Beltrami problem, Lichnerowicz-Obata conjecture and applications of integrable systems in differential geometry

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Abstract

This is a short survey on new results in the theory of geodesically equivalent metrics. We emphasise the role of integrable systems in obtaining this results.

1 Introduction

Let (M^n, g) be a Riemannian manifold of dimension $n \geq 2$.

Definition 1. *A metric $\bar{g}$ on M^n is called* **geodesically equivalent** *(***affine equivalent***, respectively) to g, if every geodesic of g, considered as an unparameterised curve (as a parameterised curve, respectively), is a geodesic of $\bar{g}$.*

A diffeomorphism $F : M^n \to M^n$ is called **a projective transformation** *(***affine transformation***, respectively) of (M^n, g), if the pull-back metric F^*g is geodesically (affine, respectively) equivalent to g.*

1.1 Questions

The notions of geodesically equivalent metrics and projective transformations are natural, and therefore classical. Here are the most famous problems:

> **Problem 1:** To describe all possible geodesically equivalent metrics.
>
> This problem was formulated by Beltrami [5] in 1865 (for dimension 2).
>
> **Problem 2:** *For what (M^n, g) the group of projective transformations is bigger than the group of isometries? (or the group of affine transformations?)*

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This problem was formulated by Lie [46] in 1882 (for dimension 2 and locally) and Schouten [74] in 1924 (for all dimensions and assuming that the manifold is closed or complete):

The goal of the present paper is to comment these problems under the additional assumption that the manifold is closed. Under this assumption, the Problem 2 was recently solved; we will give the answer in Section 3.1.

The situation with Problem 1 is more complicated. Problem 1 naturally splits into two subproblems.

Geodesic Rigidity Problem: On what manifolds every two geodesically equivalent metrics are proportional.

Geometric Beltrami Problem: Given a closed manifold, to describe all nonproportional geodesically equivalent metrics on it.

Note that the answer on the Geodesic Rigidity Problem is not empty: there do exist *geodesically rigid* manifolds, in the sense that the metrics on such manifolds are determined by their (unparameterised) geodesics. For example, in dimension 2 the surfaces of genus ≥ 2 are geodesically rigid [50].

It is also expected that the answer on Geometric Beltrami Problem depends on the topology of the manifold. It can be seen in dimension 2 already: the description of geodesically equivalent metrics on the torus is different from the description of geodesically equivalent metrics on the sphere, see [9, 10, 11] for details. If we pass to bigger dimensions, the difference increase. For example, in dimension 3, on certain lens spaces geodesically equivalent nonproportional metrics must admit two-dimensional group of isometries (although on the 3-sphere generic geodesically equivalent metrics do not admit infinitesimal symmetries).

Starting from dimension 4, certain manifolds are in a sense half-rigid, i.e.

- geodesically equivalent metrics on them are immediately affine equivalent,
- and they can carry affine equivalent nonproportional metrics.

The simplest example of a half-rigid manifold is the direct product of two surfaces of genus ≥ 2.

1.2 Plan of the paper

In Section 2 we give a few examples of geodesically equivalent metrics and projective transformations.

In Section 3.1 we give an answer on Problem 2 assuming that the manifold is complete.

In Section 3.2 we conjecture the answer on the Geodesic Rigidity Problem and give a list of results that confirm our conjecture.

Concerning the Geometric Beltrami Problem, we will describe in Section 3.4 all geodesically equivalent metrics on the torus (under the additional assumption that the metrics are strictly-non-proportional at least at one point of the torus).

Recall that two metrics g and $\bar{g}$ given on one manifold M^n are called *strictly-non-proportional* at $x \in M^n$, if the (1,1)-tensor $\bar{g}^{-1}\, g$ has n different eigenvalues at x.

The new methods that allowed us to prove all these results came from the theory of integrable systems. In Section 4 we explain the relation between integrable systems and geodesically equivalent metrics (Theorem 9 there).

In Section 4.1 we describe the geometric origin of the integrals. It will be clear that the integrals could be generalised for pseudo-Riemannian metrics (actually, it was already done in [12, 85]) and for Finsler metrics.

In Section 4.2 we show probably the most important application of the integrals: with the help of integrals we show that if two geodesically equivalent metrics are not proportional somewhere, then they are nonproportional at almost every point.

In Section 5 we describe the main ideas of the proofs of the results from Section 3.

1.3 Local versions of Problems 1,2 and generalisations

Geodesic rigidity problem as we formulated it above has sense only if the manifold is closed, since it is expected that open manifolds are never geodesically rigid. This expectation is a theorem in dimensions 2,3.

But still, if we require that the metrics satisfy additional assumptions, the geodesic rigidity problem can be formulated also for open manifolds, and even locally. These questions were a very popular in 20^{th}–40^{th} (see, for example, Eisenhart [25], Weyl [90], Thomas [81], Douglas [24]) and in 50^{th}–80^{th} (see, for example, Sinjukov [73], Venzi [89], Mikes [65]).

An interesting generalisation of Problem 2 is to describe all metrics admitting sufficiently big group of projective transformations. This problem also has sense locally. For dimensions ≥ 3 it was implicitly solved by Solodovnikov [75]. The methods of dimension ≥ 3 do not work in dimension 2. In dimension 2, the problem was actively studied by Cartan [18]. See also [2, 42, 72].

One can generalise Problems 1,2 for pseudo-Riemannian metrics (see, for example, Weyl [91], Petrov [71], Aminova [1], Barnes [4], Hall [29, 30, 31]). We refer to survey [3] for details.

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2 Examples

The goal of this section it to give examples of geodesically equivalent metrics and projective transformations. The examples 1,2, 6 are relatively simple. The other examples are less simple, and they serve the following two goals:

- to show that the set of geodesically equivalent metrics and projective transformations is quite big,
- to show that all assumptions in Theorem 3 are important.

Example 1: the spaces of constant curvature are locally geodesically equivalent

This example is due to Beltrami [5]. He constructed a diffeomorphism from the half-sphere to a plane that sends the geodesics of the round metric (i.e., the segments of the great circles) to the geodesics of the Euclidean metrics (i.e., the straight lines). This example can easily be generalised for all dimensions and for hyperbolic spaces.

