Separation of variables for spaces of constant curvature

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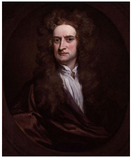
Plan

- Extended introduction
  - How (classical) physicists discover(ed) physical laws
  - Hamiltonian mechanics
  - What physicists want from mathematicians
- Orthogonal separation of coordinates and reformulation as the existence of (one) Killing tensor field.
  - Classical results (Stäckel 1889, Eisenhart 1934)
  - What is left?
- Main result: classification of separable variables for spaces of constant curvature.
- How we came to the result: Hamiltonian mechanics in the infinitely-dimensional case and corresponding integrable systems.

How physicists discover physical laws? A 3-step procedure

- Suggest a mathematical model that may describe a physical phenomenon,
- Find (partial, approximative) solutions of this model
- Compare them with known observations or experiments.

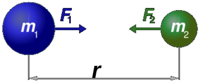
Example: How Isaac Newton found the gravitational law.



This strategy was suggested by Isaac Newton in Philosophiae Naturalis Principia Mathematica:

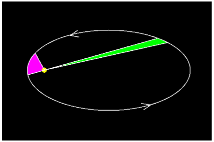
- one needs to understand the mathematical model describing the situation, and then
- find the parameters in the model using experiments or observations.

Example: Gravitational Law:



From the Newton's third law and general sense it is clear that the gravitational force between two bodies is proportional to the masses of the bodies and depends on the distance between the bodies: $F = F_1 = F_2 = m_1 m_2 \cdot f(r)$. How to find f?

Newton tried different functions for f and solved the equations of motion. He proved that for the function $f(r) = \frac{G}{r^2}$ the trajectories of the planets are ellipses which was observed by Kepler.
He concluded that the function $f(r)$ is indeed $\frac{G}{r^2}$, i.e., $F = F_1 = F_2 = G\frac{m_1 m_2}{r^2}$.



Separation of variables helps in the second step

- Mathematical models of classical physics are described by systems of ODEs or PDEs
  - The systems of ODEs are in many cases $\underbrace{\text{Hamiltonian}}_{\text{will be recalled}}$ with a special form of the Hamiltonian function
  - The systems of PDEs are in many case (Helmholtz-Schrödinger) wave equation (will be touched)
- Separation of variables method, it it works, allows to reduce solving certain systems of ODE's or PDE's to a system of algebraic equations.

Hamiltonian (ordinary differential) equations

- We consider $\mathbb{R}^{2n}$ with coordinates $x^1, ..., x^n, p_1, ..., p_n$.
- We choose a function $H = H(x, p)$
- And construct the following system of $2n$ ODEs

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial x} \end{cases}$$

  on a 2n-dimensional manifold with local coordinates $(x, p) = (x^1, ..., x^n, p_1, ..., p_n)$.
- This is a system of ODEs, not of PDEs, because $H(x, p)$ is a known function. The system describes the flow of a vector field on $2n$-dimensional manifold.
- The only freedom we have is the choice of the function H. Physicists developed a system of heuristic rules how to find a function H that can model a physical system.
- For most classical physically-relevant systems

$$H(x, p) = \frac{1}{2} g^{ij} p_i p_j + U(x)$$

  (where $g^{ij}(x)$ is a contravariant metric).

Example: Hamiltonian for the Kepler system

- We centre our coordinate system at sun, let $x(t)$ describes the position of a planet.
- The gravitational force is, by the Newton gravitaion law, $F(x) = -\,\mathrm{const}\ \frac{x(t)}{r^2}$; that is, the equations of the motion are

$$\ddot{x} = -\,\mathrm{const}\,\frac{x(t)}{r^2} \text{ with } r = \sqrt{(x^1)^2 + \ldots + (x^n)^2}$$

