Riemannian metrics with integrable geodesic flows on surfaces: local and global geometry.

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1. Introduction

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The aim of the paper is to make a review of classical and new results on integrable geodesic flows on surfaces. The main question we would like to discuss is a classification of such flows modulo the following equivalences:

1) isometry,
2) geodesic equivalence,
3) orbital equivalence,
4) Liouville equivalence.

Let us recall the main definitions. Let M^n be a smooth manifold with a Riemannian metric $g_{ij}(x)$. For simplicity we consider only C^∞-smooth Riemannian metrics, although almost all constructions work in the C^2-case. Recall, that the geodesics of the metric are smooth parameterized curves

$$\gamma(t) = (x^1(t), \dots, x^n(t)),$$

such that they are solutions of the system of ordinary differential equations

$$\nabla_{\dot\gamma}\dot\gamma = 0,$$

where $\dot\gamma = d\gamma/dt$ is the velocity vector of the curve γ, and ∇ is the operator of covariant derivation, corresponding to the symmetric connection of the metric g_{ij}.

The equations of geodesics can be considered as the Hamiltonian system

$$\left\{ \begin{array}{ccc} \frac{dx^i}{dt} & = & \frac{\partial H}{\partial p_i}, \\ \frac{dp_i}{dt} & = & -\frac{\partial H}{\partial x^i} \end{array} \right. . \qquad (1)$$

with the Hamiltonian

$$H(x,p)=\frac{1}{2}\sum g^{ij}(x)p_ip_j=\frac{1}{2}|p|^2.$$

on the cotangent bundle T^*M. In this interpretation the geodesics are the projections of the orbits of the Hamiltonian system to M. Here x and p are standard coordinates on the cotangent bundle T^*M, where $x=(x^1,\dots,x^n)$ are the position coordinates on M, and $p=(p_1,\dots,p_n)$ are the coordinates of the covector in T_x^*M in the basis $dx^1,\dots,dx^n$.

When is it possible to solve the equations explicitly, for example, in quadratures? The Liouville theorem gives a sufficient condition for it.

Let M^2 be a closed surface with a Riemannian metric g. The geodesic flow of the metric G is *completely Liouville integrable*, if it admits a smooth first integral f, such that f is functionally independent of H almost everywhere on T^*M^2. (For manifolds of arbitrary dimension n, the geodesic flow of a metric is completely Liouville integrable, if there exist n functionally independent integrals in involution.)

The following two questions are very natural and important:

1) What manifolds admit Riemannian metrics with integrable geodesic flows?

2) How to describe metrics with integrable geodesic flows?

These questions were actively discussed in many papers, see M. Adler and P. van Moerbeke [1], I. K. Babenko and N. N. Nekhoroshev [2], G. Birkhoff [7], O. I. Bogoyavlensky [8], A. V. Bolsinov, V. V. Kozlov and A. T. Fomenko [10], A. V. Brailov [12], M. L. Bialy [13], G. Darboux [16], V. V. Kozlov [27], [28], V. V. Kozlov and D. V. Treshchev [30], V. N. Kolokol'tsov [31], [32], V. V. Kozlov and N. V. Denisova [29], K. Kiyohara [24], [25], V. S. Matveev [34], [35], A.S. Mishchenko [39], A.S. Mishchenko and A. T. Fomenko [40], G. Paternain [47], G. Paternain and R. Spatzier [48], M. L. Raffy [50], A. M. Stepin [53], R. Spatzier [52], I. A. Taimanov [55], [56], A.Thimm [58].

Let us return to dimension two. It is natural to fix the level of energy $\{H=\text{const}>0\}$. Actually, the level is an invariant submanifold of the geodesic flow, and the dynamic on $Q^3=\{H=\text{const}\}$ completely determines the dynamic on T^*M^2. More precisely, a change of the energy level means the re-parameterization of the geodesics. Note, that the topology of the isoenergy surface Q^3 is very simple: Q^3 has a natural structure S^1–bundle over the surface M^2 (the fiber over an arbitrary point $x\in M^2$ is the ellipse $\{H=\text{const}>0\}$ in the tangent plane $T_x^*M^2$. In particular, for simplest surfaces: the sphere S^2, the torus T^2, the projective plane $\mathbb{R}P^2$ and the Klein bottle K^2 the isoenergy surfaces are:

1) $Q^3_{S^2}=\mathbb{R}P^3$ (the three-dimensional projective plane),

2) $Q^3_{T^2}=T^3$ (the three-dimnsional torus),

3) $Q^3_{\mathbb{R}P^2}=L_{4,1}$ (the Lenz space (4, 1)),

4) $Q^3_{K^2}=T^3$ (the three-dimensional torus), respectively.

In the geodesic flow is completely integrable, then the isoenergy surface Q^3 fibers on two-dimensional level surfaces of the additional integral $f:Q^3\to\mathbb{R}$.

The Liouville theorem states that connected regular closed level surfaces of the integral f are Liouville (two-dimensional) tori. The dynamic on each of these tori is very simple. The integral orbits of the geodesic flow on the Liouville tori are strait-line windings (rational or irrational). Moreover, the system (1) is integrable in quadratures near each Liouville torus. Thus if a geodesic flow is integrable, then there is a natural foliation of the isoenergy surface on Liouville tori and singular fibers (corresponding to critical values of the integral f). This structure is called Liouville foliation. Its fibers are by definition the connected components of the level surfaces $\{f = \text{const}\}$ (regular and singular).

Let us introduce the following four equivalence relations on the set of all geodesic flows.

DEFINITION 1. *Let (M_1, g_1) and (M_2, g_2) be surfaces with Riemannian metrics. Consider the corresponding geodesic flows v_1 and v_2 on the cotangent bundles.*

1) The metrics g_1, g_2 are isometrical, if there exists a diffeomorphism $\xi : M_1 \to M_2$, such that $g_1 = \xi^* g_2$.

2) The metrics g_1, g_2 are geodesically equivalent, if there exists a diffeomorphism that takes the geodesics of the first metric to the geodesics of the second metric. Here we consider geodesics as unparameterized curves.

3) The geodesic flows v_1, v_2 are orbitally equivalent (topologically or smoothly), if there exists a mapping (homeo- or diffeomorphism, respectively) $\eta : Q_1^3 \to Q_2^3$ that takes the oriented orbits of the flow v_1 to the oriented orbits of the flow v_2 (here the orbits are considered as unparameterized curves).

4) Assume that the geodesic flows v_1 and v_2 are integrable. Then they are Liouville equivalent, if they have the same Liouville foliations, i. e. if there exists a fiber-preserving diffeomorphism $\eta : Q_1^3 \to Q_2^3$. Usually one requires the mapping η to preserve the orientation of the isoenergy surface and critical circles of the integral.