The half-sphere is

$$S^2 := \{(x, y, z) \in \mathbb{R}^3 : \quad x^2 + y^2 + z^2 = 1, \ z < 0\},$$

and the plane is $E^2 := \{(x, y, z) \in \mathbb{R}^3 : \quad z = -1\}$. The mapping is the stereographic projection from the point $O = (0, 0, 0)$. Recall that the stereographic projection f takes a point X of the half-sphere to the intersection of the straight line that goes through O and X with the plane, see Figure 1.

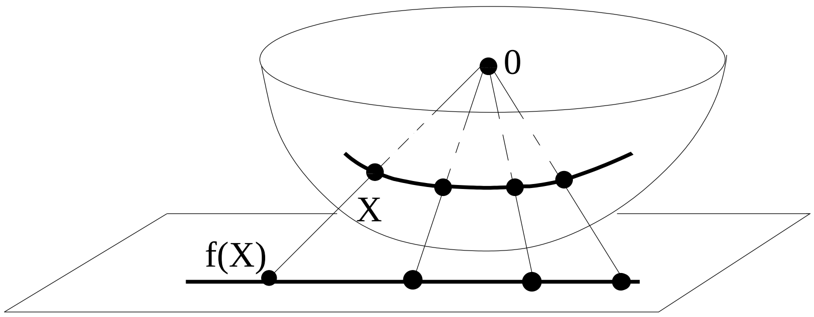

Figure 1: Stereographic projection sends great circles to straight lines

Example 2: the group of projective transformations of the standard sphere is bigger than the group of isometries

One can generalise Example 1 to obtain (nonisometric) projective transformation of the standard sphere. For dimension 2, it was done by Beltrami [5]; the idea of Beltrami works in all dimensions.

Consider the sphere

$$S^n \stackrel{\text{def}}{=} \{(x_1, x_2, ..., x_{n+1}) \in R^{n+1} : \ x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1\}$$

with the restriction of the Euclidean metric. Next, consider the mapping $a : S^n \to S^n$ given by $a : v \mapsto \frac{A(v)}{\|A(v)\|}$, where A is an arbitrary non-degenerate linear transformation of R^{n+1}.

The mapping is clearly a diffeomorphism taking geodesics to geodesics. Indeed, the geodesics of g are great circles (the intersections of planes that go through the origin with the sphere). Since A is linear, it takes planes to planes. Since the normalisation $w \mapsto \frac{w}{\|w\|}$ takes punctured planes to their intersections with the sphere, a takes great circles to great circles. Thus, a is a projective transformation. Evidently, if A is not proportional to an orthogonal transformation, a is not isometry.

Example 3: Levi-Civita's Theorem

The metrics in the above examples have constant sectional curvature. The next example/theorem shows that it is not always the case. This is a simplified version of the famous Levi-Civita's Theorem [45]. For dimension 2, it was proved by Dini in [23].

Theorem 1 (Levi-Civita [45]). *Consider two Riemannian metrics on an open subset $U^n \subset M^n$. Suppose the metrics are strictly-non-proportional at every point $x \in U^n$.*

Then the metrics are geodesically equivalent on U^n, if and only if for every point $x \in U^n$ there exist coordinates $x_1, x_2, ..., x_n$ in some neighbourhood of x such that in these coordinates the metrics have the following model form:

$$ds^2_{g_{model}} = \Pi_1 dx_1^2 + \Pi_2 dx_2^2 + \cdots + \Pi_n dx_n^2, \tag{1}$$

$$ds^2_{\bar{g}_{model}} = \rho_1 \Pi_1 dx_1^2 + \rho_2 \Pi_2 dx_2^2 + \cdots + \rho_n \Pi_n dx_n^2, \tag{2}$$

where the functions Π_i and ρ_i are given by

$$\Pi_i \stackrel{\text{def}}{=} (\lambda_i - \lambda_1)(\lambda_i - \lambda_2) \cdots (\lambda_i - \lambda_{i-1})(\lambda_{i+1} - \lambda_i) \cdots (\lambda_n - \lambda_i), \tag{3}$$

$$\rho_i \stackrel{\text{def}}{=} \frac{1}{\lambda_1 \lambda_2 ... \lambda_n} \frac{1}{\lambda_i}.$$

where, for any i, the function λ_i is a smooth function of the variable x_i.

Example 4: a closed manifold with discrete group of projective nonisometric transformations

Levi-Civita's Theorem allows one to construct examples of (nonisometric) projective transformations of a closed manifold. Consider the torus $T^2 := \mathbb{R}^2/_{\mathbb{Z}^2}$

with the standard coordinates $x, y \in (\mathbb{R} \bmod 1)$ and a positive smooth nonconstant function $f : (\mathbb{R} \bmod 1) \to \mathbb{R}$ such that the metric

$$ds^2 \quad := \quad \left(f(x) - \frac{1}{f(y)}\right)\left(\sqrt{f(x)}\, dx^2 + \frac{1}{\sqrt{f(y)}}\, dy^2\right)$$

is positive definite. Then, the diffeomorphism $F : T^2 \to T^2$ given by $F(x, y) := (y, x)$ takes the original metric to the metric

$$\left(f(y) - \frac{1}{f(x)}\right)\left(\frac{\sqrt{f(x)}}{f(x)}\, dx^2 + \frac{f(y)}{\sqrt{f(y)}}\, dy^2\right).$$

Hence, it is a projective transformation by Levi-Civita's Theorem 1.

Example 5: metric geodesically equivalent to the metric of the ellipsoid

Consider the ellipsoid $\sum_{i=1}^{n} \frac{(x^i)^2}{a_i} = 1$, where $a_i > 0$, $i = 1, ..., n$.

Theorem 2 ([48, 87, 53, 83, 84, 79]). *The restriction of the metric* $\sum_{i=1}^{n}(dx^i)^2$ *to the ellipsoid*
$\sum_{i=1}^{n} \frac{(x^i)^2}{a_i} = 1$ *is geodesically equivalent to the restriction of the metric*

$$\frac{1}{\sum_{i=1}^{n}\left(\frac{x^i}{a_i}\right)^2}\left(\sum_{i=1}^{n} \frac{(dx^i)^2}{a_i}\right)$$

to the same ellipsoid.

