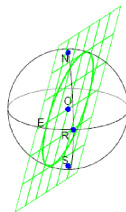
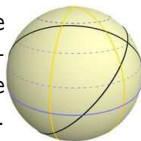
Yano-Obata conjecture

Japan (Obata, Yano)	France (Lichnerowicz)	USSR (Sinjukov)
Yano, Hiramatu 1981: constant scalar curvature	Akbar-Zadeh 1988: Ricci-flat	Mikes 1978: locally symmetric

In S^n and $CP(n)$ the groups of projective resp. h -projective transformations are much bigger than the groups of isometries.

We consider the standard $S^n \subset R^{n+1}$ with the induced metric.

Fact. Geodesics of the sphere are the great circles, that are the intersections of the 2-planes containing the center of the sphere with the sphere.

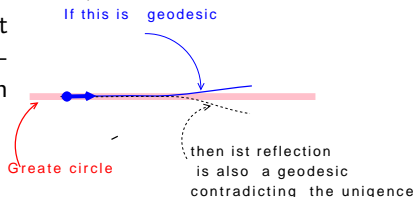


Proof. We consider the reflection with respect to the corresponding 2-plane. It is an isometry of the sphere; its sets of fixed points is the great circle and is totally geodesics. Indeed,

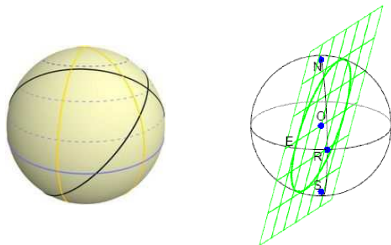
would a geodesic tangent to the great circle leave it, it would give a contradiction with the uniqueness theorem for solutions of ODE

$$\frac{d^2 x^a}{dt^2} + \Gamma_{bc}^a \frac{dx^b}{dt} \frac{dx^c}{dt} = \alpha(t) \frac{dx^a}{dt}$$

(with any fixed α).



Beltrami example



Beltrami (1865) observed:

For every $A \in SL(n+1)$ $\xrightarrow{\text{we construct}}$ $a : S^n \rightarrow S^n$, $a(x) := \frac{A(x)}{|A(x)|}$

- ▶ a is a diffeomorphism
- ▶ a takes great circles (geodesics) to great circles (geodesics)
- ▶ a is an isometry iff $A \in O(n+1)$.

Thus, $Sl(n+1)$ acts by projective transformations on S^n . We see that $Proj_0$ is bigger than $Iso_0 = SO(n+1)$

In $CP(n)$, the situation is essentially the same

Fact. A curve on $(CP(n), g_{FS}, J)$ is h -planar, if and only if it lies on a projective line.

Proof. For every projective line, there exists an isometry of $CP(n)$ whose space of fixed points is our projective line. Then, every h -planar curve whose tangent vector is tangent to a projective line stays on the projective line by the uniqueness of the solutions of a system of ODE $\frac{d^2 x^a}{dt^2} + \Gamma_{bc}^a \frac{dx^b}{dt} \frac{dx^c}{dt} = \alpha(p) \frac{dx^a}{dt} + \beta(p) \frac{dx^k}{dt} J_k^a$ (for fixed α, β).

From the other side, since the tangent space $TL \subset TCP(n)$ of every projective line is J -invariant, every curve lying on the projective line is h -planar (because of L is two-dimensional, so every vector is a linear combination of $\dot{\gamma}$ and $J(\dot{\gamma})$). □

Corollary (h -projective analog of Beltrami Example). The group of h -projective transformations is $SL(n+1, \mathbb{C})$ and is much bigger than the group of isometries which is $SU(n+1)$.

Question. The main results do not really require projective and h -projective structures. Why we introduced them?

Answer. Why we need them in the proof.

Plan of the proof.

Setup. Our manifold is closed and Riemannian. The projective (resp. h -projective) structure of the metric admits a infinitesimal symmetry, i.e., a vector field v whose flow preserves the projective (resp. h -projective) structure. Our goal is to show that this vector field is a Killing vector field unless g has constant sectional curvature (resp. constant holomorphic sectional curvature).