In this definition each equivalence relation is weaker than the previous one. Note, that the first three equivalent relations have sense for arbitrary geodesic flows.

2. Topological obstructions to integrability of geodesic flows on surfaces

Let us recall, what surfaces admit integrable geodesic flows.

Theorem 1. *(V. V. Kozlov [27]) Let a two-dimensional closed analytic manifold of negative Euler characteristic be equipped with a Riemannian analytic metric. Then the geodesic flow of this metric is not integrable in the class of analytic integrals.*

COMMENT. It follows immediately, that (in the analytic case) the following four closed surfaces can admit integrable geodesic flows: the sphere, the torus, the projective plane and the Klein bottle.

COMMENT. In the Kozlov theorem it is sufficient to require analyticity only of an isoenergy surface. The same result is true under the assumption that the additional integral is a polynomial in momenta (with the coefficients smoothly depending on the position), see below.

PROOF. Our proof differs from the original proof of V. V. Kozlov, and it is based on general properties of Liouville foliations of integrable systems. We hope that the suggested approach explains the topological nature of the obstructions to integrability, found by V. V. Kozlov.

Let M be a closed surface with an integrable geodesic flow. Suppose for simplicity, that the geodesic flow has a Bott integral f. Recall that a function $f : Q^3 \to \mathbb{R}$ is Bott, if the set of critical points of it can be decomposed in disjoint union of non-degenerated critical submanifolds (see [9]).

Consider a regular isoenergy surface Q. It is a bundle over M^2, the fiber is the circle. In the papers [9], [20] it was shown, that the Liouville foliation on Q can be described by so-called marked molecule W^*. The marked molecule W^* is a finite graph. Then the manifold Q^3 can be decomposed on finitely many one-parameter families of the Liouville tori and finitely many singular fibers.

Consider the natural projection p of the manifold Q^3 to the base M^2. It induces a mapping $p_* : H_1(Q^3, \mathbb{Z}) \to H_1(M^2, \mathbb{Z})$ of first homology groups with integer coefficients. It is known, that each element α of the group $H_1(M^2, \mathbb{Z})$ can be realized as a closed geodesic γ M^2. The geodesic γ, in turn, is the projection of a closed orbit γ' of the geodesic flow on Q^3. Since the geodesic flow is integrable, then γ' lies on a Liouville fiber (singular or regular).

CASE 1. Let γ' lie on a Liouville T^2. Then the image of the group $H_1(T^2, \mathbb{Z}) \approx \mathbb{Z} \oplus \mathbb{Z}$ contains α. Note, that all Liouville tori from the one-parametric family, containing the torus T^2, are isotopic. Then their first homology groups map onto the same group $p_* H_1(T^2, \mathbb{Z})$.

CASE 2. Let γ' lies on a singular Liouville fiber. The fiber consists of finitely many one-dimensional orbits of the Poisson action of the group R^2 (the action is generated by commuting functions H and f), and finitely many two-dimensional orbits, each of whose is homeomorphic to the ring $S^1 \times D^1$. In each case the first homology group is isomorphic to $\mathbb{Z}$. Since γ' lies at an orbit O_i, then the element α lies in the image of the group $H_1(O_i, \mathbb{Z}) \approx \mathbb{Z}$.

The group $H_1(M^2, \mathbb{Z})$ is the set-theoretic union of a finite number of subgroups, each of these subgroups has no more than two generators. Since $H_1(M^2, \mathbb{Z})$ is isomorphic either $\mathbb{Z}^{2g}$(orientable case) or $\mathbb{Z}^k \oplus \mathbb{Z}_2$ (non-orientable case), then the group of the first homology of the surface M^2 is one of the following four groups: $\{e\}, \mathbb{Z} \oplus \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z} \oplus \mathbb{Z}$. All the other groups are too big.

Thus, the surface M^2 is either the sphere, or the projective plain, or the torus, or the Klein bottle. Theorem 1 is proved for Bott integrable geodesic flows.

It is clear, that the condition "the integral is Bott function" is not necessary here. A similar proof can be given, if we assume that the foliation of the manifold Q on the orbits of the Poisson action contains only FINITELY many singular orbits and only FINITELY many one-parametric families of one-parameter families of regular Liouville tori. Or if the integral is a so-called tame function. This case was investigated in the paper [36].

It is known, that analytic integrals are tame. The theorem is proved.

3. Local results

Let us recall two simplest examples of integrable geodesic flows. Consider the surface of revolution M^2, given by the equation $r = r(z)$ in $\mathbb{R}^3$ in the standard cylindric coordinate system r, φ, z.

Variables z and φ can be considered as a local coordinate system on M^2. Consider the Euclidean metric on $\mathbb{R}^3$. Then the induced Riemannian metric on M^2 is given by

$$ds^2 = (1 + (r')^2)dz^2 + r^2(z)d\varphi^2.$$

Let $z = z(t), \varphi = \varphi(t)$ be the parametric equation of a geodesic on M^2. Let ψ be the angle between the velocity vector of the geodesic and the parallel of the surface of revolution.

Theorem 2. *(Clairaut) The geodesic flow of a surface of revolution in R^3 is completely Liouville integrable. The function $r\cos\psi$ is constant on each geodesic, i.e. it is an integral. The geodesics (considered as unparameterized curves) are given by the following explicit formula.*

$$\varphi(z) = \int c\frac{\sqrt{1 + r'^2}}{\sqrt{r(r^2 - c^2)}}dz.$$

The second example is so-called Liouville metrics. Recall, that a Riemannian metric on the surface M^2 is Liouville, if in the appropriate local coordinates x y it is given by

$$ds^2 = (f(x) + g(y))\big(dx^2 + dy^2\big),$$

where $f(x)$ and $g(y)$ are arbitrary smooth positive functions.

Theorem 3. *(Liouville) The geodesic flow of a Liouville metric is completely Liouville integrable. The geodesics (considered as unparameterized curves) are given by*

$$\int \frac{dx}{\sqrt{f(x) + a}} \pm \int \frac{dy}{\sqrt{g(y) - a}} = c,$$

where a and c are constants.

Note, that in these two examples the integrals are either linear or quadratic in momenta. It turns out, that if a geodesic flow admits an integral, linear or quadratic in momenta, then the metric is (locally) a Liouville metric or a metric of revolution.