It was not very easy to construct this example, and to find new examples of such kind. Two different constructions leading to such examples are described in [48, 83, 80]. But once the example is found, it is easy to check that the metrics are geodesically equivalent: one should find coordinates such that the metrics have the form from Levi-Civita's Theorem. For ellipsoid, such coordinates are standard elliptic coordinates.

Example 6: reducible metrics have affine equivalent metrics

This example shows that every complete reducible metric admits nonproportional affine equivalent (and, therefore, geodesically equivalent) one. Recall that a metric is *reducible*, if the natural representation of the loop group in $SO(n)$ is reducible. But in our construction we will use not this definition, but the following description of all complete reducible metrics (obtained by De Rham [22]); see also [38]).

Take the Riemannian product $(M_1 \times M_2, g_1 + g_2)$ of two complete Riemannian manifolds (M_1, g_1) and (M_2, g_2). Denote by $Iso(M_i, g_i)$ the isometry group of (M_1, g_1). Consider a discrete subgroup

$$\Gamma \subset Iso(M_1, g_1) \times Iso(M_2, g_2)$$

such that its natural action on $M_1 \times M_2$ is free. Since the group acts free on $M_1 \times M_2$ and preserves the product metric $g_1 + g_2$, the quotient space

$$(M_1 \times M_2, g_1 + g_2)/\Gamma$$

is a Riemannian manifold $(\tilde{M}, \tilde{g})$. Every such Riemannian metric $\tilde{g}$ is reducible, and every complete reducible metric can be constructed in such a way.

It is easy to see why a reducible metric always admits a metric which is geodesically equivalent and nonproportional to it. Indeed, in the model above consider the product metric $g_1 + 2g_2$ on $M_1 \times M_2$. It is geodesically equivalent to $g_1 + g_2$, because it is even affine equivalent to $g_1 + g_2$. Since the metric $g_1 + 2g_2$ is invariant with respect to the group of transformations, it generates a metric on $\tilde{M}$ which is affine equivalent to $\tilde{g}$ and is not proportional to it.

Example 7: for certain closed manifolds the index of the group of isometries in the group of affine transformation is infinite.

Consider the torus $\mathbb{R}^2/_{\mathbb{Z}^2}$, where the action of the group $\mathbb{Z}^2$ is generated by the shifts along the vectors $(1, 0)$ and $(0, 1)$. The Euclidean metric on $\mathbb{R}^2$ is invariant with respect to the action of $\mathbb{Z}^2$, and, therefore, generates a metric on the torus.

On $\mathbb{R}^2$, consider the standard action of the group $\mathbb{Z}$ generated by the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{Z}).$$

The group $\mathbb{Z}$ acts by affine transformations which are not isometries. Since the action preserves the lattice $\mathbb{Z}^2$, the action of $\mathbb{Z}$ on $\mathbb{R}^2$ generates an action on the torus $\mathbb{R}^2/_{\mathbb{Z}^2}$. Thus, on the standard flat torus the index of the group of isometries in the group of affine transformation is infinite.

Example 8: projective transformations of metrics of non-constant curvature

This example was essentially obtained in [63]. One can generalise it for all dimensions ≥ 2. Note that the metric is not complete, and for $C \neq 0, 2$ one can not extend the manifold by glueing additional pieces, since the scalar curvature goes to infinity when we approach the boarder. For $C = 0, 2$, the metric has constant sectional curvature.

The metric (on $\mathbb{R}^2$ with coordinates (x, y)) is

$$\frac{\exp(Cx)}{(\exp(x) + \exp(-x))^2} dx^2 + \frac{\exp(-x)}{(\exp(x) + \exp(-x))} dy^2.$$

The projective vector fields (=fields whose flows act by projective transformations) are $v := (1, \frac{1}{2}Cy)$, $u := (0, 1)$.

It is easy to see that the flow of u acts by projective transformations, since the flow of u acts by isometries. It is slightly tricky to see that the flow of v also acts by projective transformations. Probably the simplest way is to calculate first the geodesics explicitly, which is possible since the metric has a Killing vector field.

We see that for certain C the metric is projectively homogen (in the sense that the group of projective transformations acts transitively on it).

Example 9: projective vector fields for superintegrable metrics

If the geodesic flow of a two-dimensional metric g admits three independent integrals, such that one of them is linear in velocities, the second is quadratic in velocities, and the third is the energy integral, then one can construct two projective vector fields. Indeed, by [48, 49], the existence of the quadratic integral allows us to construct a metric $\bar{g}$ geodesically equivalent to g. The linear integral gives us a Killing vector field for g, which allows us to construct a Killing vector field for $\bar{g}$, see for example [88]. Since this Killing vector field preserved the geodesics of $\bar{g}$, it preserved the geodesics of g, i.e. it is a projective vector field.

First examples of the metrics such that their geodesic flows admits such three integrals are known since Königs [41]. Recently, they were studied by Kalnins, Kress, Miller and Winternitz [34, 35]. We will give only one example; the other examples can be constructed by the algorithm described above.

The metric is (γ is a positive constant, x, y are standard coordinates on $\mathbb{R}^2$):

$$(x^2 + y^2 + \gamma)(dx^2 + dy^2).$$

The independent integrals (in the standard coordinates (x, y, p_x, p_y) on $T^*\mathbb{R}^2$)):

$$\begin{aligned}
H &:= \frac{{p_x}^2 + {p_y}^2}{x^2 + y^2 + \gamma}, \\
F_1 &:= x p_y - y p_x \\
F_2 &:= \frac{{p_x}^2 y^2 - \left(x^2 + \gamma\right) {p_y}^2}{x^2 + y^2 + \gamma} \\
F_3 &:= xyH - p_x p_y
\end{aligned}$$

And the projective vector fields are

$$\begin{aligned}
v_1 &:= (y, -x) \\
v_2 &:= \left((x^2 + \gamma)y, y^2 x)\right) \\
v_3 &:= \left((x(y^2 - x^2 - \gamma), x(y^2 + \gamma - x^2)\right).
\end{aligned}$$

(The first vector field is the Killing vector field corresponding to the integral F_1.)

Note that the vector fields v_2, v_3 are not complete and generate no diffeomorphism of $\mathbb{R}^2$. In fact, they can not be complete by Corollary 1.