Def. The degree of the mobility of the projective (resp. h -projective) structure $[\Gamma]$ is the dimension of the space of solutions of the equation $(*)$ (resp. $(**)$).

The proof depends on the degree of mobility of the projective (resp. h -projective) structure.

If the degree of mobility of the projective structure is 1, every two projective (h -projectively, resp.) metrics are proportional. Then, a projective (h -projective) vector field is a infinitesimal homothety. Since our manifold is closed, every homothety is isometry so our vector field is a Killing.

If the degree of mobility is at least three, then the following (nontrivial) theorem works.

Theorem (Proved in [Matveev 2003/Kiosak-Matveev 2010/Matveev-Mounoud 2011 for projective structures](#); [Fedorova-Kiosak-Matveev-Rosemann for \$h\$ -projective structures](#)).

If the degree of mobility ≥ 3 , the LO and YO conjectures hold (even in the pseudo-Riemannian case).

Remark. The methods of proof are very different from the methods of my talk.

Thus, the only interesting case in when the degree of mobility is 2

The case degree of mobility =2

Let Sol be the space of solutions of the equation (*) or (**); it is a two-dimensional vector space. Let v is a projective (resp. h -projective) vector field.

Important observation. $L_v : Sol \rightarrow Sol$, where L_v is the Lie derivative.

Proof. The equations (*) (resp. (**)) are projective (resp. h -projective) invariant. Then, in a coordinate system such that $v = \frac{\partial}{\partial x^1}$ the coefficients in the equations do not depend on the x^1 -coordinate. Then, for every solution σ^{ij} its x^1 -derivative $\frac{\partial}{\partial x^1} \sigma^{ij}$, which is precisely the Lie derivative, is also a solution, □

Thus, in a certain basis $\sigma, \bar{\sigma}$ the Lie derivative is given by the following matrices (where $\lambda, \mu \in \mathbb{R}$):

$$\begin{bmatrix} L_v \sigma & = \lambda \sigma \\ L_v \bar{\sigma} & = \mu \bar{\sigma} \end{bmatrix} \quad \begin{bmatrix} L_v \sigma & = \lambda \sigma & + \mu \bar{\sigma} \\ L_v \bar{\sigma} & = -\mu \sigma & + \lambda \bar{\sigma} \end{bmatrix} \quad \begin{bmatrix} L_v \sigma & = \lambda \sigma & + \bar{\sigma} \\ L_v \bar{\sigma} & = & \lambda \bar{\sigma} \end{bmatrix}$$

We obtained that the derivatives of $\sigma, \bar{\sigma}$ along the flow of v are given by

$$\begin{bmatrix} L_v \sigma &= \lambda \sigma \\ L_v \bar{\sigma} &= \mu \bar{\sigma} \end{bmatrix} \quad \begin{bmatrix} L_v \sigma &= \lambda \sigma & + \mu \bar{\sigma} \\ L_v \bar{\sigma} &= -\mu \sigma & + \lambda \bar{\sigma} \end{bmatrix} \quad \begin{bmatrix} L_v \sigma &= \lambda \sigma & + \bar{\sigma} \\ L_v \bar{\sigma} &= & \lambda \bar{\sigma} \end{bmatrix}.$$

Thus, the evolution of the solutions along the flow ϕ_t of v are given by the matrices

$$\begin{bmatrix} \phi_t^* \sigma &= e^{\lambda t} \sigma \\ \phi_t^* \bar{\sigma} &= e^{\mu t} \bar{\sigma} \end{bmatrix}$$

$$\begin{bmatrix} \phi_t^* \sigma &= e^{\lambda t} \cos(\mu t) \sigma & + e^{\lambda t} \sin(\mu t) \bar{\sigma} \\ \phi_t^* \bar{\sigma} &= -e^{\lambda t} \sin(\mu t) \sigma & + e^{\lambda t} \cos(\mu t) \bar{\sigma} \end{bmatrix}$$

$$\begin{bmatrix} \phi_t^* \sigma &= e^{\lambda t} \sigma & + t e^{\lambda t} \bar{\sigma} \\ \phi_t^* \bar{\sigma} &= & e^{\lambda t} \bar{\sigma} \end{bmatrix}.$$

We will consider all these three cases separately.

