On degree of mobility for complete metrics

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Abstract

The degree of mobility of a Riemannian metric g is the dimension of the space of Riemannian metrics sharing the same geodesics with g. We prove that the degree of mobility of an irreducible Riemannian metric on a closed manifold is at most two, unless the sectional curvature is positive constant.

1 Introduction

1.1 Main question

Let (M^n, g) be a Riemannian manifold.

Definition 1. *A* **BM-structure** *on* (M^n, g) *is a smooth self-adjoint* $(1,1)$-*tensor* L *such that, for every point* $x \in M^n$, *for every vectors* $u, v, w \in T_x M^n$, *the following equation holds:*

$$g((\nabla_u L)v, w) = \frac{1}{2} g(v, u) \cdot d\mathrm{trace}_L(w) + \frac{1}{2} g(w, u) \cdot d\mathrm{trace}_L(v),$$

where trace_L *is the trace of* L.

Definition 2. *Let* g, $\bar{g}$ *be Riemannian metrics on* M^n. *They are* **projectively equivalent**, *if they have the same (unparameterized) geodesics.*

The relation between BM-structures and projectively equivalent metrics is given by

Theorem 1 ([9]). *Let* g *be a Riemannian metric. Suppose* L *is a self-adjoint positive-definite* $(1,1)$-*tensor. Consider the metric* $\bar{g}$ *defined by*

$$\bar{g}(\xi, \eta) = \frac{1}{\det(L)} g(L^{-1}\xi, \eta) \tag{1}$$

(for every tangent vectors ξ *and* η *with the common foot point.)*

Then, the metrics g *and* $\bar{g}$ *are projectively equivalent, if and only if* L *is a BM-structure on* (M^n, g).

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The set of all BM-structures on (M^n, g) will be denoted by $\mathcal{B}(M^n, g)$. It is a linear vector space. The dimension of $\mathcal{B}(M^n, g)$ is called the **degree of mobility** of the metric. It is at least one, since $\mathcal{B}(M^n, g)$ contains the identity tensor $\mathrm{Id} \stackrel{\mathrm{def}}{=} \mathrm{diag}(1, 1, 1, ..., 1)$.

Main question: *How big can be the dimension of the space $\mathcal{B}(M^n, g)$?*

In other words, how big is the space of the metrics projectively equivalent to the given one?

1.2 History of the question

The history of the theory of projectively equivalent metrics goes back to works of Beltrami, Dini, Levi-Civita and Weyl.

The question how big can be the space of the metrics projectively equivalent to a given one was first considered in Lie [31]. Globally, i.e. when the manifold is complete, the question was asked in Schouten [49].

It is known that, locally, the degree of mobility of a metric is less than $\frac{(n+1)(n+2)}{2} + 1$, and is equal to $\frac{(n+1)(n+2)}{2}$ for spaces of constant curvature only, see [63, 51, 28]. The most power tools in the local study of the degree of mobility are the theory of concircular vector fields developed in Yano [63], and the theory of $V(K)$ spaces developed in Solodovnikov [52, 53, 54, 55]. Combining these two theories, Shandra [50] obtained that, locally, if the dimension n of the manifold is greater than two, the degree of mobility of a metric of nonconstant curvature can take the values

$$\frac{m(m+1)}{2} + l$$

only, where $1 \le m \le n$ and $1 \le l \le \left[\frac{n+1-m}{3}\right]$. For every such "admissible" value $D_{mobility}$ there exists a metric on the disk such that the degree of mobility is precisely $D_{mobility}$. For dimension two, it follows from [28, 33] that the degree of mobility can take the values $1, 2, 3, 4, 6$.

A more detailed historical overview of the local side of the question can be found in the surveys [1, 47].

The goal of this paper is to study the degree of mobility globally, i.e. when the manifold is closed or complete. Most results on the degree of mobility of closed manifolds require additional geometric assumptions written as a tensor equation. A typical result is that, under certain tensor assumptions, the degree of mobility is precisely 1, see, for example, [14, 61, 62, 20, 48].

1.3 Main Result

Theorem 2. *Let (M^n, g) be a connected complete Riemannian manifold of dimension greater than one. Suppose* $\dim(\mathcal{B}(M^n, g)) \ge 3$.

Then, if a complete Riemannian metric $\bar g$ is projectively equivalent to g, then g has positive constant sectional curvature, or $\bar g$ is affine equivalent to g.

Recall that two metrics are said to be **affine equivalent**, if their Levi-Civita's connections coincide.

All assumption in the theorem are important: we can construct counterexamples, if one of the assumptions is omitted.

It is easy to understand whether a complete metric admits affine equivalent one which is not proportional to it. In this case, the holonomy group of the manifold must be reducible [30, 25], which implies that the universal cover of the manifold with the lifted metric is the Riemannian product of two Riemannian manifolds. Thus, a direct consequence of Theorem 2 is the following

Corollary 1. *Let* (M^n, g) *be a closed connected Riemannian manifold with irreducible holonomy group. Suppose* $\dim(\mathcal{B}(M^n, g)) \geq 3$. *Then,* g *has constant positive sectional curvature.*

In dimension 2, in view of Theorem 3, Corollary 1 follows from results of [26, 27, 24].

It is worse to mention that the converse of Corollary 1 is not always true. Of cause, the space $\mathcal{B}$ is huge for the round sphere. But for certain quotients of the round sphere, the space $\mathcal{B}$ can have dimension one. This phenomena appears already in dimension 3, see [42].

In the present paper we will prove Theorem 2 assuming that the dimension n of the manifold is greater than 2. If $n = 2$, in view of Theorem 3, under the assumption that the manifold is closed, Theorem 2 follows from [27, 24]. Without this assumption, Theorem 2 (for dimension 2) is nontrivial. It is announced in [44, 45]. Its proof uses methods from the global theory of Liouville metrics developed in [7, 8, 22], and will be published in [46].

Our prove of Theorem 2 (for dimension ≥ 3) uses the following methods:

- The classical one is the local theory of projectively equivalent metrics. It is due to Beltrami [2], Dini [16], Levi-Civita [29], Fubini [18], Eisenhart [17], Cartan [13] , Weyl [59, 60] and Solodovnikov [52]. We will formulate a part of their results in Theorems 4,5,6,7,8.

- The newer one were introduced in [32, 56, 57, 36, 35]: the main observation is that, for a given Riemannian metric g, the existence of a projectively equivalent metric allows one to construct commuting integrals for the geodesic flow of g, see Theorem 3 in Section 2.1. This technique has been used quite successfully in finding topological obstruction that prevent a closed manifold from possessing (nontrivial) BM-structure, see [34, 39, 40, 37, 42, 43], and for the study of the degree of mobility for the metric of ellipsoid, see [41].

- And the general idea came from the singularity theory. The role of singularities play the points where the eigenvalues of the BM-structure bifurcate. In Section 3.1, we describe behavior of the metric near the simplest singular points. In Sections 3.2 and 4, we will show that the simplest singular points always exist. In Section 3.3, we will explain how the structure near singular points can be extended to the whole manifold.

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2 New and classical instruments of the proof

2.1 Integrals for geodesic flows of metrics admitting BM-structure.

The relation between BM-structures and integrable geodesic flows is observed on the level of geodesic equivalence in [32] and is as follows:

Let L be a self-adjoint $(1,1)$-tensor on (M^n, g). Consider the family S_t, $t \in \mathbb{R}$, of $(1,1)$-tensors

$$S_t \stackrel{\mathrm{def}}{=} \det(L - t\,\mathrm{Id})\,(L - t\,\mathrm{Id})^{-1}. \tag{2}$$

Remark 1. *Although $(L - t\,\mathrm{Id})^{-1}$ is not defined for t lying in the spectrum of L, the tensor S_t is well-defined for every t. Moreover, S_t is a polynomial in t of degree $n-1$ with coefficients being (1,1)-tensors.*

We will identify the tangent and cotangent bundles of M^n by g. This identification allows us to transfer the natural Poisson structure from T^*M^n to TM^n.

Theorem 3 ([56, 32, 57]). *If L is a BM-structure, then, for every $t_1, t_2 \in \mathbb{R}$, the functions*

$$I_{t_i} : TM^n \to \mathbb{R}, \quad I_{t_i}(v) \stackrel{\mathrm{def}}{=} g(S_{t_i}(v), v) \tag{3}$$

are commuting integrals for the geodesic flow of g.