3 Answers assuming the manifold is closed

We will give partial answers to the Problems 1,2. The new technique that allows us to obtain these answers came from the theory of integrable Hamiltonian systems; we will explain this technique and how it works in the next sections.

3.1 Answer for the Problem 2

Theorem 3. *Let (M^n, g) be complete Riemannian manifold of dimension $n \geq 2$. Suppose the group of affine transformations has infinite index in the group of projective transformations. Then, g has positive constant sectional curvature.*

An immediate consequence of Theorem 3 is

Corollary 1 (Projective Lichnerowicz Conjecture). *Let a connected Lie group G act on a complete connected Riemannian manifold (M^n, g) of dimension $n \geq 2$ by projective transformations. Then, it acts by affine transformation, or g has constant positive sectional curvature.*

For dimension 2, Corollary 1 was announced in [62, 60] and proved in [63]. For arbitrary dimension, Corollary 1 was proved in [64].

It is known [94] that the connected component of the group of affine transformations of a closed manifold consists of isometries. Thus, an immediate consequence of Corollary 1 is

Corollary 2 (Projective Obata Conjecture). *Let a connected Lie group G act on a closed connected Riemannian manifold (M^n, g) of dimension $n \geq 2$ by projective transformations. Then, it acts by isometries, or g has constant positive sectional curvature.*

For dimension 2, Corollary 2 was announced in [62] and proved in [60]. For arbitrary dimension, Corollary 2 was proved in [64].

Projective transformations, and in particular projective Lichnerowicz-Obata conjecture were extremely popular objects of investigation in 50th–80th. There were at least three big differential-geometrical schools trying to prove it. In the table below we mention one important result of each school.

School of Lichnerowicz	School of Yano	School of Raschewskii
Couty [19, 20] proved the conjecture assuming that g is Einstein or Kähler	Yamauchi [92, 93] proved the conjecture assuming that the scalar curvature is constant	Solodovnikov [75, 78] proved the conjecture assuming that all objects are real analytic and that the dimension $n \geq 3$.

3.2 Conjectured answer on the Geodesic Rigidity Problem

Conjecture 1. *Suppose M^n is a closed connected manifold. Let Riemannian metrics g and $\bar{g}$ on M^n be geodesically equivalent and not proportional. Then, the manifold can be covered by the sphere or admits a reducible metric.*

In Section 2, we show that the manifolds admitting reducible metrics also admit geodesically equivalent nonproportional metrics, see Example 6 there.

Beltrami's example of projective transformations of the sphere (Example 2 in Section 2) and Example 5 show that the sphere can have two nonproportional metrics which are geodesically equivalent. Note that not every manifold cover by the sphere can have two nonproportional metrics which are geodesically equivalent. For example, the Poincare homology sphere (which certainly can be covered by the 3-sphere) is geodesically rigid [58].

Let us give a list of results confirming this conjecture.

In dimension 2, the conjecture was proved in [49]. In dimension 3, the conjecture was proved in [56, 58]. Actually, the following stronger statement was proved there:

Theorem 4. *Let the nonproportional Riemannian metrics g and $\bar{g}$ be geodesically equivalent on a closed connected three-dimensional manifold M^3. Then the manifold is homeomorphic either to a lens space or to a Seifert manifold with zero Euler number.*

In every dimension, the conjecture was proved on the level of fundamental group:

Theorem 5. *Suppose M^n is a closed connected manifold. Let Riemannian metrics g and $\bar{g}$ on M^n be geodesically equivalent and not proportional. Then, the manifold has a finite fundamental group or admits a reducible metric.*

3.3 Geodesic equivalence and zero entropy

All results of this section are joint with Kruglikov [44].

It is expected, that an integrable geodesic flow has zero topological entropy. This is not always the case. There are examples [13, 14, 15, 16, 17, 43] of integrable flows with non-zero entropy. But still it is possible to show that if the integrals come from strictly-non-proportional geodesically equivalent metrics by applying Theorem 9, the topological entropy of the geodesic flow must be zero.

Theorem 6. *Suppose the Riemannian metrics g, $\bar{g}$ on a closed connected M^n are geodesically equivalent. Suppose there exists a point where the metrics are strictly non-proportional. Then, the topological entropy of the geodesic flow of g is zero.*

Combining this theorem with the famous Yomdin's Theorem [28], we obtain

Corollary 3. *Suppose the Riemannian metrics g, $\bar{g}$ on a closed connected M^n are geodesically equivalent. Suppose there exists a point where the metrics are strictly non-proportional. Then, the manifold is finitely covered by the product of a rational-elliptic manifold and the torus.*

For small dimensions, Corollary 3 follows from [66, 67, 68, 69, 70, 47].

3.4 Geodesically equivalent strictly-non-proportional metrics on the torus

In this section we give a complete answer on Geometric Beltrami Problem assuming that the manifold is (diffeomorphic to) the n-torus

$$T^n := \underbrace{S^1 \times ... \times S^1}_{n \text{ times}}$$

and that there exists at least one point of the manifold such that the metrics are *strictly-non-proportional* at this point.

Consider $\mathbb{R}^n$ with the standard coordinates $x_1, x_2, ..., x_n$. Consider an $n-$lattice G on $\mathbb{R}^n$. By an $n-$lattice we mean the set of the vectors $k_1 v_1 + k_2 v_2 + ... + k_n v_n$, where $v_1, ..., v_n \in \mathbb{R}^n$ are linearly independent vectors and $k_1, k_2, ..., k_n \in \mathbb{Z}$.

Let λ_i, $i = 1, ..., n$, be smooth functions on $\mathbb{R}^n$ satisfying the following three conditions:

(i) For every i, the function λ_i depends on the the variable x_i only.

(ii) $0 < \lambda_i(x) < \lambda_j(y)$ for all $i < j$ and for all x, y.

(iii) The functions λ_i are invariant modulo the lattice so that for every vector $v = (v^1, v^2, ..., v^n) \in G$, and for all $x_i \in \mathbb{R}$

$$\lambda_i(x_i + v^i) = \lambda(x_i).$$

Consider the Riemannian metrics g_{model} and $\bar{g}_{model}$ on $\mathbb{R}^n$ given by the formulae (1,2). By Levi-Civita's Theorem, the metrics are geodesically equivalent; by definitions, they are invariant modulo the lattice so they generate two geodesically equivalent metrics on the torus $\mathbb{R}^n/_G$. We will call such metrics *model* metrics corresponding to the lattice G and to the functions λ_i.