The simplest case is when the evolution is given by

$$\begin{bmatrix} \phi_t^* \sigma &= e^{\lambda t} \cos(\mu t) \sigma &+ e^{\lambda t} \sin(\mu t) \bar{\sigma} \\ \phi_t^* \bar{\sigma} &= -e^{\lambda t} \sin(\mu t) \sigma &+ e^{\lambda t} \cos(\mu t) \bar{\sigma} \end{bmatrix}.$$

Suppose our metrics correspond to the element $a\sigma + b\bar{\sigma}$.
Its evolution is given by

$$\begin{aligned} \phi_t^*(a\sigma + b\bar{\sigma}) &= a(e^{\lambda t} \cos(\mu t) \sigma + e^{\lambda t} \sin(\mu t) \bar{\sigma}) \\ &\quad + b(-e^{\lambda t} \sin(\mu t) \sigma + e^{\lambda t} \cos(\mu t) \bar{\sigma}) \\ &= e^{\lambda t} \sqrt{a^2 + b^2} (\cos(\mu t + \alpha) \sigma + \sin(\mu t + \alpha) \bar{\sigma}), \end{aligned}$$

where $\alpha = \arccos(a/(\sqrt{a^2 + b^2}))$.

Now, we use that the metric is Riemannian. Then, for any point x there exists a basis in $T_x M$ such that σ and $\bar{\sigma}$ are given by diagonal matrices: $\sigma = \text{diag}(s_1, s_2, \dots)$ and $\bar{\sigma} = \text{diag}(\bar{s}_1, \bar{s}_2, \dots)$.

Then, $\phi_t^*(a\sigma + b\bar{\sigma})$ at this point is also diagonal with the i th element $e^{\lambda t} \sqrt{a^2 + b^2} (\cos(\mu t + \alpha) s_i + \sin(\mu t + \alpha) \bar{s}_i)$.

Clearly, for a certain t we have that $\phi_t^*(a\sigma + b\bar{\sigma})$ is degenerate which contradicts the assumption, □

The proof is is similar when the evolution is given by

$$\begin{bmatrix} \phi_t^* \sigma & = e^{\lambda t} \sigma & + t e^{\lambda t} \bar{\sigma} \\ \phi_t^* \bar{\sigma} & = & e^{\lambda t} \bar{\sigma} \end{bmatrix}.$$

We again suppose that our metrics correspond to the element $a\sigma + b\bar{\sigma}$. Its evolution is given by

$$\begin{aligned} \phi_t^*(a\sigma + b\bar{\sigma}) &= a(e^{\lambda t}\sigma + e^{\lambda t}t\bar{\sigma}) + b(e^{\lambda t}\bar{\sigma}) \\ &= e^{\lambda t}(a\sigma + (b + at)\bar{\sigma}). \end{aligned}$$

We again see that unless $a \neq 0$ there exists t such that $\phi_t^*(a\sigma + b\bar{\sigma})$ is degenerate which contradicts the assumption. Now, if $a = 0$, then g corresponds to $\bar{\sigma}$ and v is its Killing vector field, □

The most complicated case is when the evolution is given by the matrix

$$\begin{bmatrix} \phi_t^* \sigma & = e^{\lambda t} \sigma \\ \phi_t^* \bar{\sigma} & = e^{\mu t} \bar{\sigma} \end{bmatrix}. \quad (1)$$

The complete proof needs technical nontrivial details, I will explain the effect and list the technical problems that should be solved.

We take two elements of Sol corresponding to two Riemannian metrics; they are linear combinations of the basis solutions σ and $\bar{\sigma}$ and have the form $a\sigma + b\bar{\sigma}$, $c\sigma + d\bar{\sigma}$.