Remark 2. *Integrable systems of slightly less general type were recently studied in [3, 4, 5, 21, 15]).*

Since L is self-adjoint, its eigenvalues are real. At every point $x \in M^n$, let us denote by $\lambda_1(x) \le ... \le \lambda_n(x)$ the eigenvalues of L at the point.

Corollary 2 ([43, 57, 38]). *Let (M^n, g) be a connected Riemannian manifold such that every two points can be connected by a geodesic. Suppose L is a BM-structure on (M^n, g). Then, for every $i \in \{1, ..., n-1\}$, for every $x, y \in M^n$, the following statements hold:*

1. $\lambda_i(x) \le \lambda_{i+1}(y)$.
2. *If $\lambda_i(x) < \lambda_{i+1}(x)$, then $\lambda_i(z) < \lambda_{i+1}(z)$ for almost every point $z \in M^n$.*

At every point $x \in M^n$, denote by $N_L(x)$ the number of different eigenvalues of the BM-structure L at x.

Definition 3. *A point $x \in M^n$ will be called typical with respect to the BM-structure L, if*

$$N_L(x) = \max_{y \in M^n} N_L(y).$$

Corollary 3 ([36]). *Let L be a BM-structure on a connected Riemannian manifold (M^n, g). Then, almost every point of M is typical with respect to L.*

2.2 Results of Beltrami, Levi-Civita and Solodovnikov

Theorem 4. *Let Riemannian metrics g and $\bar{g}$ on M^n be projectively equivalent. If g has constant sectional curvature, then $\bar{g}$ has constant sectional curvature as well.*

For dimension two, Theorem 4 was proven by Beltrami [2]. For dimension greater than two, a proof can be found in Eisenhart [17].

Corollary 4. *Let projectively equivalent metrics g and $\bar{g}$ on M^n (of dimension $n > 1$) be complete. If g has constant negative sectional curvature, $\bar{g}$ is proportional to g. If g is flat, $\bar{g}$ is affine equivalent to g.*

This statements is a folklore, in the sense that we did not find a classical reference for it, although certain authors use it as a known fact. We will be grateful if anybody gives us this reference.

Let us explain a proof of Corollary 4 by using newer methods. If both metrics are flat, Corollary 4 is equivalent to the statement that every diffeomorphism of $\mathbb{R}^n$ that takes straight lines to straight lines is a composition of linear transformation and translation. Its proof can be found in almost every advanced textbook on linear algebra and analytic geometry.

If g has constant negative sectional curvature, it is sufficient to prove Corollary 4 in dimension two only, since in every two-dimensional direction there exists a totally geodesic complete submanifold. If $\bar{g}$ is flat, the statement is trivial, since in the Euclidean space the parallel postulate of Euclid holds, and in the hyperbolic space not. If both metrics have constant negative curvature, Corollary 4 was proven in [10], see his lemma on page 59. The geometric idea behind the proof of Bonahon is the nontrivial observation from metric geometry (see [11, 12, 23] for the proof of this observation) that, for hyperbolic 2-spaces, the only isometry that preserves the boundary at infinity is the identity. Since the boundary at infinity can be defined by using unparameterized geodesics, Corollary 4 becomes to be trivial.

In view of Theorem 1, the next theorem is equivalent to the classical Levi-Civita's Theorem from [29].

Theorem 5 (Levi-Civita's Theorem). *The following statements hold:*

1. *Let L be a BM-structure on (M^n, g). Let $x \in M^n$ be typical. Then, there exists a coordinate system $\bar{x} = (\bar{x}_1, ..., \bar{x}_m)$ (in a neighborhood $U(x)$ containing x), where $\bar{x}_i = (x_i^1, ..., x_i^{k_i})$, $(1 \le i \le m)$, such that L is diagonal*
$$\mathrm{diag}(\underbrace{\phi_1, ..., \phi_1}_{k_1}, ..., \underbrace{\phi_m, ..., \phi_m}_{k_m}), \tag{4}$$
and the quadratic form of the metric g have the following form:
$$\begin{aligned} g(\dot{\bar{x}}, \dot{\bar{x}}) &= P_1(\bar{x}) A_1(\bar{x}_1, \dot{\bar{x}}_1) + P_2(\bar{x}) A_2(\bar{x}_2, \dot{\bar{x}}_2) + \cdots + \\ &+ P_m(\bar{x}) A_m(\bar{x}_m, \dot{\bar{x}}_m), \end{aligned} \tag{5}$$
where $A_i(\bar{x}_i, \dot{\bar{x}}_i)$ are positive-definite quadratic forms in the velocities $\dot{\bar{x}}_i$ with coefficients depending on $\bar{x}_i$,
$$P_i \stackrel{\mathrm{def}}{=} (\phi_i - \phi_1) \cdots (\phi_i - \phi_{i-1})(\phi_{i+1} - \phi_i) \cdots (\phi_m - \phi_i),$$
and $0 < \phi_1 < \phi_2 < ... < \phi_m$ are smooth functions such that
$$\phi_i = \begin{cases} \phi_i(\bar{x}_i), & \text{if } \ k_i = 1 \\ \text{constant}, & \text{otherwise.} \end{cases}$$
2. *Let g be a Riemannian metric and L be a (1,1)-tensor. If in a neighborhood $U \subset M^n$ there exist coordinates $\bar{x} = (\bar{x}_1, ..., \bar{x}_m)$ such that g and L are given by formulae (4, 5), then the restriction of L to U is a BM-structure for the restriction of g to U.*

Corollary 5 ([9],[39]). *The Nijenhuis torsion of a BM-structure vanishes.*

Remark 3. *In Levi-Civita's coordinates from Theorem 5, the metric $\bar{g}$ given by (1) has the form*
$$\begin{aligned} \bar{g}(\dot{\bar{x}}, \dot{\bar{x}}) &= \rho_1 P_1(\bar{x}) A_1(\bar{x}_1, \dot{\bar{x}}_1) + \rho_2 P_2(\bar{x}) A_2(\bar{x}_2, \dot{\bar{x}}_2) + \cdots + \\ &+ \rho_m P_m(\bar{x}) A_m(\bar{x}_m, \dot{\bar{x}}_m), \end{aligned} \tag{6}$$
where
$$\rho_i = \frac{1}{\phi_1^{k_1} ... \phi_m^{k_m}} \frac{1}{\phi_i}$$
The metrics g and $\bar{g}$ are affine equivalent if and only if all functions ϕ_i are constant.

Let p be a typical point with respect to the BM-structure L. Fix $i \in 1, ..., n$ and a small neighborhood U of p. At every point of U, consider the eigenspace V_i with the eigenvalue ϕ_i. If the neighborhood is small enough, it contains only typical points and V_i is a distribution. Denote by $M_i(p)$ the integral manifold containing p.

Levi-Civita's Theorem says that the eigenvalues ϕ_j, $j \neq i$, are constant on $M_i(p)$, and that the restriction of g to $M_i(p)$ is proportional to the restriction of g to $M_i(q)$, if it is possible to connect q and p by a line orthogonal to M_i. We will need the second observation later and formulate it as

Corollary 6. *Let L be a BM-structure for connected (M^n, g). Suppose the curve $\gamma : [0,1] \to M^n$ contains only typical points and is orthogonal to $M_i(p)$ at every point $p \in Image(\gamma)$. Let the multiplicity of the eigenvalue ϕ_i at every point of the curve be greater than one. Then, the restriction of the metric to $M_i(\gamma(0))$ is proportional to the restriction of the metric to $M_i(\gamma(1))$. (i.e. there exists a diffeomorphism of a small neighborhood $U_i(\gamma(0)) \subset M_i(\gamma(0))$ to a small neighborhood $U_i(\gamma(1)) \subset M_i(\gamma(1))$ taking the restriction of the metric g to $M_i(\gamma(0))$ to a metrics proportional to the restriction of the metric g to $M_i(\gamma(1))$).*

Definition 4. *Let (M^n, g) be a Riemannian manifold. We say that the metric g has a* **warped decomposition** *near $x \in M^n$, if a neighborhood U^n of x can be split in the direct product of disks $D^{k_0} \times ... \times D^{k_m}$, $k_0 + ... + k_m = n$, such that the metric g has the form*

$$g_0 + \sigma_1 g_1 + \sigma_2 g_2 + ... + \sigma_m g_m, \tag{7}$$

where the ith *metric g_i is a Riemannian metric on the corresponding disk D^{k_i}, and functions σ_i are functions on the disk D^{k_0}. The metric*

$$g_0 + \sigma_1 dy_1^2 + \sigma_2 dy_2^2 + ... + \sigma_m dy_m^2 \tag{8}$$

on $D^{k_0} \times \mathbb{R}^m$ is called the **adjusted metric***.*

We will always assume that k_0 is at least 1.