DEFINITION 2. *The geodesic flow of a metric is called linearly integrable, if the geodesic flow of it has an integral linear in momenta. The geodesic flows of a metric is called quadratically integrable, if it is not linearly integrable, and if the geodesic flow of it has an integral quadratic in momenta.*

Local properties of metrics with linearly and quadratically integrable geodesic flows were investigated in the classical papers of G. Darboux [16], U. Dini [18], M. L. Raffy [50], G. Birkhoff [7]. In modern terms the theory was reformulated by V. N. Kolokol'tsov [31], and independently obtained by K. Kiyohara [15]. See also [24], [28].

The following theorems give a local description of such metrics at a point of general position.

Theorem 4. *Let the geodesic flow of a metric ds^2 have an integral F, linear in momenta, in a neighborhood of a point P. Let the integral be not identically zero on the (co)tangent plane at the point P. Then in a neighborhood of the point P there exist regular local coordinates u, v, such that the metric has the following form:*

$$ds^2 = \lambda(u)(du^2 + dv^2).$$

Theorem 5. *Let the geodesic flow of the metric ds^2 have an integral F, quadratic in momenta, in a neighborhood of a point $P \in M$. Let the integral (considered as a 2-form) be not proportional to the Hamiltonian H on the (co)tangent plane at the point P. Then the metric is Liouville in a neighborhood of the point P, i. e. there exist local coordinates u, v, such that the metric reads*

$$ds^2 = \big(f(u) + g(v)\big)(du^2 + dv^2),$$

where $f(u)$ and $g(v)$ are smooth functions, and the integral F reads

$$F(u, v, p_u, p_v) = \frac{-f(u)p_v^2 + g(v)p_u^2}{f(u) + g(v)}.$$

These theorems describe metrics with geodesic flows, integrable linearly and quadratically in moments, near a point P of general position. What happens at a singular point? In other words, what kinds of singularities do the coordinates u, v from Theorems 4,5? The answer is given by the following theorem which includes both cases.

Theorem 6. *Let a geodesic flow in a neighborhood of a point $P \in M^2$ have an integral F, quadratic in momenta. Let the integral F be proportional to the Hamiltonian H at the point P. Then there are the following two possibilities.*

1) There exist coordinates u, v in a neighborhood of the point P, such that the metric reads

$$ds^2 = f(u^2+v^2)(du^2+dv^2),$$

where $f(t)$ is a positive smooth function. Then there exists another additional integral $\tilde{F} = vp_u - up_v$, linear in momenta.

2) There exist coordinates u, v in a neighborhood of the point P, such that the metric reads

$$ds^2 = \frac{h(r+u) - h(u-r)}{2r}(du^2+dv^2),$$

where r is equal to $\sqrt{u^2+v^2}$, and h is a function, smooth in a neighborhood of zero, such that $h'(0) > 0$. Then the integral F, quadratic in momenta, has the following form (modulo linear combination with the Hamiltonian):

$$F(u,v,p_u,p_v) = -\frac{r\big(h(u+r)+h(u-r)\big)}{h(u+r)-h(u-r)}(p_u^2+p_v^2) + (up_u^2 + 2vp_up_v - up_v^2).$$

Thus, for linear and quadratic integrability, the "canonical" coordinates can have exactly two types of singularities. These types of singularities have the polar and the elliptic coordinates.

The proof is based on an investigation of the so-called Birkhoff form. Let us recall the main construction.

Let a Riemannian metric in a neighborhood of a point of a surface be given by $ds^2 = \lambda(x,y)(dx^2+dy^2)$. In other words, let x, y be local isothermal coordinates of the metric. Suppose, that the geodesic flow of the metric has an integral

$$F = \sum b_i(x,y)p_x^{n-i}p_y^i,$$

polynomial in momenta of degree n. Consider the complex-valued function

$$R(z) = (b_0 - b_2 + b_4 - b_6 + \ldots) + i(b_1 - b_3 + b_5 - b_7 + \ldots),$$

where $z = x + iy$.

Without loss of generality we can assume, that the function R is not identically zero. (If the function is zero, then the integral can be divided by the Hamiltonian, the result is evidently an integral, polynomial in momenta of degree $n-2$.)

The following fact was noted by G. Birkhoff (for $n = 1, 2$), and generalized by V. N. Kolokol'tsov for arbitrary n. $\frac{(dz)^n}{R(z)}$ is a meromorphic form on the surface. An investigation of the form at singular and regular points leads to the results formulated above.

Besides, this fact has a very important global corollary (V. N. Kolokol'tsov): there is no polynomially integrable geodesic flow on a closed surface of negative Euler characteristic.

The last statement follows immediately from the following two facts. First, the Birkhoff form has no zeros. Second, there are no n–differentials without zeros on a closed surface of negative Euler characteristic.

What is known about the geodesic flows, integrable polynomially in momenta of degree more than two?

A kind of local answer is given by the following theorem due to V. V. Kozlov.

Theorem 7. *For any natural n there exists an integrable Hamiltonian system on the domain $D_x \times \mathbb{R}^2_p$ (where D is a disk in the plain x_1, x_2; $\mathbb{R}^2_p$ is the plain with coordinates p_1, p_2) with the Hamiltonian of form*

$$H = \frac{p_1^2 + p_2^2}{2\Lambda(x_1, x_2)},$$

such that it has an integral, polynomial (in momenta p_1, p_2) of degree n, the integral does not depend of H and there is no integral, independent of H and polynomial in momenta of degree less than n.

The proof of Theorem 7 is given in V. Ten [57].

Note, that local examples of geodesic flows, integrable by polynomial in momenta of degree three or four, can be constructed from Calogero-Moser system [14], [41] and Toda lattice [59] using Maupertuis principle [10].

4. Metrical classification of linearly and quadratically integrable geodesic flows on closed surfaces

Metrical classification of geodesic flows, integrable linearly or quadratically in momenta, was obtain in the papers of V. N. Kolokol'tsov [31], K. Kiyohara [24], I. K. Babenko [3], I. K. Babenko and N. N. Nekhoroshev [2], V. S. Matveev [34], [35].

4.1. The torus.

As it will be shown, there exist global coordinates on the covering plane, such that the metric has a canonical type, see Theorems 4,5.

Consider the plane $\mathbb{R}^2$ with Cartesian coordinates (u, v) and the Riemannian metric $ds^2 = q(v)(du^2 + dv^2)$ on it, where the function $q(v)$ has period L. Consider the lattice Γ generated by the vectors $f_1 = (1, 0)$ and $f_2 = (t, L)$, where $t \in [0, 1)$, $L > 0$. Since $q(v)$ is invariant modulo shifts along the elements of the lattice, then the metric ds^2 defines a metric on the torus $T^2 = \mathbb{R}^2/\Gamma$. The metric is constructed by three parameters: q, t, L, so we call it the (q, t, L)–metric.