- Let us see that the Hamiltonian system with $H(x,p) = \frac{1}{2} g^{ij} p_i p_j + V(x) = \frac{1}{2}\sum_i p_i^2 - \frac{\mathrm{const}}{r}$ describes the same equations of motion. We have

$$\begin{cases} \dot{x}^i = \frac{\partial H}{\partial p_i} = p_i \\ \dot{p}_i = -\frac{\partial H}{\partial x^i} = -\,\mathrm{const}\,\frac{\dot{x}}{r^2} \end{cases} \implies \ddot{x} = -\,\mathrm{const}\,\frac{x}{r^2}.$$

- We see that the equations of motion obtained from the Newton's laws coincide with that of obtained from the Hamiltonian system with $H(x,p) = \frac{1}{2} g^{ij} p_i p_j + V(x) = \frac{1}{2}\sum_i p_i^2 - \frac{\mathrm{const}}{r}$.

Example: Geodesic flow as Hamiltonian system

Consider a geodesic flow on (M, g). The initial definition goes through the standard ODE for geodesics:

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^i\dot{x}^j = 0$$

(a system of n second order ODEs).

It is a well-known fact and standard exercise that the system is naturally equivalent to the Hamiltonian system

$$\left\{ \begin{array}{r} \dot{x} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial x} \end{array} \right. \quad \text{with} \quad H = \tfrac{1}{2} g^{ij} p_i p_j \ .$$

(The system is of the first order but it has $2n$ unknown functions)

Where we are in the talk?

- What we discussed so far?
  - Many physical situations are described by Hamiltonian systems with $H(x, p) = \frac{1}{2} g^{ij} p_i p_j + V(x)$
  - Physicists want methods to solve these equations
- What we do next:
  - Hamilton-Jacobi method to solve Hamiltonian equations.

Naive suggestion to solve Hamiltonian ODEs

- Suppose we would like to solve the Hamiltonian equations

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial x} \end{cases} \tag{1}$$

  on a 2n-dimensional manifold.

- **Naive suggestion:** let us find a coordinate system $(c_1(x,p), ..., c_n(x,P), P^1(x,p), ..., P^n(x,p))$ such that $c_1 = H = \text{the initial Hamiltonian}$ (and such that the Hamiltonian equation still have their "canonical" form (1) in new coordinates)
- **Good news:** The new "canonical" coordinate system exists by the Darboux Theorem and the Hamiltonian equation is easy to solve in it
- **Bad news:** Finding such a coordinate system for general Hamiltonian (as it done in e.g. a standard proof of the Darboux Theorem) requires solving the system (1) and therefore does not help.
- On the next slide I recall for you the classical **Hamilton-Jacobi method**; it helps to find canonical coordinate system

Hamilton-Jacobi method

We look for a function $W(x^1, \dots, x^n, c_1, \dots, c_n)$ such that:

(a) The $n \times n$-matrix $\frac{\partial^2 W}{\partial c_i \partial x^j}$ is nondegenerate

(b) $H(x, \frac{\partial W}{\partial x}) = c_1$ (Hamilton-Jacobi equation)

Consider two local mappings:

$$\begin{array}{ll} \phi : \mathbb{R}^{2n}(x,c) \to \mathbb{R}^{2n}(x,p), & \phi(x,c) = \left(x, \frac{\partial W}{\partial x}\right) \\ \psi : \mathbb{R}^{2n}(x,c) \to \mathbb{R}^{2n}(P,c), & \psi(x,c) = \left(\frac{\partial W}{\partial c}, c\right). \end{array} \tag{2}$$

By (a), the mappings are local diffeomorphisms so $\psi \circ \phi^{-1}$ is a local diffeomorphism. By (b), the Hamiltonian has the form $H_{new} = c_1$