Comparing formulae (5,7), we see that if L has at least one simple eigenvalue at a typical point, Levi-Civita's Theorem gives us a warped decomposition near every typical point of M^n: the metric g_0 collects all $P_i A_i$ from (5) such that ϕ_i has multiplicity one, the metrics g_1,...,g_m coincide with A_j for multiple ϕ_j, and $\sigma_j = P_j$.

Definition 5 ([52, 53]). *Let K be a constant. A metric g is called a $V(K)$-***metric** *near $x \in M^n$ $(n \geq 3)$, if there exist coordinates in a neighborhood of x such that g has the Levi-Civita form (5) such that the adjusted metric has constant sectional curvature K.*

The definition above is independent of the choice of the presentation of g in Levi-Civita's form:

Theorem 6 ([52, 53]). *Suppose g is a $V(K)$-metric near $x \in M^n$. Assume $n \geq 3$. The following statements hold:*

1. *If there exists another presentation of g (near x) in the form (5), then the sectional curvature of the adjusted metric constructed for this other decomposition is constant and is equal to K.*
2. *Consider the metric (5). For every $i = 1, ..., m$, denote*

$$\frac{g(grad(P_i), grad(P_i))}{4\,P_i} + KP_i \tag{9}$$

by K_i. Then, the metric (7) has constant sectional curvature if and only if for every $i \in 0, ..., m$ such that $k_i > 1$ the metric A_i has constant sectional curvature K_i. More precisely, if the metric (7) is a $V(K)$-metric, if $k_1 > 1$ and if the metric A_1 has constant curvature K_1, then the metric $g_0 + P_1 A_1$ has constant curvature K.

3. *For a fixed presentation of g in the Levi-Civita form (5), for every i such that $k_i > 1$, K_i is a constant.*

Since the papers [52, 53] are not widely accessible, we will comment the proof of this theorem. The first statement of Theorem 6 is proven in §3 of [52]. In the form sufficient for our paper, it appeared already in [58]; although it is hidden there.

The second statement is in §8 of [52]. One can understand the second statement with the help of projective Weyl tensor from [60]. We will give the definition in Section 3.3, see formula (18) there. It is known [60], that (in dimension ≥ 3) the projective Weyl tensor vanishes if and only if the metric has constant sectional curvature. Now, it is possible to show by direct calculations that the projective Weyl tensor vanishes for a metric of form (5), if and only if the sectional curvatures of all A_i are equal to the corresponding K_i. In Section 3.3, we will do these calculations for one component of the projective Weyl tensor; the calculations for the other components are similar.

The third statement can be found in §8 of [52]. Its proof is similar to the standard proof of the fact that (for dimensions ≥ 3) if a metric has constant sectional curvature at every point, then the constant does not depend on the point.

The relation between $V(K)$-metrics and BM-structures is given by

Theorem 7 ([52, 54, 55]). *Let (D^n, g) be a disc of dimension $n \geq 3$ with two BM-structures L_1 and L_2 such that every point of the disc is typical with respect to both structures and the BM-structures* Id, *L_1, L_2 are linearly independent. Then, g is a $V(K)$-metric near every point.*

Its corollary is

Theorem 8 (Fubini's Theorem) . *Let (D^n, g) be a disc of dimension $n \geq 3$ with two BM-structures L_1 and L_2 such that $N_{L_1} = N_{L_2} = n$ at every point. If the BM-structures* Id, *L_1, L_2 are linearly independent, then g has constant sectional curvature.*

For dimension 2, Fubini's Theorem is wrong. First counterexamples can be found in [28]. We will give new counterexamples in [46]. Fubini's Theorem was proven by Fubini [18] for dimension 3, and was announced there and in [19] for arbitrary dimension ≥ 3. One can check that Fubini's proof for dimension 3 can be applied to every dimension ≥ 3 without essential changes.

In Section 3.4, we will explain how Solodovnikov's Theorem 7 follows from Fubini's Theorem 8.

3 Singularity theory for BM-structures

We will need the following technical lemma. For every fixed $v = (\xi_1, \xi_2, ..., \xi_n) \in T_xM^n$, the function (3) is a polynomial in t. Consider the roots of this polynomial. From the proof of Lemma 1, it will be clear that they are real. We denote them by

$$t_1(x,v) \leq t_2(x,v) \leq ... \leq t_{n-1}(x,v).$$

Lemma 1. *Suppose λ is an eigenvalue of L of multiplicity k at $x \in M^n$. Then, for every $v \in T_xM^n$, λ is a root of $I_t(v)$ of multiplicity at least $k-1$.*

Proof: By definition, the tensor L is self-adjoint with respect to g. Then, for every $x \in M^n$, there exist "diagonal" coordinates in T_xM^n where the metric g is given by the diagonal matrix $\mathrm{diag}(1, 1, ..., 1)$ and the tensor L is given by the diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$. Then, the tensor (2) reads:

$$\begin{aligned} S_t &= \det(L - t\mathrm{Id})(L - t\mathrm{Id})^{(-1)} \\ &= \mathrm{diag}(\Pi_1(t), \Pi_2(t), ..., \Pi_n(t)), \end{aligned}$$

where the polynomials $\Pi_i(t)$ are given by the formula

$$\Pi_i(t) \stackrel{\mathrm{def}}{=} (\lambda_1 - t)(\lambda_2 - t)...(\lambda_{i-1} - t)(\lambda_{i+1} - t)...(\lambda_{n-1} - t)(\lambda_n - t).$$

Hence, for every $v = (\xi_1, ..., \xi_n) \in T_xM^n$, the polynomial $I_t(x, v)$ is given by

$$I_t = \xi_1^2\Pi_1(t) + \xi_2^2\Pi_2(t) + ... + \xi_n^2\Pi_n(t). \qquad (10)$$

We see that, if λ is an eigenvalue of multiplicity k, every Π_i contains the factor $(\lambda - t)^{k-1}$. Lemma is proven.

3.1 Behavior of BM-structure near simplest non-typical points.

Within this section we assume that L is a BM-structure on a connected (M^n, g). As in Section 2.1, we denote by $\lambda_1(x) \leq ... \leq \lambda_n(x)$ the eigenvalues of L, and by $N_L(x)$ the number of different eigenvalues of L at $x \in M^n$.

Theorem 9. *Suppose the eigenvalue λ_1 is not constant, the eigenvalue λ_2 is constant and $N_L = 2$ in a typical point. Let p be a non-typical point. Then, the following statements hold:*

1. *The spheres of small radius with center in p are orthogonal to the eigenvector of L corresponding to λ_1, and tangent to the eigenspace of L corresponding to λ_2. In particular, the points where $\lambda_1 = \lambda_2$ are isolated.*

2. *The restriction of the metric to the spheres has constant sectional curvature.*

Proof: Since λ_1 is not constant, it is a simple eigenvalue in every typical point. Since $N_L = 2$, the roots $\lambda_2, \lambda_3, ..., \lambda_n$ coincide at every point and are constant. We denote this constant by λ. By Lemma 1, at every point $(x, \xi) \in T_xM^n$, the number λ is a root of multiplicity at least $n-2$ of the polynomial $I_t(x, \xi)$. Then,

$$I'_t(x,\xi) := \frac{I_t(x,\xi)}{(\lambda - t)^{n-2}}$$

is a linear function in t and, for every fixed t, is an integral of the geodesic flow of g. Denote by $\tilde{I} : TM \to \mathbb{R}$ the function

$$\tilde{I}(x,\xi) := I'_\lambda(x,\xi) := (I'_t(x,\xi))_{|t=\lambda}\,.$$

Since λ is a constant, the function $\tilde{I}$ is an integral of the geodesic flow of g. At every tangent space T_xM^n, consider the coordinates such that the metric is given by $\mathrm{diag}(1, ..., 1)$ and L is given by $\mathrm{diag}(\lambda_1, \lambda, ..., \lambda)$. By direct calculations we see that the restriction of $\tilde{I}$ to T_xM^n is given by (we assume $\xi = (\xi_1, \xi_2, ..., \xi_n)$)

$$\tilde{I}_{|T_xM^n}(\xi) = (\lambda_1(x) - \lambda)(\xi_2^2 + ... + \xi_n^2).$$

Thus, for every geodesic γ passing through p, the value of $\tilde{I}(\gamma(\tau), \dot{\gamma}(\tau))$ is zero. Then, for every typical point of such geodesic, since $\lambda_1 < \lambda$, the components $\xi_2, ..., \xi_n$ of the velocity vector vanish. Then, the velocity vector is an eigenvector of L with the eigenvalue λ_1.