Theorem 8. *Let the geodesic flow of a metric ds^2 on the torus T have an integral, linear in momenta. Then the metric ds^2 is either flat or is isometric to the (q, t, L)–metric (for appropriate parameters q, t, L).*

Consider now the case of Riemannian metric that admits an integral, quadratic in momenta, and that does not admit an integral, linear in momenta.

Consider the plane $\mathbb{R}^2$ with Cartesian coordinates (x, y) and the Riemannian metric

$$ds^2 = (f(x) + g(y))(dx^2 + dy^2)$$

on it, where f and g are smooth positive functions with periods 1 and L respectively. Consider the factorspace of the plane modulo the lattice Γ, generated by the vectors $f_1 = (m, 0)$, $f_2 = (k, L)$. Since the metric ds^2 is invariant modulo shifts along the elements of Γ, then we obtain a smooth metric on the torus $T^2 = \mathbb{R}^2/\Gamma$. This metric is called the $(L, f, g, k/m)$–metric.

Theorem 9. *If the geodesic flow of a metric on the torus has an integral, quadratic in momenta, and has no integral, linear in momenta, then the metric is isometrical to the $(L, f, g, k/m)$–metric (for appropriate parameters $L, f, g, k/m$).*

COMMENT. $(L, f, g, k/m)$–metric with $k = 0$ are globally Liouville metrics. In this case the coordinates x and y are global periodic coordinates on the torus. If $k \neq 0$, then the metric is a finite-sheeted Liouville metric. A finite-sheeted Liouville metric becomes a Liouville on a finite covering.

4.2. Klein bottle.

Recall, that the Klein bottle can be double covered by the torus.

It is clear, that if we cover the Klein bottle (with an integrable geodesic flow on it) by the torus, then the corresponding geodesic flow on the torus is also integrable. If the integral of the geodesic flow on the Klein bottle is polynomial in momenta, then the integral on the torus is also polynomial in momenta. But the minimal degree of the polynomial (in momenta) integral can be changed. For example, a quadratically integrable geodesic flow on the Klein bottle can generate a linearly integrable geodesic flow on the torus, see I. K. Babenko [3], V. S. Matveev [35], [36]. So that there are the following four series of the integrable geodesic flows on the Klein bottle:

1) linearly integrable (there exists an integral, linear in momenta);

2) quasi-linearly integrable (there exists an integral, quadratic in momenta, there is no integral, linear in momenta, and on the covering torus there exist an integral, linear in momenta);

3) quadratically integrable (three exists an integral, quadratic in momenta, and even on the covering torus, there exists no integral, linear in momenta);

4) quasi-quadratically integrable (there exists an integral, forth degree in momenta, there exists no integral of degree less than four, and on the covering torus there exists an integral, quadratic in momenta).

Let us describe linearly, quasi-linearly, quadratically and quasi-quadratically integrable geodesic flows. All of them can be constructed explicitly. Consider the plain $\mathbb{R}^2$ with the Cartesian coordinates x, y, and the orthogonal lattice Γ, generated by the basic $f_1 = (1, 0)$, $f_2 = (0, L)$, where L is a positive number.

Consider the metric
$$\big(f(x)+g(y)\big)(dx^2+dy^2)$$
on it, where $f(x)$ is a function with period $1/2$, $g(y)$ is an even function with period L, and for any x, y $f(x)+g(y)>0$.

The metric evidently generates a smooth metric on the torus $T^2=\mathbb{R}^2/\Gamma$. Consider now an involution ξ, given by the formula:

$$\xi(x,y)=(x+1/2,-y).$$

Clearly, the function $f(x)+g(y)$ is invariant modulo the involution ξ. Then the factorspace modulo the involution is the Klein bottle with a smooth metric. We call this metric the (L,f,g)–metric.

If the function f is identically zero, then the geodesic flow is linearly integrable. The linear integral on the torus is the function p_x. Since the involution ξ preserves the function p_x, then the function p_x generates an integral, linear in momenta, of the geodesic flow of the $(L,0,g)$–metric on the Klein bottle.

If the function g is identically zero, then the geodesic flow of the (L,f,g)–metric is quasi-linearly integrable. More precisely, the corresponding geodesic flow on the covering torus has an integral p_y, linear in momenta, which is not preserved by the involution ξ. The involution ξ changes the sign of the function p_y. Then the involution ξ preserves the function p_y^2, so that the function p_y^2, quadratic in momenta, is an integral of the geodesic flow of the $(L,f,0)$–metric.

Finally, if the functions f and g are not constant, then the geodesic flow of the (L,f,g)–metric (for any positive L) is quadratically integrable.

Theorem 10.

1) If the geodesic flow of a metric on the Klein bottle is linearly integrable, then the metric is isometric to the $(L,0,g)$–metric (for appropriate parameters L and g).

2) If the geodesic flow of a metric on the Klein bottle is quasi-linearly integrable, then the metric is isometrical to the $(L,f,0)$–metric (for appropriate parameters L and f).

3) If the geodesic flow of a metric on the Klein bottle is quadratically integrable, then the metric is isometric to the (L,f,g)–metric (for appropriate parameters L, f and g).

Now consider quasi-quadratically integrable geodesic flows on the Klein bottle. The integral F of such geodesic flows locally has the form $F=\tilde{F}^2$, where $\tilde{F}$ is a polynomial in momenta of second degree. On the covering torus the geodesic flow is quadratically integrable.

Let m and n be natural, relatively prime numbers. Let f be a positive nonconstant function with period 1. Consider the standard Cartesian coordinates x, y on the plane $\mathbb{R}^2$, and consider the metric

$$ds^2=\big(f(x)+f(y+n/2)\big)(dx^2+dy^2).$$

Let Γ be an orthogonal lattice on the plane, generated by the vectors $f_1 = (m, -m), f_2 = (n, n)$. Easy to see, that the metric is invariant modulo the lattice Γ. Then the metric defines a metric on the torus $\mathbb{R}^2/\Gamma$.

Consider the involution $\xi : \mathbb{R}^2/\Gamma \to \mathbb{R}^2/\Gamma$, given by the formula $\xi(x, y) = (y + n/2, x + n/2)$. The factorspace of the torus $\mathbb{R}^2/\Gamma$ modulo the involution is the Klein bottle. Clearly, the metric on the torus is preserved by the involution ξ, so that the metric on the torus defines a metric on the Klein bottle. Denote this metric on the Klein bottle by $ds^2_{m,n,f}$.

Theorem 11.
1) The geodesic flow of the metric $ds^2_{m,n,f}$ on the Klein bottle has an integral, polynomial in momenta of fourth degree, and has no integral, polynomial in momenta of degree less than four.