(Known) Theorem. $\psi \circ \phi^{-1}$ is a canonical transformation, in the sense the Hamiltonian system still has the "canonical Hamiltonian" form

$$\begin{pmatrix} \dot c_1 \\ \vdots \\ \dot c_n \\ \dot P_1 \\ \vdots \\ \dot P_n \end{pmatrix} = \begin{pmatrix} \frac{\partial H_{new}}{\partial P_1} \\ \vdots \\ \frac{\partial H_{new}}{\partial P_n} \\ -\frac{\partial H_{new}}{\partial c_1} \\ \vdots \\ -\frac{\partial H_{new}}{\partial c_n} \end{pmatrix} \stackrel{H_{new}=c_1}{=} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ \vdots \\ 0 \end{pmatrix}.$$

It can be immediately solved:

$$\begin{pmatrix} c_1(t) \\ \vdots \\ c_n(t) \\ P_1(t) \\ \vdots \\ P_n(t) \end{pmatrix} = \begin{pmatrix} \text{const}_1 \\ \vdots \\ \text{const}_n \\ \mathrm{Const}_1 - t \\ \vdots \\ \mathrm{Const}_n \end{pmatrix}.$$

Where we are in the talk?

- What we discussed so far?
  - Physicists want methods to solve Hamiltonian systems with $H(x,p) = \frac{1}{2} g^{ij} p_i p_j + U(x)$
  - Hamilton-Jacobi method reduces solving Hamiltonian equations to solving a system of functional equations (i.e., constructing the coordinate change $\psi \circ \phi^{-1}$). In order to apply the Hamilton-Jacobi method, one should find a function $W(x,c)$.
- What we do next:
  - Orthogonal separation of variables is a way to find such functions for some metrics g.

What is (orthogonal) separation of variables?

- It is a differential geometric method to find the function $W(x, c)$ from the Hamilton-Jacobi method
  - Most known explicit solutions of physically interesting systems were obtained by it
- My next goal is to re-formulate the problem as a differential geometric one

Definition of orthogonal separation of variables

- We consider for simplicity the geodesic flow of of an n-dimensional pseudo-Riemannian manifold (M, g); that is, our Hamiltonian is $H = \frac{1}{2} g^{ij} p_i p_j$.
- By *orthogonal separation of variables* we understand the existence of a Killing tensor K_{ij} such that the operator field K^i_j has n different eigenvalues and such that the Haantjes torsion of K^i_j vanishes.
- Recall that a symmetric $(0, 2)$- tensor is a *Killing tensor* for g, if the quadratic in velocities function $I : TM \to \mathbb{R}\,, \quad I(\xi) = K(\xi, \xi)$ is constant along the orbits of the geodesic flow of g (we call the function I *integral* corresponding to K). Alternatively, $\nabla_{(i} K_{jk)} = 0$.
- Vanishing of the Haantjes torsion for operators with n different eigenvalues is equivalent to the local existence of a coordinate system such that K^i_j is diagonal. In this coordinate system, K_{ij} and g_{ij} are also automatically diagonal.
- These variables are called *separating variables* (for g corresponding to K).
- Later, I will speak about spaces of constant curvature, they are real analytic. By Kruglikov-Matveev 2018/ Gover-Leistner 2019, the Killing tensors are also real-analytic so the local existence of separating coordinates implies their existence almost everywhere.

Why only one Killing tensor?

- **Theorem (Benenti 1992).** Assume $x^1, ..., x^n$ are separating coordinates for g. Then, in addition to the Killing tensor K, there exist $n-2$ additional Killing tensors $\overset{1}{K}_{ij} = g_{ij}, \overset{2}{K}_{ij} = K_{ij}, \overset{3}{K}_{ij}, ..., \overset{n}{K}_{ij}$ such that:
  - They are linearly independent
  - They are diagonal in the same coordinate system.
  - Moreover, the integrals corresponding to these tensors mutually commute.
- Moreover, in the separating coordinate system $(x^1, ..., x^n)$, the tensors $\overset{1}{K}_{ij}, ..., \overset{n}{K}_{ij}$ have the so-called Stäckel form by Eisenhart 1934; I will explain it on the next slide.