Then, the points where $\lambda_1 = \lambda$ are isolated: otherwise we can pick two such points p_1 and p_2 lying in a ball with radius less than the radius of injectivity. Then, for almost every point q of the ball, the geodesics connecting this point with the points p_1 and p_2 intersect transversally at q. Then, the point q is non-typical; otherwise the eigenspace of λ_1 contains the velocity vectors of geodesics and is no more one-dimensional. Finally, almost every point of the ball is not typical, which contradicts Corollary 3. Thus, the points where $\lambda_1 = \lambda$ are isolated.

It is known (Lemma of Gauß), that the geodesics passing through p intersect the spheres of small radius with center in p orthogonally. Since the velocity vectors of such geodesics are eigenvectors of L with eigenvalue λ_1, then the eigenvector with eigenvalue λ_1 is orthogonal to the spheres of small radius with center in p. Since L is self-adjoint, the spheres are tangent to the eigenspaces of λ. The first statement of Theorem 9 is proven.

The second statement of Theorem 9 is trivial, if $n = 2$. In order to prove the second statement for $n \geq 3$, we will use Corollary 6. The role of the curve γ from Corollary 6 plays the geodesic passing through p. We put $i = 2$. By the first statement of Theorem 9, $M_i(x)$ are spheres with center in p. Then, by Corollary 6, for every sufficiently small spheres S_{ϵ_1} and S_{ϵ_2} with center in p, the restriction of g to the first sphere is proportional to the restriction of g to the second sphere. Since for very small ϵ the metric in a ϵ-ball is very close to the Euclidean metric, the restriction of g to the ϵ-sphere is close to the round metric of the sphere. Thus, the restriction of g to every (sufficiently small) sphere has constant sectional curvature. Theorem 9 is proven.

Theorem 10. *Suppose $N_L = 3$ at a typical point and there exists a point where $N_L = 1$. Then, there exist points p_1, p_n such that $\lambda_1(p_1) < \lambda_2(p_1) = \lambda_n(p_1)$ and $\lambda_1(p_n) = \lambda_2(p_n) < \lambda_n(p_n)$.*

Proof: Suppose $\lambda_1(p_2) = \lambda_2(p_2) = ... = \lambda_n(p_2)$ and the number of different eigenvalues of L at a typical point equals three. Then, by Corollary 2, the eigenvalues $\lambda_2 = ... = \lambda_{n-1}$ are constant. We denote this constant by λ. Take a ball B of small radius with center in p_2. We will prove that this ball has a point p_1 such that $\lambda_1(p_1) < \lambda_2 = \lambda_n(p_1)$; the proof that there exists a point where $\lambda_1 = \lambda_2 < \lambda_n$ is similar. Take $p \in B$ such that $\lambda_1(p) < \lambda$ and $\lambda_1(p)$ is a regular value of the function λ_1. Denote by $\check{M}_1(p)$ the connected component of $\{q \in M^n : \lambda_1(q) = \lambda_1(p)\}$ containing the point p. Since $\lambda_1(p)$ is a regular value, $\check{M}_1(p)$ is a submanifold of codimension 1. Then, there exists a point $p_1 \in \check{M}_1(p)$ such that the distance from this point to p_2 is minimal over all points of $\check{M}_1(p)$.

Let us show that $\lambda_1(p_1) < \lambda = \lambda_n(p_1)$. The inequality $\lambda_1(p_1) < \lambda$ is fulfilled by definition, since $p_1 \in \check{M}_1(p)$. Let us prove that $\lambda_n(p_n) = \lambda$.

Consider the shortest geodesic γ connecting p_2 and p_1. We will assume $\gamma(0) = p_1$ and $\gamma(1) = p_2$. Consider the values of the roots $t_1 \leq ... \leq t_{n-1}$ of the polynomial I_t at points of the geodesic orbit $(\gamma, \dot{\gamma})$. Since I_t are integrals, the roots t_i are independent of the point of the orbit. Since the geodesic pass through the point where $\lambda_1 = ... = \lambda_n$, by Lemma 1, we have

$$t_1 = ... = t_{n-1} = \lambda. \tag{11}$$

Since the distance from p_1 to p_2 is minimal over all points of $\check{M}_1$, the velocity vector $\dot{\gamma}(0)$ is orthogonal to $\check{M}_1$. In view of Corollary 5, the sum of eigenspaces of L corresponding to λ and λ_n is tangent to $\check{M}_1$. Hence, the vector $\dot{\gamma}(0)$ is an eigenvector of L with eigenvalue λ_1.

At the tangent space $T_{p_1}M^n$, choose a coordinate system such that L is diagonal $\text{diag}(\lambda_1, ..., \lambda_n)$ and g is Euclidean $\text{diag}(1, ..., 1)$. In this coordinate system, $I_t(\xi)$ is given by (we assume $\xi = (\xi_1, ..., \xi_n)$)

$$(\lambda - t)^{n-3}\left((\lambda_n - t)(\lambda - t)\xi_1^2 + (\lambda_n - t)(\lambda_1 - t)(\xi_2^2 + ... + \xi_{n-1}^2) + (\lambda_1 - t)(\lambda - t)\xi_n^2\right).$$

Since $\dot{\gamma}(0)$ is an eigenvector of L with eigenvalue λ_1, the last $n - 1$ components of $\dot{\gamma}(0)$ vanish, so that $t_{n-1} = \lambda_n$. Comparing this with (11), we see that $\lambda_n(p_1) = \lambda$. Theorem 10 is proven.

3.2 Splitting Lemma

Definition 6. *A local-product structure on M^n is the triple (h, B_r, B_{n-r}), where h is a Riemannian metrics and B_r, B_{n-r} are transversal foliations of dimensions r and $n - r$, respectively (it is assumed that $1 \leq r < n$), such that every point $p \in M^n$ has a neighborhood $U(p)$ with coordinates*

$$(\bar{x}, \bar{y}) = \big((x_1, x_2, ...x_r), (y_{r+1}, y_{r+2}, ..., y_n)\big)$$

such that the x-coordinates are constant on every leaf of the foliation $B_{n-r} \cap U(p)$, the y-coordinates are constant on every leaf of the foliation $B_r \cap U(p)$, and the metric h is block-diagonal such that the first $(r \times r)$ block depends on the x-coordinates and the last $((n-r) \times (n-r))$ block depends on the y-coordinates.

A model example of manifolds with local-product structure is the direct product of two Riemannian manifolds (M_1^r, g_1) and (M_2^{n-r}, g_2). In this case, the leaves of the foliation B_r are the products of M_1^r and the points of M_2^{n-r}, the leaves of the foliation B_{n-r} are the products of the points of M_1^r and M_2^{n-r}, and the metric h is the product metric $g_1 + g_2$.

Below we assume that

(a) L is a BM-structure for a connected (M^n, g).

(b) There exists r, $1 \leq r < n$, such that $\lambda_r < \lambda_{r+1}$ at every point of M^n.

We will show that (under the assumptions (a,b)) we can naturally define a local-product structure (h, B_r, B_{n-r}) such that (the tangent spaces to) the leaves of B_r and B_{n-r} are invariant with respect to L, and such that the restrictions $L_{|B_r}$, $L_{|B_{n-r}}$ are BM-structures for the metrics $h_{|B_r}$, $h_{|B_{n-r}}$, respectively.

At every point $x \in M^n$, denote by V_x^r the subspaces of T_xM^n spanned by the eigenvectors of L corresponding to the eigenvalues $\lambda_1, ..., \lambda_r$. Similarly, denote by V_x^{n-r} the subspaces of T_xM^n spanned by the eigenvectors of L corresponding to the eigenvalues $\lambda_{r+1}, ..., \lambda_n$. By assumption, for every i, j such that $i \leq r < j$, we have $\lambda_i \neq \lambda_j$ so that V_x^r and V_x^{n-r} are two smooth distributions on M^n. By Corollary 5, the distributions are integrable so that they define two transversal foliations B_r and B_{n-r} of dimensions r and $n-r$, respectively.