Theorem 7 ([57, 61]). *Let Riemannian metrics g and $\bar g$ on the torus T^n be geodesically equivalent and strictly non-proportional at least at one point. Then, there exist an $n-$lattice G, functions $\lambda_1, ..., \lambda_n$ satisfying conditions (i-iii) and a diffeomorphism $\phi : T^n \to \mathbb{R}^n/_G$ such that $g = \phi^* g_{model}$, $\bar g = \phi^* \bar g_{model}$, where g_{model} and $\bar g_{model}$ are model metrics with respect to the lattice G and the functions λ_i.*

Thus, every pair of geodesically equivalent Riemannian metrics on the $n-$torus which are strictly non-proportional at least at one point is given by an $n-$lattice G and by functions λ_i satisfying conditions (i-iii). The following theorem answers when two of such sets of data define the same pair of metrics:

Theorem 8 ([57, 61]). *Let G^1 and G^2 be two $n-$lattices on $\mathbb{R}^n$. Suppose the functions $\lambda^1_1, ..., \lambda^1_n$ satisfy conditions (i-iii) with respect to the lattice G^1; suppose the functions $\lambda^2_1, ..., \lambda^2_n$ satisfy conditions (i-iii) with respect to the lattice G^2. Consider the pairs of model geodesically-equivalent metrics $g^1_{model}, \bar g^1_{model}$ (which are model metrics with respect to the lattice G^1 and the functions λ^1_i) and $g^2_{model}, \bar g^2_{model}$ (which are model metrics with respect to the lattice G^2 and the functions λ^2_i).*

Then, there exists a diffeomorphism $\phi : \mathbb{R}^n/_{G^1} \to \mathbb{R}^n/_{G^2}$ such that $g^1_{model} = \phi^* g^2_{model}$ and $\bar g^1_{model} = \phi^* \bar g^2_{model}$, if and only if there exists $\alpha_1, ..., \alpha_n \in \mathbb{R}$ and $\epsilon_1, ..., \epsilon_n \in \{+1, -1\}$ such that the coordinate change

$$x_i \mapsto \epsilon_i x_i + \alpha_i, \quad i = 1, 2, ..., n,$$

takes the lattice G^1 to the lattice G^2 and the functions λ^1_i to the functions λ^2_i.

4 Integrable systems in the theory of geodesically equivalent metrics.

Let $g = (g_{ij})$ and $\bar g = (\bar g_{ij})$ be Riemannian metrics on a manifold M^n. Consider the (1,1)-tensor L given by the formula

$$L^i_j \stackrel{\text{def}}{=} \left(\frac{\det(\bar g)}{\det(g)}\right)^{\frac{1}{n+1}} \bar g^{i\alpha} g_{\alpha j}. \tag{4}$$

Then, L determines the family S_t, $t \in R$, of $(1,1)$-tensors

$$S_t \stackrel{\text{def}}{=} \det(L - t \ \mathrm{Id}) \, (L - t \ \mathrm{Id})^{-1}. \tag{5}$$

We will identify the tangent and cotangent bundles of M^n by g. This identification allows us to transfer the natural Poisson structure from T^*M^n to TM^n.

Theorem 9 ([48, 87]). *If g, $\bar g$ are geodesically equivalent, then, for every $t_1, t_2 \in R$, the functions*

$$I_{t_i} : TM^n \to R, \quad I_{t_i}(v) \stackrel{\text{def}}{=} g(S_{t_i}(v), v) \tag{6}$$

are commuting integrals for the geodesic flow of g.

Remark 1. *Although* $(L - t\,\mathrm{Id})^{-1}$ *is not defined for* t *lying in the spectrum of* L*, the tensor* S_t *is well-defined for every* t*. Moreover,* S_t *is a polynomial in* t *of degree* $n-1$ *with coefficients being (1,1)-tensors. Consequently, the function* I_t *is a polynomial in* t *of degree* $n-1$*.*

Note that the converse of Theorem 9 is not always true: If the functions I_t are integrals, the metric $\bar g$ must not be geodesically equivalent to g, a counterexample could be found in [51]. In [53] it was found under what additional assumption the converse of Theorem 9 is true.

Remark 2. *Such integrable systems were recently actively studied in mathematical physics, in the theory of separation of variables, see [6, 7, 8, 21, 32].*

4.1 Geometric sense of the integrals from Theorem 9

We will recall a construction that, given an orbit-preserving diffeomorphism between two Hamiltonian systems allows one to construct an integral of the system. The construction is essentially due to [82], see also [48, 87, 88]. Independently, it was discovered in [79].

Geodesically equivalent metrics provide such diffeomorphism: the manifold is TM^n, the symplectic structures come from T^*M^n via identification of T^*M^n and TM^n by g and $\bar g$, and the orbit-preserving diffeomorphism is the reparametrisation m given by

$$m : TM^n \to TM^n, \quad m(\xi) := \frac{\sqrt{\bar g(\xi,\xi)}}{\sqrt{g(\xi,\xi)}} \cdot \xi. \tag{7}$$

Applying this construction to this diffeomorphism we obtain the integrals from Theorem 9.

Let v and $\bar v$ be Hamiltonian systems on symplectic manifolds (N^{2n}, ω) and $(\bar N^{2n}, \bar\omega)$ with Hamiltonians H and $\bar H$, respectively. Consider the isoenergy surfaces

$$Q^{2n-1} \stackrel{\mathrm{def}}{=} \left\{p \in N^{2n} : H(x) = h\right\}, \quad \bar Q^{2n-1} \stackrel{\mathrm{def}}{=} \left\{p \in \bar N^{2n} : \bar H(x) = \bar h\right\},$$

where h and $\bar h$ are regular values of the functions H, $\bar H$, respectively. A diffeomorphism $\phi : Q^{2n-1} \longrightarrow \bar Q^{2n-1}$ is *orbital*, if it takes the orbits (considered as unparameterised curves) of the system v to the orbits of the system $\bar v$.