2) If the geodesic flow of a metric on the Klein bottle is quasi-quadratically integrable, then the metric is isometric to $ds^2_{m,n,f}$ (for appropriate m, n, f)

4.3. Sphere.

A description of the geodesic flows on the sphere, integrable linearly or quadratically in momenta, is due to Kolokol'tsov [31], and was independently obtained by K. Kiyohara [24], [25]. In the paper [44] N. T. Zung, E. N. Selivanova and L. S. Polyakova suggested a geometric formulation of the results. The geometrical approach seems to be more convenient for our description of a global topology of integrable geodesic flows on the sphere.

First we describe linearly integrable geodesic flow on the sphere. Let us consider the sphere as a compactified complex plane. The global coordinates on the sphere are real and imaginary parts of the standard complex variable on the complex plane.

Theorem 12. *The geodesic flow of a metric on the sphere has an integral, linear in momenta, iff in global coordinates x, y the metric is*

$$ds^2 = f(x^2 + y^2)(dx^2 + dy^2),$$

where $f(t)$ is a positive smooth function on the ray $[0, +\infty)$, such that $\dfrac{f(1/t)}{t^2}$ is also a positive smooth function on the ray $[0, +\infty)$.

The second condition is equivalent to the metric being smooth at the infinity point.

We call such metrics f-metrics on the sphere.

An equivalent formulation of Theorem 12 is the following: The geodesic flow of a metric on the sphere has an integral, linear in momenta, iff the metric is isometric to the metric $ds^2 = h(z)ds_0^2$ for an appropriate smooth positive function $h(z) : [-1, 1] \to \mathbb{R}$.

Here ds_0^2 denotes the induced metric on the standard sphere $\{x^2+y^2+z^2 = 1\}$ in the Euclidean space $\mathbb{R}^3$ (with the standard metric $dx^2 + dy^2 + dz^2$).

COROLLARY. The geodesic flow of a metric on the sphere has an integral, linear in momenta, iff there exists a smooth action of the circle $S^1 = SO(2)$, such that the action preserves the metric and the action is conjugate to the standard action of the group $SO(2)$ by the rotations around the axis z.

The following metrics are model metrics with quadratically integrable geodesic flows. Later on we will show, that if the geodesic flow of a metric is quadratically integrable, then the metric is isometric to an appropriate model one.

Consider the torus T^2 as a factorspace of $\mathbb{R}^2$ (with the Cartesian coordinates x, y) modulo orthogonal lattice Γ, generated by vectors $f_1 = (1, 0)$ and $f_2 = (0, L)$, where L is a positive number. Consider the involution $\sigma : T^2 \to T^2$, given by

$$\sigma(x, y) = (-x, -y).$$

Clearly the involution preserves the lattice Γ, so that σ is an involution on the torus. It is easy to see, that the factorspace T^2/σ is homeomorphic to the sphere, and that the projection $\varkappa : T^2 \to S^2 = T^2/\sigma$ can be considered as a smooth (and even complex-analytic) branched covering with exactly four branch points, each having degree 2.

Let $f(x)$, $g(y)$ be smooth functions, such that

a) the function $f(x)$ is non-negative, even, with the period 1.

b) the function $g(y)$ is non-negative, even, with the period L.

c) the function $f(x)$ is zero exactly at the points $x = m/2$, $m \in \mathbb{Z}$. The function $g(y)$ is zero exactly at the points $y = Lk/2$, $k \in \mathbb{Z}$. For any point $(m/2, kL/2) \in \mathbb{R}$ there exists a function $h(t)$ smooth in a neighborhood of zero, such that $h'(0) \neq 0$ and

$$f(m/2 + t) = h(t^2),$$
$$g(kL/2 + t) = -h(-t^2).$$

If we want the metric to be real-analytic, then the functions f, g and h must be real-analytic.

Consider the following degenerate metric $\big(f(x) + g(y)\big)(dx^2 + dy^2)$. The degenerate metric is not positively definite exactly at the branch points of $\varkappa$. Easy to see, that the degenerate metric is invariant on the torus T^2 modulo the involution σ. Therefore it induces a Riemannian metric on the sphere without four points. It turns out, that it is possible to define the form in the four points, such that the result is a smooth Riemann metric on the whole sphere.

Thus, given a triple (L, f, g) (satisfying the conditions a,b,c), we construct a smooth Riemannian metric ds^2 on the sphere S^2. We called the metric the (L, f, g)–metric.

Theorem 13. *The geodesic flow of a metric on the sphere is quadratically integrable, iff the metric is isometric to the (L, f, g)–metric (for appropriate parameters L, f, g).*

4.4. Projective plain.

Let us describe a family of metrics on $\mathbb{R}P^2$, whose geodesic flows are integrable, linear in momenta. Evidently $\mathbb{R}P^2$ is the factorspace $S^2/\mathbb{Z}_2$. Consider the sphere S^2 as a compactified complex plane $\mathbb{C}\cup\{\infty\}$. Let τ be an involution $\tau(z) = -1/\bar{z}$. Clearly the involution has no fixed points. Consider the metric

$$ds^2 = f(|z|^2)dzd\bar{z},$$

on the sphere $S^2 = \mathbb{C} + \{\infty\}$.

Here $f(t)$ is a smooth positive function $f : [0,\infty) \to \mathbb{R}$, such that $f(t) = f(1/t)/t^2$. Clearly the metric is preserved by the involution τ. Then the metric induced a metric on the projective space S^2/τ. The resulting metric on S^2/τ is constructed by the function f, we call it the f–metric .

Theorem 14. *The geodesic flow of a metric on $\mathbb{R}P^2$ has an integral, linear in momenta, iff the metric is isometric to the f-metric (for an appropriate f).*

Let us describe the metrics on $\mathbb{R}P^2$, whose geodesic flows are quadratically integrable. As above, consider the sphere as the factorspace of the torus modulo an involution σ, and consider the (L, f, g)–metrics,
where the functions f and g satisfy the conditions a,b,c. Consider the projective plane as the factorspace of the sphere modulo the involution τ given by:

$$\tau((x, y), (-x, -y)) = ((1/2 - x), L/2 + y), (x - 1/2, -L/2 - y)).$$

It is easy to see, that the involution τ has no fixed point. Evidently $S^2/\tau \approx \mathbb{R}P^2$. Now take the (L, f, g)–metrics, such that they are preserved by the involution τ. Evidently an (L, f, g)–metric is preserved by τ, if the functions f and g have periods $1/2$ and $L/2$, respectively.

Thus if the functions f and g have periods $1/2$ and $L/2$, respectively, then the (L, f, g)–metric on the sphere induces a metric on the projective plane. We call it the (L, f, g)–metric on the projective plane.