By construction, the distributions V_r and V_{n-r} are invariant with respect to L. Let us denote by L_r, L_{n-r} the restrictions of L to V_r and V_{n-r}, respectively. We will denote by χ_r, χ_{n-r} the characteristic polynomials of L_r, L_{n-r}, respectively. Consider the (1,1)-tensor

$$C \stackrel{\text{def}}{=} ((-1)^r \chi_r(L) + \chi_{n-r}(L))$$

and the metric h given by the relation

$$h(u, v) \stackrel{\text{def}}{=} g(C^{-1}(u), v)$$

for every vectors u, v. (In the tensor notations, the metrics h and g are related by $g_{ij} = h_{i\alpha} C_j^\alpha$).

Lemma 2 (Splitting Lemma). *The following statements hold:*

1. *The triple (h, B_r, B_{n-r}) is a local-product structure on M^n.*
2. *For every leaf of B_r, the restriction of L to it is a BM-structure for the restriction of h to it. For every leaf of B_{n-r}, the restriction of L to it is a BM-structure for the restriction of h to it.*

Proof: First of all, h is a well-defined Riemannian metric. Indeed, take an arbitrary point $x \in M^n$. At the tangent space to this point, we can find a coordinate system such that the tensor L and the metric g are diagonal. In this coordinate system, the characteristic polynomials χ_r, χ_{n-r} are given by

$$\begin{array}{lcc} (-1)^r\chi_r & = & (t-\lambda_1)(t-\lambda_2)...(t-\lambda_r) \\ \chi_{n-r} & = & (\lambda_{r+1}-t)(\lambda_{r+2}-t)...(\lambda_n-t). \end{array} \tag{12}$$

Then, the (1,1)-tensor

$$C = ((-1)^r\chi_r(L)+\chi_{n-r}(L))$$

is given by the diagonal matrices

$$\mathrm{diag}\left(\prod_{j=r+1}^{n}(\lambda_j-\lambda_1),...,\prod_{j=r+1}^{n}(\lambda_j-\lambda_r),\prod_{j=1}^{r}(\lambda_{r+1}-\lambda_j),...,\prod_{j=1}^{r}(\lambda_n-\lambda_j)\right), \tag{13}$$

We see that the tensor is diagonal and that all diagonal components are positive. Then, the tensor C^{-1} is well-defined and h is a Riemannian metric.

By construction, B_r and B_{n-r} are well-defined transversal foliations of supplementary dimensions. In order to prove Lemma 2, we need to verify that, locally, the triple (h, B_r, B_{n-r}) is as in Definition 6, that the restriction of L to a leaf is a BM-structure for the restriction of h to the leaf.

It is sufficient to verify these two statements at almost every point of M^n. More precisely, it is known that the triple (h, B_r, B_{n-r}) is a local-product structure if and only if the foliations B_r and B_{n-r} are orthogonal and totally geodesic. Clearly, if the foliations and the metric are globally given and smooth, if the foliations are orthogonal and totally-geodesic at almost every point, then they are orthogonal and totally-geodesic at every point.

Similarly, since the foliations and the metric are globally-given and smooth, if the restriction of L satisfies Definition 1 at almost every point, then it satisfies Definition 1 at every point.

Consider Levi-Civita's coordinates $\bar{x}_1, ..., \bar{x}_m$ from Theorem 5. As in Levi-Civita's Theorem, we denote by $\phi_1 < ... < \phi_m$ the different eigenvalues of L. In Levi-Civita's coordinates, the matrix of L is diagonal

$$\mathrm{diag}\left(\underbrace{\phi_1,...,\phi_1}_{k_1},...,\underbrace{\phi_m,...,\phi_m}_{k_m}\right) = \mathrm{diag}(\lambda_1,...,\lambda_n).$$

Consider s such that $\phi_s = \lambda_r$ (clearly, $k_1 + ... + k_s = r$). Then, by constructions of the foliations B_r and B_{n-r}, the coordinates $\bar{x}_1, ..., \bar{x}_s$ are constant on every leaf of the foliation B_{n-r}, the coordinates $\bar{x}_{s+1}, ..., \bar{x}_m$ are constant on every leaf of the foliation B_r. The coordinates $\bar{x}_1, ..., \bar{x}_s$ will play the role of x-coordinates from Definition 6, and the coordinates $\bar{x}_{s+1}, ..., \bar{x}_m$ will play the role of y-coordinates from Definition 6.

Using (13), we see that, in Levi-Civita's coordinates, C is given by

$$\mathrm{diag}\left(\underbrace{\prod_{j=s+1}^{m}(\phi_j-\phi_1)^{k_j},\dots,\prod_{j=s+1}^{m}(\phi_j-\phi_1)^{k_j}}_{k_1},\dots,\underbrace{\prod_{j=s+1}^{m}(\phi_j-\phi_s)^{k_j},\dots,\prod_{j=s+1}^{m}(\phi_j-\phi_s)^{k_j}}_{k_s},\right.$$
$$\left.\underbrace{\prod_{j=1}^{s}(\phi_{s+1}-\phi_j)^{k_j},\dots,\prod_{j=1}^{s}(\phi_{s+1}-\phi_j)^{k_j}}_{k_{s+1}},\dots,\underbrace{\prod_{j=1}^{s}(\phi_m-\phi_j)^{k_j},\dots,\prod_{j=1}^{s}(\phi_m-\phi_j)^{k_j}}_{k_m}\right). \quad (14)$$

Thus, h is given by

$$\begin{array}{rcllll} h(\dot{\bar{x}},\dot{\bar{x}}) & = & \tilde{P}_1 A_1(\bar{x}_1,\dot{\bar{x}}_1) & +\dots+ & \tilde{P}_s A_s(\bar{x}_s,\dot{\bar{x}}_s) & \\ & + & \tilde{P}_{s+1} A_{s+1}(\bar{x}_{s+1},\dot{\bar{x}}_{s+1}) & +\dots+ & \tilde{P}_m A_m(\bar{x}_m,\dot{\bar{x}}_m), & \end{array} \quad (15)$$

where the functions $\tilde{P}_i$ are as follows: for $i \le r$, they are given by

$$\tilde{P}_i \stackrel{\mathrm{def}}{=} (\phi_i-\phi_1)\dots(\phi_i-\phi_{i-1})(\phi_{i+1}-\phi_i)\dots(\phi_s-\phi_i) \prod_{\substack{j=1\\ j\ne i}}^{s} |\phi_i-\phi_j|^{k_j-1}.$$

For $i > r$, the functions $\tilde{P}_i$ are given by

$$\tilde{P}_i \stackrel{\mathrm{def}}{=} (\phi_i-\phi_{s+1})\dots(\phi_i-\phi_{i-1})(\phi_{i+1}-\phi_i)\dots(\phi_m-\phi_i) \prod_{\substack{j=s+1\\ j\ne i}}^{m} |\phi_i-\phi_j|^{k_j-1}.$$

Clearly, $|\phi_i-\phi_j|^{k_j-1}$ can depend on the variables $\bar{x}_i$ only; moreover, if ϕ_i is multiple, $|\phi_i-\phi_j|^{k_j-1}$ is a constant. Then, the products

$$\prod_{\substack{j=1\\ j\ne i}}^{s} |\phi_i-\phi_j|^{k_j-1}$$

can be hidden in A_i, so that the the restriction of the metric to the leaves of B_r has the form from Levi-Civita's Theorem, and, therefore, the restriction of L is a BM-structure for it. We see that the leaves of B_r are orthogonal to leaves of B_{n-r}, and that the restriction of h to B_r (B_{n-r}, respectively) is precisely the first row of (15) (second row of (15), respectively) and depends on the coordinates $\bar{x}_1,\dots,\bar{x}_s$ ($\bar{x}_{s+1},\dots,\bar{x}_m$, respectively) only. Lemma 2 is proven.

Let p be a typical point for g with respect to BM-structure L. Fix $i \in 1, ..., n$. At every point of M^n, consider the eigenspace V_i with the eigenvalue λ_i. V_i is a distribution near p. Denote by $M_i(p)$ its integral manifold containing p.