Stäckel-Eisenhart description of metrics admitting separation of variables

- Take a nondegenerate $n \times n$ matrix $S = (S_{ij})$ whose (i, j) component S_{ij} is a function of the ith variable x^i only.
- Next, define vector-functions $I = (I_1, ..., I_n)^T$ (on T^*M) by the following condition
$$SI = p^2.$$
Above, p^2 is the vector of the squares of momenta, $p^2 = (p_1^2, p_2^2, ..., p_n^2)^T$.
- The functions I_i are quadratic in momenta; they therefore correspond to symmetric second order tensors which we call
$$\overset{1}{K}_{ij}, \overset{2}{K}_{ij} = K_{ij}, \overset{3}{K}_{ij}, ..., \overset{n}{K}_{ij} \ \ \text{(we may think } \ g_{ij} = \overset{1}{K}_{ij}\text{)}.$$
  - **Theorem (Stäckel 1897).** The functions are commuting integrals.
    - By construction, the matrices of K_{ij} are diagonal so Haanties torsion of the corresponding K^i_j vanishes.
  - **Theorem (Eisenhart 1934 and Benenti 1992).** Any pair (metric, orthogonal separating coordinates) can be constructed by the Stäckel procedure.
- The function $W(x, c)$ from the Hamilton-Jacobi equation is as follows:
$$W(x, c) = \pm \int_0^{x^1} \sqrt{\sum_j S_{1j}(\xi) c_j} d\xi \pm \int_0^{x^2} \sqrt{\sum_j S_{2j}(\xi) c_j} d\xi \pm ... \pm \int_0^{x^n} \sqrt{\sum_j S_{nj}(\xi) c_j} d\xi$$

More about Stäckel form. Is everything done?

- Stäckel-Eisenhart Theorems give an effective description of metrics in separable coordinates.
  - One chooses a matrix S such that its ith row depends on x^i only
  - One constructs $I_1 = H_g$, I_2,...,I_n using

    $$SI = P \quad (\Longleftrightarrow \quad I = S^{-1}P).$$

    (That is, $g^{11} = \Delta_1/\Delta$, $g^{22} = \Delta_2/\Delta$ etc where $\Delta = \det(S)$ and Δ_i is the determinant of the matrix obtained from S by striking out the first coloumn and the ith row).
  - By Stäckel-Eisenhart, all metrics admitting separation of variables, and their "separating" Killing tensors, can be obtained by this procedure.
- What remains?
  - GIVEN a metric, construct separating coordinates for it.
    - **Eisenhard problem (1934):** Describe all separating coordinates for metrics of constant sectional curvature.
- This is actually what physicists want: they have a model (i.e., a family of Hamiltonians of the special form $H(x,p) = \frac{1}{2}g^{ij}p_ip_j + U(x)$) and they what to see solutions
  - In many physically interesting systems, the metric g has constant curvature.

Why to study separation of variables for CCS?

- Of course, geodesics of spaces of constant curvature are known.
- Separation of variables is used though in many other problems:
  - There is no problem to introduce potential energy.
  - For spaces of constant curvature, the Carter-Robinson conditions are fulfilled and the second order differential operators

    $$\overset{\alpha}{\mathcal{K}} := \sum_{i,j} \nabla_i \overset{\alpha}{K}{}^{ij} \nabla_j : C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R})$$

    mutually commute and also commute with $\Delta_g = \overset{1}{\mathcal{K}}$.
  - This allows us to look for solutions of the Schrödinger-Helmholtz equation $\Delta_g \phi = \lambda \phi$ by a *multiplicative ansatz* $\phi = X_1(x^1)X_2(x^2)...X_n(x^n)$and reduce it to n uncoupled ordinary differential equations.
- There is a relation of orthogonal separation of variables to infinitely dimensional integrable systems which is far from being understood and which I touch in the conclusion