We consider $A := (a\sigma + b\bar{\sigma})(c\sigma + d\bar{\sigma})^{-1}$, this is an one-one tensor whose all eigenvalues are positive. We take an arbitrary point of M and consider a basis where the metrics are diagonal; in this basis σ and $\bar{\sigma}$ are diagonal as well so that A is also diagonal. Then,

$$\phi_t^* A := (ae^{\lambda t} \sigma + be^{\mu t} \bar{\sigma})(ce^{\lambda t} \sigma + de^{\mu t} \bar{\sigma})^{-1}.$$

We take an arbitrary point x of M and consider a basis where the metrics are diagonal; in this basis σ and $\bar{\sigma}$ are diagonal as well:

$$\sigma = \text{diag}(s_1, \dots, s_{2n}), \quad \bar{\sigma} = \text{diag}(\bar{s}_1, \dots, \bar{s}_{2n}).$$

Then, the eigenvalues $a_1(t), \dots, a_{2n}(t)$ of $\phi_t^* A$ at the point x are given by

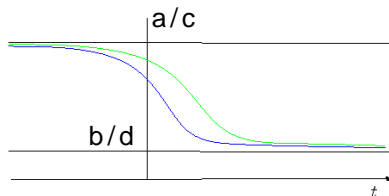
$$a_i := \frac{ae^{\lambda t} s_i + be^{\mu t} \bar{s}_i}{ce^{\lambda t} s_i + de^{\mu t} \bar{s}_i}.$$

Let us consider the functions

$$a_i(t) = \frac{ae^{\lambda t}s_i + be^{\mu t}\bar{s}_i}{ce^{\lambda t}s_i + de^{\mu t}\bar{s}_i}$$

in details.

Easy “first semester calculus exercise” shows, then the function qualitatively look as on the picture.



Suppose for simplicity that A does not have constant eigenvalue equal to b/d . Let us now exchange b by 0. We obtain a bilinear form such that it is a Riemannian metrics at the points such that $a_i \neq b/d$. The vector field v is a homothety of this metric. Then, the metric is flat implying (by the classical result of Beltrami-Schur and by its natural generalization by Otsuki-Tashiro to the h -projective structures) that the metric g has constant curvature.

Since they are eigenvalues of $(1,1)$ -tensor connecting two metrics,

- ▶ they must be positive and
- ▶ they must be bounded.

What to do if some constant eigenvalues are equal to b/d ?

One can generalize the above proof for this setting.

The way to do it is to prove that the product of all eigenspaces all other eigenvalues is a J -invariant totally geodesics submanifolds (will be explained).

The flow of g preserves the distribution $Eigenspace_A(b/d)$ and its orthogonal complement (in any metric in the h -projective class).

After doing this, by the argument above the restriction of the metrics to them has constant curvature and with by additional work one can show that the initial metrics are projectively or h -projectively flat.

Proof that for constant eigenvalues of A the distribution orthogonal to the eigenspace is totally geodesic

Recall that $A = (a\sigma + b\bar{\sigma})(c\sigma + d\bar{\sigma})^{-1}$, where $(a\sigma + b\bar{\sigma})$ and $(c\sigma + d\bar{\sigma})$ are two solutions corresponding to two h -projectively equivalent metrics:

$$(a\sigma + b\bar{\sigma}) = g^{-1} \otimes (Vol_g)^{2/(n+1)} = g^{ij} \det(g)^{1/(n+1)}.$$
$$(c\sigma + d\bar{\sigma}) = \bar{g} \otimes (Vol_{\bar{g}})^{-2/(n+1)} = \bar{g}_{ij} \frac{1}{\det(\bar{g})^{1/(n+1)}}.$$

Integrals of the geodesic flow

Theorem (Matveev–Topalov 1998) In the projective case, the functions $F_t : TM \rightarrow \mathbb{R}$, $t \in \mathbb{R}$,

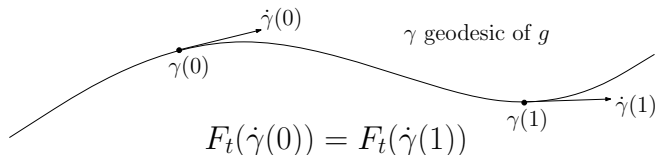
$$F_t(\xi) = \det(A - t\text{Id}) g((A - t\text{Id})^{-1}\xi, \xi)$$

are a family of commuting integrals for the geodesic flow of g .