Remark 4. *The following statements hold:*

1. *If $\lambda_i(p)$ is multiple, the restriction of g to $M_i(p)$ is proportional to the restriction of h to $M_i(p)$.*
2. *The restriction of L to B_r does not depend on the coordinates $y_{r+1}, ..., y_n$ (which are coordinates $\bar{x}_{s+1}, ..., \bar{x}_m$ in the notations in proof of Lemma 2). The restriction of L to B_{n-r} does not depend on the coordinates $x_1, ..., x_r$ (which are coordinates $\bar{x}_1, ..., \bar{x}_s$ in the notations in proof of Lemma 2).*

Combining Lemma 2 with Theorem 9, we obtain

Corollary 7. *Let L be BM-structure on connected (M^n, g). Suppose there exist $i \in 1, ..., n$ and $p \in M^n$ such that:*

- *λ_i is multiple (with multiplicity $k \geq 2$) at a typical point.*
- *$\lambda_{i-1}(p) = \lambda_i(p) < \lambda_{i+k}(p)$,*
- *The eigenvalue λ_{i-1} is not constant.*

Then, for every typical point $q \in M^n$ which is sufficiently close to p, $M_i(q)$ is diffeomorphic to the sphere and the restriction of g to $M_i(q)$ has constant sectional curvature.

Indeed, take a small neighborhood of p and apply Splitting Lemma 2 two times: for $r = i + k - 1$ and for $r = i - 1$. We obtain a metric h such that locally, near p, the manifold with this metric is the Riemannian product of three discs with BM-structures, and BM-structure is the direct sum of these BM-structures. The second component of such decomposition satisfies the assumption of Theorem 9; applying Theorem 9 and Remark 4 we obtain what we need.

Arguing as above, combining Lemma 2 with Theorem 10, we obtain

Corollary 8. *Let L be a BM-structure for connected (M, g). Suppose the eigenvalue λ_i has multiplicity k at a typical point. Suppose there exists a point where the multiplicity of λ_i is greater than k. Then, there exists a point where the multiplicity of λ_i is precisely $k + 1$.*

Combining Lemma 2 with Corollary 2, we obtain

Corollary 9. *Let L be a BM-structure for connected (M^n, g). Suppose the eigenvalue λ_i has multiplicity k_i at a typical point and multiplicity $k_i + d$ at a point $p \in M^n$. Then, there exists a point $q \in M^n$ in a small neighborhood of p such that the eigenvalue λ_i has multiplicity $k_i + d$ in p, and such that*

$$N_L(q) = \max_{x \in M^n} (N_L(x)) - d.$$

We saw that under hypotheses of Theorems 9,10, the set of typical points is connected. As it was shown in [34], in dimension 2 the set of typical points is connected as well. Combining these observations with Lemma 2, we obtain

Corollary 10. *Let L be a BM-structure on connected (M^n, g). Then, the set of typical points of L is connected.*

3.3 If ϕ_i is not isolated and if $\dim(\mathcal{B}(M^n, g)) \geq 3$, then A_i has constant sectional curvature K_i.

In this section we assume that (M^n, g) is connected and complete and L is a BM-structure for M^n. As usual, we denote by $\lambda_1(x) \leq ... \leq \lambda_n(x)$ the eigenvalues of L at $x \in M^n$.

Definition 7. *An eigenvalue λ_i is called isolated, if $\lambda_i(p_1) = \lambda_j(p_1)$ implies $\lambda_i(p_2) = \lambda_j(p_2)$ for every point p_2.*

As in Section 3.2, at every point $p \in M^n$, we denote by V_i the eigenspace of L with the eigenvalue $\lambda_i(p)$. V_i is a distribution near every typical point; by Corollary 5, it is integrable. We denote by $M_i(p)$ the connected component containing p of the intersection of the integral manifold with a small neighborhood of p.

Theorem 11. *Suppose λ_i is a non-isolated eigenvalue. Then, for every typical point p, the restriction of g to $M_i(p)$ has constant sectional curvature.*

It could be easier to understand this Theorem using the language of Levi-Civita's Theorem 5: denote by $\phi_1 < \phi_2 < ... < \phi_m$ the different eigenvalues of L at a typical point. Theorem 11 says that, if ϕ_i is non-isolated, then A_i from Levi-Civita's Theorem has constant sectional curvature.

Proof of Theorem 11: If eigenvalue λ_i is simple at a typical point, M_i is one dimensional and the statement is trivial; below we assume that λ_i is multiple. Let $k_i > 1$ be the multiplicity of λ_i at a typical point. Then, λ_i is constant. Take a typical point p. We assume that λ_i is not isolated; without loss of generality, we can suppose $\lambda_i(p_1) = \lambda_{i+k_i-1}(p_1)$ for some point p_1. By Corollary 8, without loss of generality, we can assume $\lambda_{i-1}(p_1) = \lambda_i(p_1) < \lambda_{i+k_i}(p_1)$. By Corollary 9, we can also assume that $N_L(p_1) = N_L(p) - 1$.

Consider a geodesic $\gamma : [0,1] \to M^n$ connecting p_1 and p, $\gamma(0) = p$ and $\gamma(1) = p_1$. Since it is sufficient to prove Theorem 11 at almost every typical point, arguing as in proof of Corollary 2 in [43], without loss of generality, we can assume that p_1 is the only non-typical point of the geodesic segment $\gamma(\tau)$, $\tau \in [0,1]$.

Take a point $q := \gamma(1-\epsilon)$ of the segment, where $\epsilon > 0$ is small enough. By Corollary 7, the restriction of g to $M_i(q)$ has constant sectional curvature.

Let us prove that the geodesic segment $\gamma(\tau)$, $\tau \in [0, 1-\epsilon]$ is orthogonal to $M_i(\gamma(\tau))$ at every point.

Indeed, consider the function

$$\tilde{I}: TM^n \to \mathbb{R};\ \tilde{I}(x,\xi) := \left(\frac{I_t(x,\xi)}{(\lambda_i - t)^{k_i-1}}\right)_{|t=\lambda_i}.$$

Since the multiplicity of λ_i at every point is at least k_i, the function $\left(\frac{I_t(x,\xi)}{(\lambda_i - t)^{k_i-1}}\right)$ is polynomial in t of degree $n-k_i$ and is an integral for every fixed t; since λ_i is a constant, the function $\tilde{I}$ is an integral.

At the tangent space to every point of geodesic γ consider the coordinates such that $L = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ and $g = \mathrm{diag}(1, ..., 1)$. In this coordinates, $I_t(\xi)$ is given by (10). Then, the integral $\tilde{I}(\xi)$ is the sum (we assume $\xi = (\xi_1, ..., \xi_n)$)

$$\left(\sum_{\alpha=i}^{i+k_i-1}\left(\xi_\alpha^2 \prod_{\substack{\beta=1\\ \beta\neq i,i+1,...,i+k_i-1}}^{n}(\lambda_\alpha-\lambda_i)\right)\right) \tag{16}$$

$$+\left(\sum_{\substack{\alpha=1\\ \alpha\neq i,i+1,...,i+k_i-1}}^{n}\left(\xi_\alpha^2 \prod_{\substack{\beta=1\\ \beta\neq i+1,...,i+k_i-1\\ \beta\neq\alpha}}^{n}(\lambda_\alpha-\lambda_i)\right)\right). \tag{17}$$

Since the geodesic passes through the point where $\lambda_{i-1} = \lambda_i = ... = \lambda_{i+k_i-1}$, all products in the formulae above contain the factor $\lambda_i - \lambda_i$, and, therefore, vanish, so that $\tilde{I}(\gamma(0), \dot{\gamma}(0)) = 0$. Since $\tilde{I}$ is an integral, $\tilde{I}(\gamma(\tau), \dot{\gamma}(\tau)) = 0$ for every τ. Let us show that it implies that the geodesic is orthogonal to M_i at every typical point, in particular, at points lying on the segment $\gamma(\tau)$, $\tau \in [0,1]$.

Clearly, every term in the sum (17) contains the factor $\lambda_i - \lambda_i$, and, therefore, vanishes. Then, the integral $\tilde{I}$ is equal to (16).

At a typical point, we have

$$\lambda_1 \leq ... \leq \lambda_{i-1} < \lambda_i = ... = \lambda_{i+k_i-1} < \lambda_{k_i} \leq ... \leq \lambda_n.$$

Then, all products

$$\prod_{\substack{\beta=1\\ \beta\neq i,i+1,...,i+k_i-1}}^{n}(\lambda_\alpha-\lambda_i)$$

have the same sign and are nonzero. Then, all components ξ_α, $\alpha \in i, ..., k_i - 1$ vanish. Thus, γ is orthogonal to M_i at every typical point.

Finally, by Corollary 6, the restriction of g to $M_i(p)$ is proportional to the restriction of g to $M_i(q)$ and, hence, has constant sectional curvature. Theorem is proven.