The main problem which I discuss in this talk

- **Eisenhart problem (1934):** Describe all diagonal metrics

$$g_{ij} = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + \ldots + g_{nn}(dx^n)^2$$

  of arbitrary signature such that
  - they have constant curvature and
  - There exists a (0,2) tensor of the form $g_{ij} = \rho_1 g_{11}(dx^1)^2 + \rho_2 g_{22}(dx^2)^2 + \ldots + \rho_n g_{nn}(dx^n)^2$ with mutially different ρ_i which is Killing
- In our paper, we also give answers to the most natural questions about these separation coordinates:
  - **(Schöbel-Veselov question 2012:)** Describe the moduli space of separating coordinates modulo isometries (alternative version of the question (**McLenaghan-Smirnov-The 2003)**: what are the "types" of separating coordinates).
  - **(Blaszak-Marciniak-Sergyeyev problem 2006:)** Write a formula of transformation to flat coordinates
  - **(Kalnins-Miller-Reid task 1984:)** Construct the Stäckel matrix

Previous results and our result

- In dimension 2 and 3 the answer was known in the 19th/beginning of 20th century
- E. Kalnins, W. Miller and A. Reid gave an answer (1984) to the problem above, for all dimensions and all signatures.
  - They gave a recursive description of separating coordinates for metrics of constant curvature
- They did not publish a proof. The proof in the Riemannian case appeared in Kalnins-Miller 1986 (nonnegative curvature) and Kalnins 1986 (positive, zero and negative curvature).
- Our main result: we proved that the description of Kalnins et al is correct and wrote an explicit formula for it

Most nondegenerate example – Benenti systems/ projectively equivalent metrics

- Take a polynomial $P(t) = a_0 + a_1 t + \dots + a_{n+1} t^{n+1}$ of degree $\leq n+1$. Consider the metric $g_{Levi-Civita}$

$$\frac{\prod_{s\neq 1}(x^1 - x^s)}{P(x^1)}(dx^1)^2 + \frac{\prod_{s\neq 2}(x^2 - x^s)}{P(x^2)}(dx^2)^2 + \dots + \frac{\prod_{s\neq n}(x^n - x^s)}{P(x^n)}(dx^n)^2$$

- **Fact.** The metric has constant curvature $-\frac{1}{4}a_{n+1}$. The coordinates x^i are separating coordinates for it.
- Some participants of the seminar, e.g., MikE and Thomas saw this form of the metric of constant curvature, it appears in the description of projectively equivalent metrics of Tullio Levi-Civita 1896.

Most nondegenerate separating coordinates for CCS

Theorem (Eisenhart 1934). Let the coordinates $(x^1, ..., x^n)$ be separating for g of constant curvature. Assume in addition that for every $j \neq i$ we have $\frac{\partial}{\partial x^j} g_{ii} \neq 0$. Then, by a coordinate change of the form

$$x^i_{new} = x^i_{new}\left(x^i_{old}\right)$$

the metric is $g_{Levi-Civita}$.

Remark. The coordinate changes of the form $x^i_{new} = x^i_{new}\left(x^i_{old}\right)$ change nothing:

- the coordinate remains separating
- the condition $\frac{\partial}{\partial x^j} g_{ii} \neq 0$ is not affected.

Remark. The freedom in the choice of the metric g is the choice of the polynomial $P(t) = a_0 + a_1 t + ... + a_{n+1} t^{n+1}$. (The a_{n+1}-coefficient is responsible for the curvature by $a_{n+1} = -4K$).

Remark The metrics $g_{Levi-Civita}$ will be "building blocks" in the general case.

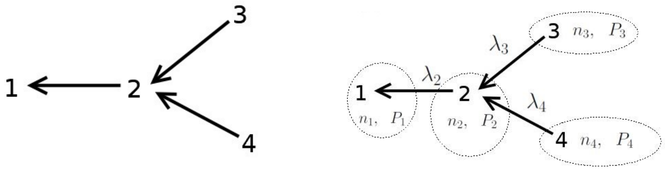
The general case

- Eisenhart theorem gives a complete answer under the following condition on g: he assumed in addition that in the separating coordinates for every $j \neq i$ we have $\frac{\partial}{\partial x^j} g_{ii} \neq 0$.
- The main result of my talk is to give an answer in the general case.