Theorem (Topalov 2003) In the h -projective case, the functions $F_t : TM \rightarrow \mathbb{R}$, $t \in \mathbb{R}$,

$$F_t(\xi) = \sqrt{\det(A - t\text{Id})} g((A - t\text{Id})^{-1}\xi, \xi)$$

are a family of commuting integrals for the geodesic flow of g .



Remark. Integrals quadratic in velocities are essentially the same as Killing $(0, 2)$ -tensors, and the Killing equations is projectively or h -projectively invariant (if we consider them on the weighted tensors).

t -derivative of the integrals.

Then, for every t_0 and every k , the functions

$$F_{t_0}^{(k)} : TM \rightarrow \mathbb{R}, \quad F_{t_0}^{(k)}(\xi) = \frac{d^k}{dt^k} F_t(\xi)|_{t=t_0}$$

are also integrals.

Linear algebra: calculation of $F_{b/d}^{(k)}$ is the “best” basis

We will work in the more complicated h -projective case. The best basis $(e_1, \dots, e_n, \bar{e}_1, \dots, \bar{e}_n)$ at $x \in M$ is defined by the property

$g = \text{diag}(1, \dots, 1)$; $A = \text{diag}(a_1, \dots, a_n, a_1, \dots, a_n)$; $J(e_i) = \bar{e}_i$. The components of ξ in this basis will be denoted by $(\xi_1, \dots, \xi_n, \bar{\xi}_1, \dots, \bar{\xi}_n)$.

By direct calculation, we see that in this basis $F_t(\xi)$ in the h -projective case reads

$$\begin{aligned} F_t(\xi) &= \sum_{i=1}^n \left[(\xi_i^2 + \bar{\xi}_i^2) \prod_{j=1, j \neq i}^n (a_j - t) \right] \\ &= (a_2 - t) \dots (a_n - t) (\xi_1^2 + \bar{\xi}_1^2) + \dots + (a_1 - t) \dots (a_{n-1} - t) (\xi_n^2 + \bar{\xi}_n^2). \end{aligned}$$

Recall that one of the eigenvalues, say a_1 , is the “minimal” b/d ; we think that it has multiplicity $2k + 2$ so $a_1 = a_2 = \dots = a_{k+1} = b/d$.

We see by direct calculations that $F_{b/d}^{(k)}$ is given (up to a constant) by

$$F_{b/d}^{(k)}(\xi) = (a_{k+2} - t) \cdot \dots \cdot (a_n - t) (\xi_1^2 + \bar{\xi}_1^2 + \dots + \xi_{k+1}^2 + \bar{\xi}_{k+1}^2).$$

We see that $F_{b/d}^{(k)}(\xi) = 0 \iff \xi \in \text{Eigenspace}_A(b/d)^\perp$. Then, the distribution $\text{Eigenspace}_A(b/d)$ is totally geodesic, because the property of $F_{b/d}^{(k)}(\dot{\gamma}) = 0$ to be zero is preserved along every geodesic γ .

Summary and open question

- ▶ I gave a proof of the projective Lichnerowicz-Obata and h -projective Yano-Obata conjecture.
- ▶ What stays under the rock are
 - ▶ The case $\dim(Sol) \geq 3$.
 - ▶ This case was proved by completely different methods.
 - ▶ The regularity questions (for example, what happens near the points where the dimension of $Eigenspace_A(b/d)$ changes?)
 - ▶ This is a very technical, and essentially Riemannian part
- ▶ Open problem:
 - ▶ Pseudo-Riemannian case
- ▶ New research direction
 - ▶ Take a statement from the projective geometry and try to generalize it to the h -projective one.

Thank you for your attention