Theorem 12. *Suppose* $\dim(\mathcal{B}(M^n, g)) \geq 3$. *Let* ϕ_i *be a non-isolated eigenvalue of* L *such that its multiplicity at a typical point is at least two. Then, the sectional curvature of* A_i *is equal to* K_i.

Recall that the definition of K_i is in the second statement of Theorem 6.
Proof of Theorem 12: Let us denote by $\bar{K}_i$ the sectional curvature of the metric A_i. By assumptions, it is constant in a neighborhood of every typical point. Since by Corollary 10, the set of typical points is connected, $\bar{K}_i$ is independent of a typical point. Similarly, since K_i is locally-constant by Theorem 6, K_i is independent of a typical point. Thus, it is sufficient to find a point where $\bar{K}_i = K_i$.

Without loss of generality, we can suppose that there exists $p_1 \in M^n$ such that $\lambda_r(p_1) = \lambda_{r+1}$.

By Corollary 9, without loss of generality we can assume that the multiplicity of λ_{r+1} is $k_i + 1$ in p_1, and that $N_L(p_1) = m - 1$. Take a typical point p in a small neighborhood of p_1.

Then, by Corollary 7, the submanifold $M_{r+1}(p)$ is homeomorphic to the sphere. Since it is compact, there exists a set of local coordinates charts on it such that there exist constants const and CONST such that, in every chart $(x_i^1, ..., x_i^{k_i})$, for every $\beta \in \{1, .., k_i\}$, the entry $(A_i)_{\beta\beta}$ lies between const and CONST, (i.e. CONST $\geq A_i(\frac{\partial}{\partial x_i^\beta}, \frac{\partial}{\partial x_i^\beta}) \geq$ const.)

By shifting these local coordinates along the vector fields $\frac{\partial}{\partial x_j^k}$, where $j \neq i$, for every typical point p' in a neighborhood of p_1, we obtain coordinate charts on $M_{r+1}(p')$ such that CONST $\geq (A_i)_{\beta\beta} \geq$ const.

Let us calculate the projective Weyl tensor W for g in these local coordinate charts. Recall that the projective Weyl tensor is given by the formula

$$W^i_{jkl} := R^i_{jkl} - \frac{1}{n-1}\left(\delta^i_l \, R_{jk} - \delta^i_k \, R_{jl}\right) \tag{18}$$

We will be interested in the components (actually, in one component) of W corresponding to the coordinates $\bar{x}_i$. In what follows we reserve the Greek letter α, β for the coordinates from $\bar{x}_i$, so that, for example, $g_{\alpha\beta}$ will mean the component of the metric staying on the intersection of column number $r + \beta$ and row number $r + \alpha$.

As we will see below, the formulae will include only the components of A_i. To simplify the notations, we will not write subindex i near A_i, so for example, $g_{\alpha\beta}$ is equal to $P_i \, A_{\alpha\beta}$.

Let calculate the component $W^\alpha_{\beta\beta\alpha}$. In order to do it by formula (18), it is necessary to calculate $R^\alpha_{\beta\beta\alpha}$ and $R_{\beta\beta}$. These was done in §8 of [52]. Rewriting the results of Solodovnikov in our notations, we obtain

$$R^\alpha_{\beta\beta\alpha} = \left(\bar{K}_i - (K_i - K \, P_i)\right) A_{\beta\beta},$$

$$R_{\beta\beta} = \left((k_i - 1)\,\bar{K}_i + K\,(n-1)\,P_i - (k_i - 1)\,K_i\right)\;A_{\beta\beta}.$$

Substituting these expressions in (18), we obtain

$$W^{\alpha}_{\beta\beta\alpha} = (\bar{K}_i - K_i)\frac{n - k_i}{n-1}A_{\beta\beta}.$$

We see that, if $\bar{K}_i \neq K_i$, the component $W^{\alpha}_{\beta\beta\alpha}$ is bounded from zero.

Now if we consider a sequence of typical points converging to p_1, the component $W^{\alpha}_{\beta\beta\alpha}$ converge to zero, since the length of $\frac{\partial}{\partial x_i^{\beta}}$ goes to zero. Finally, $\bar{K}_i = K_i$. Theorem is proven.

3.4 Geometric sense of the adjusted metric

Consider the metric (8) on the product

$$D^{k_0} \times ... \times D^{k_m}.$$

Take a point $P = (p_0, ..., p_m) \in D^{k_0} \times ... \times D^{k_m}$. At every disk $D^{k_i},\ i = 1, ..., m$, consider a geodesic segment $\gamma_i \in D^{k_i}$ passing through p_i. Consider the product

$$M_A := D^{k_0} \times \gamma_1 \times \gamma_2 \times ... \times \gamma_m$$

as a submanifold of $D^{k_0} \times ... \times D^{k_m}$. As it easily follows from Definition 4,

- M_A is a totally geodesic submanifold.
- The restriction of the metric (8) to M_A is (isometric to) the adjusted metric.

Now let us explain how one can proof Theorem 7. Our proof is slightly different from the original proof of Solodovnikov [52] (which is correct and very good written).

If the dimension of M_A is one, Theorem 7 follows from Definition 4. Suppose the dimension of M_A is two. Consider two BM-structures L_1 and L_2 such that L_1, L_2 and Id are linearly independent, and such that the number of different eigenvalues of each BM-structure at each point is precisely two. Then, without loss of generality, locally there exists a coordinate system $(x_1, ..., x_n)$ such that

$$L = \operatorname{diag}(\lambda_1(x_1), \underbrace{\lambda_2, ..., \lambda_2}_{n-1}),$$

and g is given by the formula

$$(\lambda_1(x_1) - \lambda_2)(dx_1^2 + A_2), \tag{19}$$

where A_2 is a metric on the disk of dimension $(n-1)$ with coordinates $(x_2, ..., x_n)$, λ_2 is a constant, and λ_1 is a function of x_1. Consider the Ricci-tensor R^i_j of the metric (19). By direct calculation, it is possible to see that

- At every point, R^i_j has at most two different eigenvalues.
- If R^i_j has two eigenvalues, one eigenvalue has multiplicity 1. The corresponding eigenvector is $\frac{\partial}{\partial x_1}$.
- If R^i_j has precisely one eigenvalue in a neighborhood of a point, then the sectional curvature of the adjusted metric is constant near the point.

Combining these three observation, we see that the sectional curvature of the adjusted metric is constant, or L_1 and L_2 are diagonal in the same coordinate system. In the latter case, the formula (19) for the metric shows that L_1, L_2 and Id are linear dependent. Theorem 7 is proven under the assumption that M_A is two-dimensional.

Now let the dimension of M_A be greater than 2. Consider two BM-structures L_1 and L_2 such that L_1, L_2 and Id are linearly independent and have only typical points on (D^n, g). Without loss of generality, since Id is also a BM-structure, we can think that L_1 and L_2 are positive-definite. If $N_{L_1} = n$, Theorem 7 follows from Theorem 8. Suppose $N_{L_1} < n$. Then, as we already explained after Definition 4, Theorem 5 applied to the BM-structure L_1 gives us a warped decomposition $D^n = D^{k_0} \times ... \times D^{k_m}$. Consider the constructed above submanifold

$$M_A := D^{k_0} \times \gamma_1 \times \gamma_2 \times ... \times \gamma_m$$

for this warped decomposition.

By construction, every tangent space to M_A is invariant with respect to L_1. By the second part of Levi-Civita's Theorem 5, the restriction of L_1 to M_A is a BM-structure for the restriction $g_{|M_A}$ of g to M_A. The number of its different eigenvalues at P coincides with the number of different eigenvalues of L_1 and, therefore, equals the dimension of M_A.

Let us show that L_2 generates one more BM-structure on M_A. Since L_2 is positive-definite, by Theorem 1, it generates a metric g_2 projectively equivalent to g. Since M_A is totally geodesic, $g_{2|M_A}$ is geodesically equivalent to $g_{|M_A}$. Then, by Theorem 1, it generates one more BM-structure for $g_{|M_A}$. We denote this BM-structure by $\bar L_2$.

Thus, in view of Fubini's Theorem 8, our goal is to prove that, for a certain choice of geodesic segments $\gamma_1, ..., \gamma_m$, these two BM-structures (on M_A) and the trivial BM-structure Id are linearly independent.