Given an orbital diffeomorphism, we can invariantly construct an integral for the system v. Denote by σ, $\bar\sigma$ the restrictions of the forms $\omega, \bar\omega$ to Q^{2n-1}, $\bar Q^{2n-1}$, respectively. Consider the pull-back $\phi^*\bar\sigma$ which is a skew-symmetric two-form on Q^{2n-1}.

Lemma 1 ([82]). *The flow of* v *preserves the forms* σ *and* $\phi^*\bar\sigma$*.*

Proof: Since v preserves Q^{2n-1} and ω, it preserves σ. Let us prove that v preserves $\phi^*\bar\sigma$. We need to prove that the Lie derivative $L_v\phi^*\bar\sigma$ of the form $\phi^*\bar\sigma$ along the vector field v vanishes.

By Cartan's formula, the Lie derivative is given by

$$L_v\phi^*\bar\sigma = \mathrm{d}\left[\imath_v\phi^*\bar\sigma\right] + \imath_v\mathrm{d}\left[\phi^*\bar\sigma\right].$$

On the right-hand side both terms vanish. Indeed, the term $\imath_v\mathrm{d}\left[\phi^*\bar\sigma\right]$ vanishes since the form $\bar\omega$ is closed and, therefore, the form $\phi^*\bar\sigma$ is also closed. Let us show that the term $\mathrm{d}\left[\imath_v\phi^*\bar\sigma\right]$ vanishes as well. We will even show that $\imath_v\phi^*\bar\sigma$ vanishes.

By definition of the vector field $\bar v$, for any vector u we have $\omega(\bar v, u) = d\bar H(u)$. Then, the kernel of the form $\bar\sigma$ coincides (in the space $T_q\bar Q^{2n-1}$ at each point $q \in \bar Q^{2n-1}$) with the linear span of the vector $\bar v$. Since the mapping ϕ takes orbits to orbits, the differential $d\phi$ takes the vector field v to a vector field which is proportional to $\bar v$. Therefore, v lies in the kernel of $\phi^*\bar\sigma$, and $\imath_v\phi^*\bar\sigma$ vanishes. Finally, the Lie derivative vanishes, and the form is preserved by v. Lemma 1 is proved.

Since both forms are preserved by the flow, a function constructed invariantly by using these forms must automatically be an integral. The construction could be as follows. Take the eigenvalues of one form with respect to the other, (i.e., for every point $p \in Q^{2n-1}$ take all $\lambda \in \mathbb{C}$ such that $\mathbf{rank}(\sigma - \lambda\phi^*\bar\sigma) < 2n-2$). None of them are zero or infinite. The multiplicity of such λ is $n - 1 - \frac{1}{2}\mathbf{rank}(\sigma - \lambda\phi^*\bar\sigma)$. Take the symmetric products of all such λ's counted with multiplicities. They are a smooth real-valued function on Q^{2n-1}. Since construction is coordinate-independent, and since the forms are preserved by v, the functions are preserved by v as well, i.e., they are integrals of the Hamiltonian system v.

By direct calculations (which are not very complicated and can be found, for example, in [48, 88]), one can see that these integral constructed for the orbital diffeomorphism m of the geodesic flows for geodesically equivalent metrics are essentially the coefficients of the polynomial I_t from Theorem 9. Thus, for every t the function I_t is an integral.

4.2 First applications of the integrals

Suppose that g and $\bar g$ are complete geodesically equivalent metrics on the same manifold M^n. As in the previous section, consider the tensor L given by (4). The next statement shows that the structure of the eigenvalues of L is the same at almost every point of the manifold.

Since L is clearly self-adjoint with respect to $\bar g$, the eigenvalues of L are real. At every point $x \in M^n$, let us denote by $\lambda_1(x) \le ... \le \lambda_n(x)$ the eigenvalues of L at the point.

Corollary 4 ([59]). *For every $i \in \{1, ..., n-1\}$, for every $x, y \in M^n$, the following holds:*

1. $\lambda_i(x) \le \lambda_{i+1}(y)$.
2. *If $\lambda_i(x) < \lambda_{i+1}(x)$, then $\lambda_i(z) < \lambda_{i+1}(z)$ for almost every point $z \in M^n$.*

Corollary 4 was proved in [52, 54, 55, 59, 88].

We will prove this Corollary assuming that the dimension of the manifold is 2. The proof contains all ideas of the proof for arbitrary dimensions. Consider the integral $I_0 := (I_t)_{|t=0}$. By direct computations, in dimension 2

$$I_0(\xi) = \left(\frac{\det(g)}{\det(\bar g)}\right)^{2/3} \bar g(\xi,\xi).$$

Proof of Corollary 4 in dimension 2: Consider $TM^2 \setminus (TM^2)_0$, where $(TM^2)_0$ is the zero-section of the tangent space TM^2, and the function $I_0(\xi)/E(\xi)$, where $E(\xi)$ is the energy-integral $g(\xi,\xi)$. The function $I_0(\xi)/E(\xi)$ is well-defined on $TM^2 \setminus (TM^2)_0$. Note, that for every $P \in M^2$ and $\xi \in T_PM^2$, $\xi \neq 0$, the following inequality holds:

$$\lambda_1(P) \le I_0(\xi)/E(\xi) \le \lambda_2(P). \tag{8}$$

Indeed, take coordinates on T_PM^2 such that the metrics g, $\bar g$ have matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{\lambda_2(P)} & 0 \\ 0 & \frac{1}{\lambda_1(P)} \end{pmatrix}$$

In this coordinates, the function $I_0(\xi)/E(\xi)$ is given by:

$$\begin{aligned} \frac{I_0(\xi)}{E(\xi)} = \frac{\lambda_1(P)\xi_1^2 + \lambda_2(P)\xi_2^2}{\xi_1^2+\xi_2^2} &= \lambda_2(P) - \frac{(\lambda_2(P)-\lambda_1(P))\xi_1^2}{\xi_1^2+\xi_2^2} \\ &= \lambda_1(P) + \frac{(\lambda_2(P)-\lambda_1(P))\xi_2^2}{\xi_1^2+\xi_2^2}. \end{aligned}$$

Using that $\frac{\lambda_2(P)-\lambda_1(P)}{\xi_1^2+\xi_2^2}$ is nonnegative, we obtain the inequality (8).