Theorem 15. *The geodesic flow of a Riemannian metric on the projective plane is quadratically integrable, iff the metric is isometric to the (L, f, g)–metric on a projective plane (for appropriate parameters L, f, g).*

We gave a complete list of geodesic flows, integrable linearly or quadratically in momenta. Each metric in the list is given by a finite set of parameters. The natural question is the following: what parameters correspond isometric metrics? The answer is given in [35]. Roughly speaking, it is not hard to know, given two sets of parameters, whether the corresponding metrics are isometric.

5. The metrics on the sphere, generated by Kovalevskaya and Goryachev-Chaplygin cases of rigid body motion.

The following two examples are constructed by using Maupertuis principle from the Kovalevskaya and Goryachev-Chaplygin cases of the rigid body motion.

Here we give only the explicit formulae of the resulting Riemannian metrics on the sphere. The complete construction is given in [10].

The metric of Goryachev-Chaplygin is the restriction of the metric:

$$ds^2 = \frac{h - u_1}{4} \cdot \frac{du_1^2 + du_2^2 + 4du_3^2}{u_1^2 + u_2^2 + u_3^2/4}$$

to the standard unit sphere. The geodesic flow of the metric is integrable, polynomial of third degree.

The metric of Kovalevskaya is the restriction of the metric

$$ds^2 = \frac{h - u_1}{2} \cdot \frac{du_1^2 + du_2^2 + 2du_3^2}{u_1^2 + u_2^2 + u_3^2/2}$$

to the standard unit sphere. The geodesic flow of the metric is integrable, polynomial of fourth degree.

S. A. Chaplygin [15] and D. N. Goryachev [17] constructed a four-parameter family of potentials on the sphere, such that the corresponding Maupertuis geodesic flows are integrable, polynomial of degree four. The family includes the Kovalevskaya metric.

Thus there exist metrics on the sphere, integrable polynomially of degree 1,2,3,4.

We have completely described Riemannian metrics, whose geodesic flows are quadratically or linearly integrable. The question of a complete description of metrics, integrable polynomial of degree more than two, is still open. Moreover, there is no known example, except for metrics of Kovalevskaya, Goryachev and Chaplygin. There is no known example of metric on the torus, integrable polynomial of degree more than two.

CONJECTURE A.(A. T. Fomenko, V. V. Kozlov). If the geodesic flow of a Riemannian metric on the torus T^2 has an integral, polynomial in momenta, then it has also an integral, linearly or quadratically in momenta.

CONJECTURE B. (A. T. Fomenko, V. V. Kozlov). If the geodesic flow of a Riemannian metric on the sphere S^2 has an integral, polynomial in momenta, then it has also an integral, linear, quadratic, third degree or fourth degree in momenta.

There are many results, confirming Conjecture A, see the papers V. V. Kozlov and N. V. Denisova [29], M. L. Bialy [13], V. V. Kozlov and D. V. Treshchev [30]. See also the review paper by A. V. Bolsinov, V. V. Kozlov and A. T. Fomenko [10] and the book by V. V. Kozlov [28].

6. Liouville classification of geodesic flows on surfaces, integrable linearly or quadratically in momenta

Here we review results of V. V. Kalashnikov (jun.) [23], N. T. Zung, L. S. Polyakova, E. N. Selivanova [44], V. S. Matveev [35], describing the topology of the Liouville foliations of integrable geodesic flows on surfaces. Following the general construction [9], [20], we give an answer in terms of so-called marked molecules. The molecules can be constructed explicitly by the parameters (of type f, g, L). Here we give an answer only in the simplest case, when the Liouville foliation has a minimal number of singularities. The results are given in Table 1.

Comments to Table 1.

1. The functions f and g have the minimal possible number of singular points. All singular points are non-degenerate.

2. We use the standard notations for singularities of Liouville foliation. The list of the standard notations see in [9], [20].

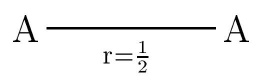
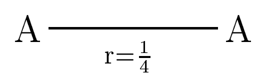
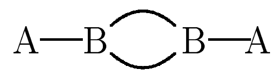
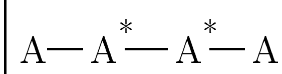
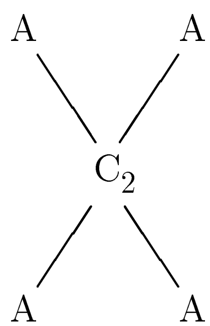
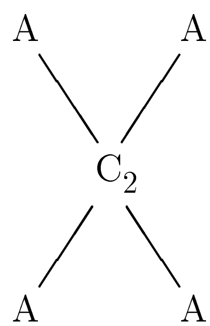
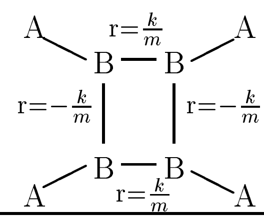
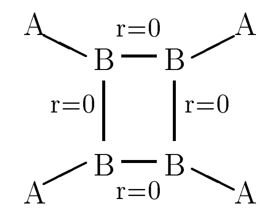
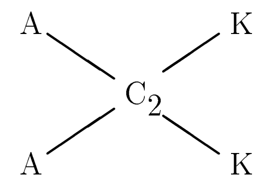
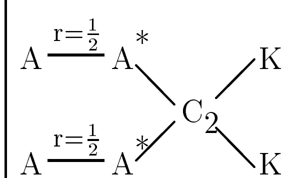
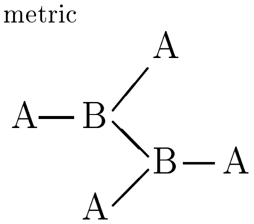
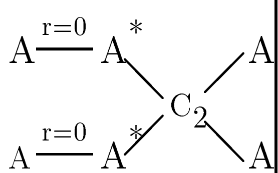
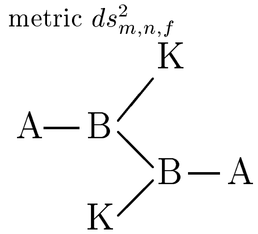
3. There are numerical marks, adjoint to the edges of the molecules, see [9], [20]. Here we put the marks only in the cases, when they are essential to distinguish topologically non-equivalent Liouville foliations. The reason is that in most cases the molecule (without numerical marks) is sufficient, to distinguish topologically non-equivalent Liouville foliations. Only for a few systems the molecules (without numerical marks) coincide, although the Liouville foliation is different. As an example, the molecule (without numerical marks) for the Kovalevskaya metric, and the molecule (without numerical marks) for metric of the simplest quadratically integrable geodesic flow on the Klein bottle are the same. That means that locally the corresponding Liouville foliations coincide. But the marks are different, so that globally the Liouville foliations are different.