Parameters describing separating coordinates

1. Natural number $B \leq n := \dim M$ (="number of blocks").
2. Natural numbers $n_1, ..., n_B$ (="dimensions of blocks") with $\sum_{\alpha=1}^{B} n_\alpha = n$.
3. In-directed rooted forest F (with B vertexes which we denote by numbers $1, ..., B$), that is, an oriented graph such that each connected component is a rooted tree such that for any vertex there exists a necessary unique oriented way towards a root (1 is root on Fig.). The induced partial order on $\{1, ..., B\}$ will be denoted by $\prec$. On Fig. , $1 \prec 2 \prec 3$ and $2 \prec 4$.
4. Each edge $\vec{\beta\alpha}$ of the in-directed rooted forest is labeled by a number λ_β.
5. Polynomials P_α, $\alpha = 1, ..., B$ of degree at most $n_\alpha + 1$, whose coefficients are denoted as follows:

$$P_\alpha(t) = \overset{\alpha}{a}_0 + \overset{\alpha}{a}_1 t + ... + \overset{\alpha}{a}_{n_\alpha+1} t^{n_\alpha+1}.$$



Conditions on the parameters

Further, we assume that the polynomials P_α and the numbers λ_β satisfy the following restrictions:

(i) If F has more than one connected component and therefore more that one root, then for any root α we have $\overset{\alpha}{a}_{n_\alpha+1} = 0$, i.e., $\deg P_\alpha \leq n_\alpha$.

(ii) If $\alpha = \text{next}(\beta)$, then λ_β is a root of P_α and $\overset{\beta}{a}_{n_\beta+1} = P'_\alpha(\lambda_b)$, where $P'(t)$ denotes the derivative of $P(t)$.

(iii) If $\alpha = \text{next}(\beta)$ and $\alpha = \text{next}(\gamma)$ with $\lambda_\beta = \lambda_\gamma = \lambda$, $\beta \neq \gamma$, then λ is a double root of P_α (in view of (ii) this automatically implies $\overset{\beta}{a}_{n_\alpha+1} = \overset{\gamma}{a}_{n_\gamma+1} = 0$).

Form of the metric

- We divide our coordinates into B blocks of dimensions $n_1, ..., n_B$:

$$(\underbrace{x_1^1, ..., x_1^{n_1}}_{X_1}, ..., \underbrace{x_B^1, ..., x_B^{n_B}}_{X_B}). \tag{3}$$

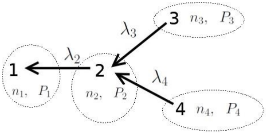
- For every $\alpha = 1, ..., B$ we consider the n_α-dimensional metric g_α^{LC} and n_α-dimensional operator ($=$ $(1,1)$-tensor) L_α given by:

$$g_\alpha^{\text{LC}} = \sum_{s=1}^{n_\alpha} \tfrac{1}{P_\alpha(x^i)} \left(\prod_{j \neq s} (x_\alpha^s - x_\alpha^j) \right) \left(dx_\alpha^i \right)^2 \quad , \quad L_\alpha = \text{diag}(x_\alpha^1, ..., x_\alpha^{n_\alpha}).$$

- We introduce a diagonal metric g given by

$$g = \text{diag}(g_1, \ldots, g_B) \quad \text{with } g_\alpha = f_\alpha \cdot g_\alpha^{\text{LC}}, \quad \text{where} \tag{4}$$

$$f_\alpha = \prod_{\substack{s \overset{\prec}{=} \alpha \\ s \text{ not a root}}} \det(\lambda_s \cdot \text{Id}_{n_s} - L_{\text{next}(s)}).$$