Since (M^2, g) is complete, for every two points $P, Q \in M^2$, there exists a connecting geodesic: $\gamma(0) = P$, $\gamma(1) = Q$. Since the functions $I_0(\xi)$ and $E(\xi)$ are integrals, the function $I_0(\xi)/E(\xi)$ is also an integral, so that $I_0(\dot\gamma(0))/E(\dot\gamma(0)) = I_0(\dot\gamma(1))/E(\dot\gamma(1))$. Combining with (8), we obtain:

$$\lambda_1(P) \le I_0(\dot\gamma(0))/E(\dot\gamma(0)) = I_0(\dot\gamma(1))/E(\dot\gamma(1)) \le \lambda_2(Q).$$

Corollary 4 is proved in dimension 2.

5 Main ideas of proof of Theorems 3,5

5.1 Sketch of the proof of Theorem 5

The following two cases are possible:

Case 1: There exists $r \in \{1, ..., n-1\}$ such that, for every $x \in M^n$

$$\lambda_r(x) < \lambda_{r+1}(x).$$

Case 2: For every $r \in \{1, ..., n-1\}$ and $x \in M^n$

$$\lambda_r(x) < \lambda_{r+1}(x).$$

In [59] it was proved that in the first case the manifold admits a reducible metric. We will explain only part of this result, namely we will show why the manifold admits two transversal foliations of supplementary dimensions (a reducible metric defines such two foliations; in the description of the reducible metrics in Section 2 the leaves of the first foliation came from the products of M_1 and the points of M_2, the leaves of the second foliation came from the the points of M_1 and M_2.)

We will need the following statement.

Corollary 5. *Suppose the Riemannian metrics $g, \bar{g}$ are geodesically equivalent. Then the Nijenhuis torsion of the tensor L vanishes.*

Under the assumption that the metrics are strictly non-proportional at least at one point, Corollary 5 follows from Corollary 4 and from Levi-Civita's Theorem 1. Indeed, under this assumption, in a neighbourhood of almost every point, in the Levi-Civita coordinates the tensor L is diagonal $\mathrm{diag}(\lambda_1(x_1), ..., \lambda_n(x_n))$, and its Nijenhuis torsion vanishes. Without this assumption, Corollary 5 was proved in [12] (or follows from the original, more general version of Levi-Civita's Theorem from [45].)

Now let us continue to explain the proof of Theorem 5 (under the assumptions of Case 1).

Consider the distribution spanned over the eigenspaces of the first r eigenvalues of L. By Corollary 5, it is integrable so that it generates an $r-$dimensional foliation B_r. Similarly, the distribution spanned over the eigenspaces of the last $n-r$ eigenvalues of L generates an $(n-r)-$dimensional foliation B_{n-r}. Levi-Civita's Theorem also gives us a local description of geodesically equivalent metrics in a neighbourhood of almost every point. From this description, it is easy to see how, given these foliations B_r and B_{n-r} one can change g in order to obtain reducible $\tilde{g}$. In [59] it was shown how the change can be done invariantly and globally.

Now let us explain why under the assumption of Case 2 the manifold has finite fundamental group. We will do it for dimension 2 only.

Suppose the surface M^2 carries two geodesically equivalent metrics g and $\bar{g}$ which are not proportional at least at one point. Then, the metrics are not proportional at almost every point. Suppose there exists a point x such that the metrics are proportional. We will show that in this case the fundamental group can not be infinite.

Indeed, suppose the fundamental group is infinite. Without loss of generality we can assume that at the point x the metrics g and $\bar{g}$ coincide.

Consider the universal cover U of M^2. The lifted metrics on it are geodesically equivalent and coincide at infinitely many points. Take five points $x_1, ..., x_5$ such that the lifted metrics coincide. There exists a open neighbourhood such

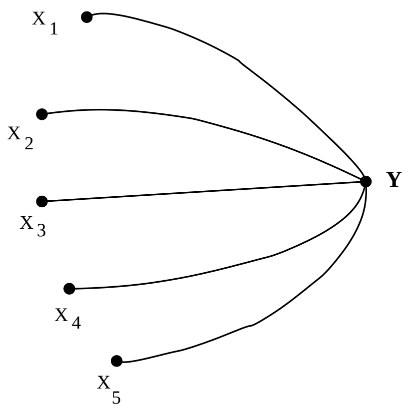

Figure 2: The points $y, x_1, ..., x_5$ and the geodesic $\gamma_1, ..., \gamma_5$

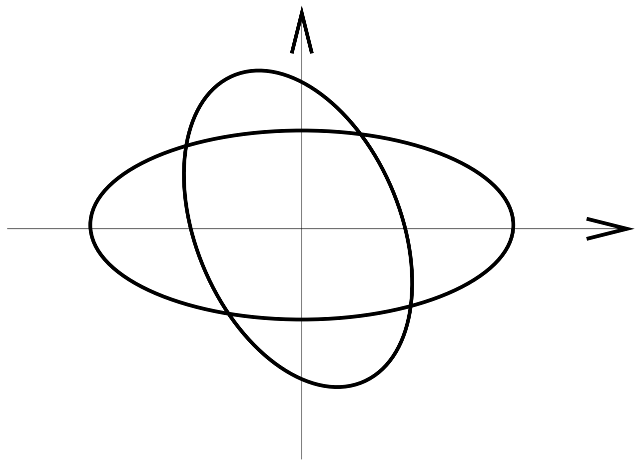

Figure 3: Since the ellipses $E(\xi) = 1$ and $I_0(\xi) = 1$ have no more than 4 points of intersections, the velocity vectors of the geodesics $\gamma_1, ..., \gamma_5$ can not be mutually transversal

that for every point y of this neighbourhood there exist geodesics $\gamma_1, ..., \gamma_5$ such that $\gamma_i(0) = x_i$, $\gamma_i(1) = y$, and such that their velocity vectors at y are mutually transversal, see Figure 2. Then, the metrics coincide at all such y. Indeed, since the geodesics contain points such that $g = \bar{g}$, the integral $I_0(\xi)$ coincides with $E(\xi) := g(\xi, \xi)$. From the other side, if the metrics do not coincide at y, the set of the vectors $\xi \in T_yU$ such that the $E(\xi) = 1$ and $I_0(\xi) = 1$ contains at most 4 point. Thus, the velocity vectors of the geodesics can not be mutually transversal at y, see Figure 3. Thus, the metrics coincide at every point of some open subset. Finally, by Corollary 4, they coincide everywhere.