By construction, the metric $g_{1|M_A}$ does not depend on the choice of geodesic segments γ_k: the results are isometric. Suppose the BM-structures $L_{1|M_A}$, $\bar L_2$ and Id are linearly dependent for every choice of the geodesic segments. Then, for every choice of the geodesic segments, $\bar L_2$ is a linear combination of $L_{1|M_A}$ and Id. Clearly, the coefficients of the linear combination do not depend on the choice of geodesic segments γ_k. (To see it, it is sufficient to consider the length of the integral curve of the eigenvector v_i corresponding to a nonconstant λ_i. The integral curve lies in M_A and its length does not depend on the choice of the geodesic segments γ_k.) Then, the eigenspaces of L are invariant with respect to the BM-structure L_2. Hence, the metrics g_1, g_2 have the form from

Remark 3 in the same coordinate system. Then, L_2 is linear dependent of L_1 and Id. We obtained a contradiction. Thus, the adjusted metric has constant sectional curvature. Theorem 7 is proven.

4 Proof of Theorem 2

Assume $\dim(\mathcal{B}(M^n, g)) \geq 3$, where (M^n, g) is a connected complete Riemannian metric of dimension $n \geq 3$. Suppose a complete Riemannian metric $\bar{g}$ is projectively equivalent to g. Denote by L the BM-structure from Theorem 1. By Theorem 7, for every typical point, the sectional curvature of the adjusted metric is constant.

Denote by m the number of different eigenvalues of L in a typical point. The number m does not depend on the typical point. If $m = n$, Theorem 2 follows from Fubini's Theorem 8 and Corollary 4.

Thus, we can assume $m < n$. Denote by m_0 the number of simple eigenvalues of L at a typical point. By Corollary 2, the number m_0 does not depend on the typical point. Then, by Levi-Civita's Theorem 5, the metric g has the following warped decomposition near every typical point p:

$$g = g_0 + \left| \prod_{i=1}^{m_0} (\phi_{m_0+1} - \phi_i) \right| g_{m_0+1} + \dots + \left| \prod_{i=1}^{m_0} (\phi_m - \phi_i) \right| g_m. \tag{20}$$

Here the coordinates are $(\bar{y}_0, ..., \bar{y}_m)$, where $\bar{y}_0 = (y_0^1, ..., y_0^{m_0})$ and for $i > 1$ $\bar{y}_i = (y_i^1, ..., y_i^{k_i})$. For $i > 0$, every metric g_{m_0+i} depends on the coordinates $\bar{y}_i$ only. Every function ϕ_i depends on y_0^i for $i \leq m_0$ and is constant for $i > m_0$.

Let us explain the relation between Theorem 5 and the formula above. The term g_0 collects all one-dimensional terms of (5). The coordinates $\bar{y}_0 = (y_0^1, ..., y_0^{m_0})$ collect all one-dimensional $\bar{x}_i$ from (5). For $i > m_0$, the coordinate $\bar{y}_i$ is one of the coordinates $\bar{x}_j$ with $k_j > 1$. Every metric g_{m_0+i} for $i > 1$ came from one of the multidimensional terms of (5) and is proportional to the corresponding A_j. The functions ϕ_i are eigenvalues of L; they must not be ordered anymore: the indexing can be different from (4). Note that, by Corollary 2, this re-indexing can be done simultaneously in all typical points.

Since the dimension of the space $\mathcal{B}(M^n, g)$ is greater than two, by Theorem 7, g is a $V(K)$ metric.

According to Definition 7, a multiple eigenvalue ϕ_i of L is **isolated**, if there exists no nonconstant eigenvalue ϕ_j such that $\phi_j(q) = \phi_i$ at some point $q \in M^n$. If every multiple eigenvalue of L is non-isolated, then applying Theorems 11,12,6 we obtain that g has constant sectional curvature.

Thus, we can assume that there exist isolated eigenvalues. Without loss of generality, we can assume that (at every typical point) the re-indexing is made in such a way that the first multiple eigenvalues $\phi_{m_0+1}, ..., \phi_{m_1}$ are non-isolated and the last multiple eigenvalues $\phi_{m_1+1}, ..., \phi_m$ are isolated. By assumption, $m_1 < m$.

We will prove that in this case all eigenvalues of L are constant. By Remark 3, it implies that the metrics g, $\bar{g}$ are affine equivalent.

Let us show that the sectional curvature of the adjusted metric g is nonpositive. We suppose that it is positive and will find a contradiction.

At every point q of M^n, denote by $V_0 \subset T_qM^n$ the direct product of the eigenspaces of L corresponding to the eigenvalues $\phi_1, ..., \phi_{m_1}$. Since the eigenvalues $\phi_{m_1+1}, ..., \phi_m$ are isolated by the assumptions, the dimension of V_0 is constant, and V_0 is a distribution. By Corollary 5, V_0 is integrable. Take a typical point $p \in M^n$ and denote by M_0 the integral manifold of the distribution containing this point. The restriction $g_{|M_0}$ of the metric g to M_0 is complete.

Consider the direct product $M_0 \times \mathbb{R}^{m-m_1}$ with the metric

$$g_{|M_0} + \left|\prod_{i=1}^{m_0}(\phi_{m_1+1} - \phi_i)\right| dt_{m_1+1}^2 + ... + \left|\prod_{i=1}^{m_0}(\phi_m - \phi_i)\right| dt_m^2, \tag{21}$$

where $(t_{m_1+1}, ..., t_m)$ are the standard coordinates on $\mathbb{R}^{m-m_1}$. Since the eigenvalues $\phi_{m_1+1}, ..., \phi_m$ are isolated, (21) is a well-defined Riemannian metric. Since $g_{|M_0}$ is complete, the metric (21) is complete. By definition, the metric is the adjusted metric for the warped decomposition (20). Hence, the sectional curvature of the adjusted metric is positive constant. Then, the product $M_0 \times \mathbb{R}^{m-m_1}$ must be compact, which contradicts the fact that $\mathbb{R}^{m-m_1}$ is not compact. Finally, the sectional curvature of the adjusted metric is not positive.

Now let us prove that all eigenvalues of L are constant. Without loss of generality, we can assume that the manifold is simply connected. We will construct a totally geodesic submanifold M_A, which is a global analog of the submanifold M_A from Section 3.4. At every point $x \in M^n$, consider $V_{m_1+1}, ..., V_m \subset T_xM^n$, where V_{m_1+i} is the eigenspace of the eigenvalue ϕ_{m_1+i}. Since the eigenvalues ϕ_{m_1+i} are isolated, $V_{m_1+1}, ..., V_m$ are distributions. By Corollary 5, they are integrable. Denote by $M_{m_1+1}, M_{m_1+2}, ..., M_m$ the corresponding integral submanifolds.

Since the manifold is simply connected, then, by [6], it is homeomorphic to the product $M_0 \times M_{m_1+1} \times M_{m_1+2} \times ... \times M_m$. Clearly, the metric g on

$$M^n \simeq M_0 \times M_{m_1+1} \times M_{m_1+2} \times ... \times M_m$$

has the form

$$g_{|M_0} + \left|\prod_{i=1}^{m_0}(\phi_{m_1+1} - \phi_i)\right| g_{m_1+1} + ... + \left|\prod_{i=1}^{m_0}(\phi_m - \phi_i)\right| g_m, \tag{22}$$

where every g_k is a metric on M_k. Take a point

$$P = (p_0, p_{m_1+1}, ..., p_m) \in M_0 \times M_{m_1+1} \times M_{m_1+2} \times ... \times M_m.$$

On every M_{m_1+k}, $k = 1, ..., m - m_1$, pick a geodesic γ_{m_1+k} (in the metric g_{m_1+k}) passing through p_k. Denote by M_A the product

$$M_0 \times \gamma_{m_1+1} \times ... \times \gamma_m.$$

M_A is an immersed totally geodesic manifold. More precisely, the natural immersion of $M_0 \times \mathbb{R}^{m-m_1}$ (endowed with the metric (21)) into M^n is isometric and totally geodesic. Locally, in a neighborhood of every point, M_A coincides with M_A from Section 3.4 constructed for the warped decomposition (22). The restriction of the metric g to M_A is isometric to the adjusted metric and, therefore, has nonpositive constant sectional curvature. Then, by Corollary 4, the restriction of $\bar{g}$ to M_A is affine equivalent to the restriction of g to M_A. Then, by Remark 3, all ϕ_i are constant. Then, the metric g is affine equivalent to the metric $\bar{g}$. Theorem 2 is proven.

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