$$
\begin{aligned}
P_1(L_1)\cdot g_1^{\mathsf{LC}} + & \frac{1}{\det(\lambda_2 \mathrm{Id}_{n_1}-L_1)}\cdot P_2(L_2)\cdot g_2^{\mathsf{LC}}\\
+ & \frac{1}{\det(\lambda_2 \mathrm{Id}_{n_1}-L_1)\det(\lambda_3 \mathrm{Id}_{n_2}-L_2)}\cdot P_3(L_3)\cdot g_3^{\mathsf{LC}}\\
+ & \frac{1}{\det(\lambda_2 \mathrm{Id}_{n_1}-L_1)\det(\lambda_4 \mathrm{Id}_{n_2}-L_2)}\cdot P_4(L_4)\cdot g_4^{\mathsf{LC}}
\end{aligned}
$$

Figure: An example of an in-tree, structure of its labels and the corresponding (contravariant, i.e., with upper indexes) metric.

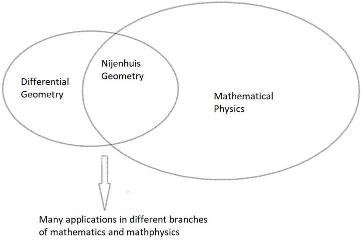
Main result

$$g = \mathrm{diag}(g_1, \ldots, g_B) \quad \text{with } g_\alpha = f_\alpha \cdot g_\alpha^{\mathrm{LC}}, \quad \text{where} \tag{5}$$
$$f_\alpha = \prod_{\substack{s \overset{\prec}{=} \alpha \\ s \text{ not a root}}} \det(\lambda_s \cdot \mathrm{Id}_{n_s} - L_{\mathrm{next}(s)}).$$

Theorem. The metrics (5) have constant sectional curvature and the coordinates $x^1, ..., x^n$ are separating coordinates for them. Moreover, every pair (metric of constant curvature, separating coordinate system for it) can be brought to this form by renumeration of the coordinates and by the coordinate changes of the form $x^i_{new} = x^i_{new}(x^i_{old})$ for every i

How/why we obtained the result

- Our main research subject since 2019 is NIJENHUIS GEOMETRY (9 joint papers; 3 organised conferences in this topic)



- We (Bolsinov-Konyaev-Matveev) observed that the existence of separating coordinates is equivalent to a system of PDEs which we studied and solved before, in "Applications of Nijenhuis geometry III".
- The motivation of "Applications of Nijenhuis geometry III" has nothing to do with separation of variables: our goal was to describe all compatible pencils of ∞-dimensional geometric Poisson structure on the loop space of the form $\mathcal{P}_3 + \mathcal{P}_1$, where $\mathcal{P}_i$ has order i, such that $\mathcal{P}_3$ is Darboux-Poisson.
- We obtained a full list of such structures and then observed that they coincide (though they are visually different) with those described in [Kalnins-Miller-Reid 1984]. In other words, we solved another problem and obtain this as a byproduct

Relation to infinitely-dimensional integrable systems

- The relation of separating variables to infinitely dimensional integrable systems seems to be very deep and we do not understand it completely; it is rather a collection of observations. In additional to the relation discovered by us, the following should be mentioned:
  - Finite-gap solutions of KdV equations (Dubrovin et al 1987) can be viewed as geodesic flow of a metric admitting separation of variables
  - Finite-dimensional reductions of certain ∞-dimensional Hamiltonian systems (Moser, Veselov in 1990th, Magri et al 1999, Blaszak et al 2022) are symplectically equivalent to geodesic flows of metrics admitting separation of variables
  - There is 1-to-1 correspondence between separable variables of metrics of any curvature and integrable weakly nonlinear infinitely-dimensional systems of hydrodynamic type (Ferapontov 1992 – Ferapontov-Fordy 1997).
- Our current project with Thomas and MikE is about projective nature of infinitely dimensional Poisson brackets; projectively equivalent metrics are natural objects within this approach and in my talk I have shown their appearance in the theory of separation of variables.

THANK YOU VERY MUCH