Remark 3. *As a by-product we obtain that in dimension 2 the set of the points such that the metrics are proportional either coincide with the manifold, or is discrete. This observation can be generalised for every dimension ≥ 2. Namely, if $\lambda_i \neq \lambda_{i+1}$ at least at one point, then the set of the points such that $\lambda_i = \lambda_{i+1}$ has codimension at least 2.*

5.2 Sketch of the proof of Theorem 3

Suppose the metrics g and $\bar g$ are geodesically equivalent on M^n of dimension $n \geq 2$. Then, we can construct an one-parameter family (we will denote this family by $G(g, \bar g)$ of the metrics geodesically equivalent to g. The construction goes as follows:

Consider the (1,1)-tensor L given by (4). Take $c \in \mathbb{R}$ and consider the symmetric (0,2)-tensor g_c given by

$$g_c(\xi, \eta) := g\left(L(\xi) - c\,\xi, \eta\right) \cdot \frac{1}{\det(L - c \cdot \mathrm{Id})}.$$

The tensor g_c is not defined for $c \in \mathrm{Spectrum}(L)$. For certain values of c (for example, for all negative c) it is well-defined, positive definite and, therefore, is a Riemannian metric on M^n. We will denote the set of all such Riemannian metrics by $G(g, \bar g)$.

Theorem 10 *Suppose g and $\bar g$ are geodesically equivalent. Then, every metric from $G(g, \bar g)$ is geodesically equivalent to them.*

We prove this theorem under the assumption, that there exists a point such that the metrics are strictly non-proportional. The proof without this assumption is similar.

Under this assumption, by Corollary 4, the metrics are strictly-non-proportionally at almost every point. Then, it is sufficient to prove the statement in Levi-Civita coordinates. By direct calculation we obtain that in Levi-Civita coordinates the metric g_c is given by

$$ds^2_{\bar g_c} = \rho_1 \Pi_1 dx_1^2 + \rho_2 \Pi_2 dx_2^2 + \cdots + \rho_n \Pi_n dx_n^2, \tag{9}$$

where the functions Π_i and ρ_i are given by

$$\begin{aligned} \Pi_i &\stackrel{\mathrm{def}}{=} (\lambda_i - \lambda_1)(\lambda_i - \lambda_2)\cdots(\lambda_i - \lambda_{i-1})(\lambda_{i+1} - \lambda_i)\cdots(\lambda_n - \lambda_i), \\ \rho_i &\stackrel{\mathrm{def}}{=} \frac{1}{(\lambda_1 - c)(\lambda_2 - c)...(\lambda_n - c)} \frac{1}{(\lambda_i - c)}. \end{aligned}$$

We see that the metric g_c and the metric g_{model} given by (1) are in the Levi-Civita form (the new functions λ are related to the initial functions λ by the formula $\lambda_i^{\mathrm{new}} = \lambda_i - c$. Finally, the metric g_c is geodesically equivalent to the metric g_{model} by Levi-Civita's Theorem. Theorem 10 is proved.

Now let us sketch the proof of Theorem 3. Suppose there exists infinitely many projective transformations $f_1, f_2, ..., f_k, ...$ such that none of projective transformations $f_i \circ (f_j)^{-1}$, $i \neq j$ are affine transformation. We will assume in addition that the pull-back metrics ${f_i}^* g$ are strictly not-proportional to the metric g. (Without this assumption the proof is much harder; the half of the paper [64] is denoted to the fight with the metrics which are not strictly non-proportional). Denote by $\bar{g}$ the metric $f_1^* g$.

The following two cases are possible:

Case 1: Every metric $f_i^* g$ is proportional to a certain metric of $G(g, \bar{g})$.

Case 2: There exists i such that the metric $f_i^* g$ is proportional to no metric from $G(g, \bar{g})$.

Note that in the proof of Theorem 10 we have shown that the Levi-Civita coordinates for metrics g , $\bar{g}$ are Levi-Civita coordinates for the metrics g, g_c. Therefore, under the assumptions of Case 1, the Levi-Civita coordinates for the metrics $g, \bar{g}$ are also Levi-Civita coordinates for all metrics $f_i^* g$ (the only freedom we have is we can multiply all coordinates by the same constant.)

By Remark 3, the set of the points where the Levi-Civita coordinates are defined is everywhere dense and connected. Then, a certain f_i in the coordinates looks $x_i \mapsto x_i^{\text{new}}(x_i)$, and $f_i^* g_{model}$ is proportional to (9). This gives a system of functional equation on the $2n$ unknown functions $\lambda_i(x_i)$, $x_i^{\text{new}}(x_i)$ of one variable and on two parameters (c and the coefficient proportionality). This equations can be solved; the solutions can be analysed, and one can show that the solutions correspond to a complete Riemannian metric if and only if they generate a metric of constant positive sectional curvature.

Now let us explain the main idea of the proof that under the assumptions of Case 2 the metric has constant sectional curvature. The local part of this explanation is due to Fubini [26, 27]. For the metrics which are not strictly-non-proportional the arguments of Fubini were generalised by Solodovnikov [75, 76, 77, 78].

The trick of Fubini/Solodovnikov works only if the dimension of the manifold is greater than 2. For dimension 2, the proof of this statement requires a special technique which could be found in [63, 64]. The technique is based on the theory of quadratically integrable geodesic flows developed in [39, 40, 36, 33, 37, 9, 10, 11].

Under the assumptions of Case 2, in view of Corollary 4, in a neighbourhood of almost every point there exist two different coordinate systems such that the metric has the Levi-Civita form (1). By direct calculation of the curvature tensor R_{ijkl} for the metric (1), one can see that the component R_{ijkl} of the curvature tensor vanishes unless $i = k,\ j = l$ or $i = l,\ j = k$. We see that at the tangent space to almost every point there exists two (essentially) different basis such that the metrics an the curvature tensor are diagonal. One can show that in this case the curvature tensor is given by the formula $R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk})$. In dimensions ≥ 3 it implies that K is a constant, so that the metric has constant sectional curvature.

Since every projective transformation of a manifold of constant nonpositive sectional curvature is an affine transformation, the metric has constant positive sectional curvature.

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