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SHORT COMMUNICATIONS

On the Degree of Geodesic Mobility for Riemannian Metrics

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Two metrics g and $\bar{g}$ on the same manifold M are said to be *geodesically equivalent* if any geodesic of the metric g can be parameterized so as to become a geodesic of $\overline{g}$. As is known [1], there is a one-to-one correspondence between geodesically equivalent metrics and nondegenerate symmetric (0,2) tensors a_{ij} satisfying the equations

$$a_{ij,k} = \frac{1}{2}(a^{\alpha}_{\alpha,i} g_{jk} + a^{\alpha}_{\alpha,j} g_{ik}), \tag{1}$$

where $a^{\alpha}_{\alpha} = a_{\alpha\beta} g^{\alpha\beta}$, and the g^{ij} are the elements of the inverse matrix to g_{ij}, and the commas denote covariant differentiation with respect to the connection on (M, g). These equations are linear, so that their solutions form a linear vector space; the dimension of this space is called the *degree of geodesic mobility* of the metric g and denoted by p. The number p is a numerical characteristic of the cardinality of the geodesic class of the given metric g_{ij} (see [1]).

It has long been known that metrics with large degree of mobility admit concircular covector fields [2]. Recall that a covector field φ_i on (M, g) is said to be *concircular* if

$$\varphi_{i,j} = \rho g_{ij} \tag{2}$$

for some function ρ on the manifold (M, g).

We denote the linear space of concircular vector fields in the metric g_{ij} by $\mathrm{Con}(M)$.

The following assertion was stated in [3].

Theorem 1. *The degree of geodesic mobility of a Riemannian* (*with positive definite metric*) *space* (M, g) *of dimension* $n \geq 3$ *of nonconstant curvature can take only the values* $p = m(m+1)/2 + l$, *where* $m = \dim \mathrm{Con}(M)$, *and* l *ranges from* 1 *to* $L = [(n + 1 - m)/3]$, *where the brackets denote the integer part of a number.*

Unfortunately, this assertion is incorrect. Indeed, in the dimension $n = 4$, it follows from this theorem that the degree of mobility is 2 if and only if the number of linearly independent concircular vector fields is 1. In other words, according to the theorem, there exist no 4-spaces with degree of mobility $p \geq 2$ that admit no concircular vector fields. However, such metric spaces exist. An example of a metric with the required property is the restriction of the standard metric $dx_1^2 + dx_2^2 + \cdots + dx_5^2$ on Euclidean space $\mathbb{R}^5$ to the standard ellipsoid

$$\left\{(x_1, x_2, \dots, x_5) \in \mathbb{R}^5 \;\middle|\; \frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \cdots + \frac{x_5^2}{a_5} = 1\right\}, \qquad \text{where} \quad 0 < a_1 < a_2 < \cdots < a_5.$$

It was shown in [4]–[6] that the metric on the ellipsoid admits a nontrivial geodesic equivalence. Therefore, the degree of mobility of this metric is at least 2. On the other hand, the metric on the

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ellipsoid admits no concircular vector fields. To show this, consider conditions (2), which are a system of differential equations for the components of the vector φ_i. Covariant differentiation and alternation with Ricci's identity taken into account yield the integrability conditions

$$\varphi_\alpha R^\alpha_{ijk} = \rho_k g_{ij} - \rho_j g_{ik} \tag{3}$$

for this system. Here R^i_{jkl} is the Riemann tensor and $\rho_i := \rho_{,i}$. Contracting Eq. (3), we obtain

$$\varphi^\alpha R_{\alpha i} = (n-1)\rho_i,$$

where R_{ij} is the Ricci tensor, $\varphi^i = \varphi_\alpha g^{\alpha i}$, and the g^{ij} are the elements of the inverse matrix of the metric. This reduces (3) to the form

$$\varphi^\alpha W^h_{\alpha ij} = 0, \tag{4}$$

where W^h_{ijk} is the Weyl projective curvature tensor. A direct calculation shows that the Weyl tensor of the ellipsoid admits no nontrivial vector fields satisfying conditions (4).

Nevertheless, the main idea of [3] is correct; using this idea, we obtain the following theorem.

Theorem 2. *The degree of mobility of a Riemannian n-space ($n \geq 3$) of nonconstant curvature can take either the values $p = m(m+1)/2 + l$, where $m = \dim \mathrm{Con}(M)$, and l ranges from 1 to $L = [(n+1-m)/3]$, where the square brackets denote the integer part of a number, or $p = 1, 2$.*

The difference between Theorem 2 and Theorem 1 is that in Theorem 2, the degree of mobility can take the values 1 and 2, among others; the case $p \leq 2$ is a special case not related to the number of concircular vector fields.

The mistake in the proof of Theorem 1 is related to this special case $p \leq 2$. The proof of Theorem 1 assumes implicitly that the space under consideration is $V_n(B)$ [7], [8]. However, not every space admitting $p = 2$ is the space $V_n(B)$. The metric on the ellipsoid mentioned above is not a $V_n(B)$ metric, because every $V_n(B)$-metric on a closed manifold must be of constant curvature [9], [10], [11].

On the other hand, as shown in [7], [8], for $p \geq 3$, any metric is a $V_n(B)$ metric.

Thus, the proof given in [3] is correct under the additional assumption $p > 2$, and the words "or $p = 1, 2$" render the theorem valid.

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