4. The atom K appears in the cases of quasi-linearly and quasi-quadratically integrable geodesic flows. The corresponding fiber is diffeomorphic to the Klein bottle. Locally in a neighborhood of each point the integral F has the form $F = \tilde{F}^2$, where $\tilde{F}$ is a function without singularities. Globally it is not possible to take the square root. The reason is that the normal bundle to the Klein bottle (in Q^3) is non-trivial.

5. If the functions f and g have an arbitrary amount of critical points, the general structure of the molecules is similar. Roughly speaking, instead of atoms A we must put some graphs, see [23], [44].

6. The question marks in Table 1 mean, that the are no known examples of the corresponding metrics. Up to the above conjectures, there is no such example at all.

TABLE 1

	S^2	RP^2	T^2	K^2
1	f-metric 	f-metric 	(q,t,L)-metric 	$(L,0,g)$-metric 
2	(L,f,g)-metric  n=2	(L,f,g)-metric  n=4	$(L,f,g,\frac{k}{m})$-metric  Globally Liouville metric: $(L,f,g,0)$-metric 	Quasi-linear case: $(L,f,0)$-metric  (L,f,g)-metric 
3	Goryachev-Chaplygin metric 	**?**	**?**	**?**
4	Kovalevskaya metric 	**?**	**?**	Quasi-quadratic case: metric $ds^2_{m,n,f}$ 

7. Orbital classification of integrable geodesic flows

An orbital classification of geodesic flows on surfaces, integrable linearly or quadratically in momenta, can be made with the help of a so-called t-molecule. t-molecule is an invariant, suggested by A. V. Bolsinov and A. T. Fomenko [9]. This invariant is essentially the marked molecule plus new invariants that describe the behavior of the orbits of the Hamiltonian system on one-parameter families of Liouville tori (i.e. on the edges of the molecule) and in the neighborhoods of singular fibers of the Liouville foliation. In the case of geodesic flows, integrable linearly and quadratically, all the invariants can be obtained explicitly, depending on the parameters of the metric, see [51]. For simplicity we give the most interesting examples.

Consider a one-parameter family of Liouville tori in the isoenergy surface $Q^3 = \{H = 1\}$. It is known, that the dynamic of the Hamiltonian system on a Liouville torus is described by the rotation number. The rotation number depends on the torus, and therefore we must consider so-called rotation function $\rho(t)$, where t is a parameter on the edge of the molecule. The class of conjugacy of the rotation function is an orbital invariant. (Two functions $\rho(t)$ and $\tilde\rho(\tilde t)$ are conjugate, if there exist a substitution $\tilde t = \tilde t(t)$, such that $\tilde\rho(\tilde t(t)) = \rho(t)$).

It can be shown, that in the simplest bifurcations of the Liouville tori (i. e. if there is no atom except for A, A^* and B), there is no atom invariant, describing the behavior of the orbits near the singular fibers. In other words, the rotation function describes the system modulo orbital equivalence. As a result we get the following theorem.

Theorem 16. *Let ds^2 and ds'^2 be two globally-Liouville metrics on the torus. Suppose there is no bifurcation of the Liouville tori of the corresponding geodesic flows v and v', except for the atoms A and B. Then the systems v and v' are topologically orbitally equivalent, iff the flows are Liouville equivalent and the rotation functions on the corresponding edges are conjugate.*

Using the theorem, it is easy to construct examples of topologically orbitally equivalent geodesic flows on the torus. For example, consider the following two globally-Liouville metrics on the torus.

$$ds^2 = \big(f(x) + g(y)\big)(dx^2 + dy^2) \qquad d\tilde s^2 = \big(\tilde f(x) + \tilde g(y)\big)(dx^2 + dy^2),$$

where the periodic functions f, g and $\tilde f, \tilde g$ have exactly one minimum and one maximum inside the period. The corresponding molecules W^* and $\tilde W^*$ have the simplest form, see Table 1.

Suppose for simplicity, that the rotation functions ρ and $\tilde\rho$ are monotone on each edge. Then the function ρ changes from zero to plus infinity on every edge with atoms B at each origins. Then all rotation functions on such edges are conjugate. The behavior of the function ρ on the edges between the atoms A and B is the following. The limit of the rotation function on the atom B is either zero or infinity, so it gives no contribution to invariants. The limit of the rotation function on the edge with the atom A is a finite number. This number is a non-trivial invariant of the geodesic flow. Denote by p this number for two upper atoms A (the number is the same for both atoms), denote by q this number for two lower atoms A.

The notations for the flow $d\tilde{s}^2$ are similar, we denote the limits by $\tilde{p}, \tilde{q}$. Using Theorem 16, we have that the flows v are $\tilde{v}$ are topologically orbitally equivalent, iff $(p, q) = (\tilde{p}, \tilde{q})$ or $(p, q) = (\tilde{q}^{-1}, \tilde{p}^{-1})$. The second possibility is because of a non-trivial symmetry of the geodesic flow, corresponding to a rearrangement of the variables x and y. The rearrangement changes the rotation function in the following way: $\rho \to 1/\rho$.

The last result has an equivalent formulation. In the assumptions (monotonicy of the rotation functions and the simplest type of the molecule) the geodesic flow has evidently exactly two stable closed geodesics, corresponding to two pairs of A.

Consider a stable geodesic γ of the geodesic flow. Consider the limit ρ_0 of the rotation function $\rho(t)$, when t goes to the point, corresponding to γ. Consider the Morse index $\mathrm{ind}(\gamma)$ of the geodesic γ. Consider the multiplier $\nu = \exp(2\pi i\varphi)$ of the geodesic γ (a multiplier is an eigenvalue of the linearization of the Poincare mapping). The following formulae are true:

$$\mathrm{ind}(\gamma) = [2\rho_0] \quad \text{(i. e. integral part of } 2\rho_0),$$

$$\nu = \exp(2\pi i\rho_0).$$

Combining the formulae with Theorem 16, we have the following corollary.

COROLLARY. Consider two quadratically integrable geodesic flows on the torus. Suppose, that the corresponding molecules have the simplest type, and that the rotation functions are monotone. Then the geodesic flows are orbitally equivalent, iff the corresponding closed stable geodesics have equal Morse indices and equal multipliers.

A similar result is true for the sphere.

Theorem 17. *Let ds^2 and $d\tilde{s}^2$ be two (L, f, g)–metric on the sphere. Suppose that the molecules of the metrics are the simplest possible, see Table 1. Then their geodesic flows are topologically orbitally equivalent, iff they are Liouville equivalent, and their rotation functions (on the corresponding edges) are topologically conjugate.*

COROLLARY. Consider two quadratically integrable geodesic flows on the sphere. Let the corresponding molecules W and W' be the simplest possible, and let the rotation functions be monotone. Then the geodesic flows are orbitally equivalent, iff the corresponding closed stable geodesics have equal Morse indices and equal multipliers.

The assumptions of the theorems are very natural. For example, geodesic flows on the ellipsoid and on the Poisson sphere satisfy them.

It can be shown, that for appropriate parameters, geodesic flows on the ellipsoid and on the Poisson sphere are topologically orbitally equivalent, but they are never smoothly orbitally equivalent.

Theorem 18. *(E. N. Selivanova [51]). Let ds^2 be a smooth Riemannian metric on the sphere or on the torus. Let the geodesic flow of the metric be integrable, linearly or quadratically in momenta. Then there exists a family of metrics (depending on a functional parameter), such that the metrics in the family are pairwise different, orbitally equivalent and the geodesic flow of each metric in the family has a non-trivial integral, quadratic in momenta.*

The theorem has the following interesting corollaries. Let us apply it to the standard metric of constant curvature on the sphere. We get a big family of metrics, whose geodesic flows are orbitally equivalent to the geodesic flow of the metric of constant curvature. In particular, all geodesics are closed, so that the metrics are so-called Zoll metrics.

This fact is well known for metrics of revolution, see A. Besse [4]. Let us consider them as integrable geodesic flows.

Consider the metric of constant curvature as a metric of revolution.

$$ds_0^2 = d\theta^2 + \sin^2\theta d\varphi^2.$$

The geodesic flow of it has an integral $F = p_\varphi$, linear in momenta. The molecule has the simplest type A——A, i. e. there exists exactly one one-parameter family of Liouville tori. The rotation function on the family is

$$\rho_0(F) = \frac{1}{\pi}\int_x^{\pi-x} \frac{F d\theta}{\sin\theta\sqrt{\sin^2\theta - F^2}},$$

where $\sin(x) = F$. Easy to check that $\rho_0(F) \equiv 1$.

Theorem 19. *(See [4]). A metric of revolution ds^2 on the sphere is a Zoll metric, iff it can be written as*

$$ds_h^2 = (1 + h(\cos\theta))^2 d\theta^2 + \sin^2\theta d\varphi^2,$$

where h is an odd function.

Now let us consider the metric of constant curvature as an (L, f, g)–metric.

Theorem 20. *There exists a family (depending on two functional parameters α and β) of C^∞–smooth (pairwise non-isometric) metrics $ds^2_{\alpha,\beta}$ on the sphere S^2, such that:*

1) The geodesic flows of the metrics are quadratically integrable,

2) all geodesic of the metrics are closed and have equal length.

The metrics $ds^2_{\alpha,\beta}$ are given by explicit formulae.

PROOF of the theorem.

To start with, the metric ds_0^2 of curvature 1 is an (L,g,h)-metric. Take the sphere-conical coordinates ν_2, ν_3. Then

$$ds_0^2 = \frac{1}{4}(\nu_2 - \nu_3)\left(-\frac{d\nu_2^2}{P(\nu_2)} + \frac{d\nu_3^2}{P(\nu_3)}\right),$$

where $P(\nu) = (a+\nu)(b+\nu)(c+\nu)$.

The function P can be considered as four separate functions on the following four domains

$U_{++} = \{x > 0, z > 0\}$,
$U_{-+} = \{x < 0, z > 0\}$,
$U_{+-} = \{x > 0, z < 0\}$,
$U_{--} = \{x < 0, z < 0\}$.

The functions $P(\nu_2)$ and $P(\nu_3)$ can be given independently, the only thing we need is that the functions must smoothly glue on the borders of the domains. The formulae for the functions $P(\nu_2)$ and $P(\nu_3)$, corresponding to the metric of constant curvature, is the same. We do not need it for the integrability of the geodesic flow. This observation allows us to construct perturbations of the standard metric, such that the rotation function does not change.

Let the metric be the following

$$ds^2 = \begin{cases} \frac{1}{4}(\nu_2 - \nu_3)\left(-\frac{d\nu_2^2}{Q_+(\nu_2)} + \frac{d\nu_3^2}{R_+(\nu_3)}\right) & \text{in the domain } U_{++}, \\ \frac{1}{4}(\nu_2 - \nu_3)\left(-\frac{d\nu_2^2}{Q_+(\nu_2)} + \frac{d\nu_3^2}{R_-(\nu_3)}\right) & \text{in the domain } U_{+-}, \\ \frac{1}{4}(\nu_2 - \nu_3)\left(-\frac{d\nu_2^2}{Q_-(\nu_2)} + \frac{d\nu_3^2}{R_+(\nu_3)}\right) & \text{in the domain } U_{-+}, \\ \frac{1}{4}(\nu_2 - \nu_3)\left(-\frac{d\nu_2^2}{Q_-(\nu_2)} + \frac{d\nu_3^2}{R_-(\nu_3)}\right) & \text{in the domain } U_{--}. \end{cases}$$

Now we can give an explicit formula for the the rotation function, and understand, when the rotation function is constant.

For example take

$$\begin{aligned} Q_+(\nu_2) &= \left(\frac{1}{\sqrt{-P(\nu_2)}} + \alpha(\nu_2)\right)^{-2}, \\ Q_-(\nu_2) &= \left(\frac{1}{\sqrt{-P(\nu_2)}} - \alpha(\nu_2)\right)^{-2}, \\ R_+(\nu_3) &= \left(\frac{1}{\sqrt{P(\nu_3)}} + \beta(\nu_3)\right)^{-2}, \\ R_-(\nu_3) &= \left(\frac{1}{\sqrt{P(\nu_3)}} - \beta(\nu_3)\right)^{-2}. \end{aligned}$$

Note, that the functions α and β are almost arbitrary. The only essential conditions is that adding the functions α and β does not change the asymptotic of the function $P(\nu)$ near the points $\nu = -a, -b, -c$, and the expression in the brackets must be positive.

The constructed perturbated metrics $ds^2_{\alpha,\beta}$ are smooth, but not analytic. Moreover, it is not possible to make such perturbations analytic.

COMMENT. A slightly different description of such metrics is given in the paper of K. Kiyohara [24]. In the paper [25] he generalized the construction for an arbitrary dimension. The idea of finding new examples of Zoll metric, starting from quadratically integrable geodesic flows, is due to V. N. Kolokol'tsov [32]. The present construction was suggested by E. N. Selivanova.

Zoll metrics are evidently integrable. It is interesting to understand types of the integrals. Are they polynomials in momenta? There are no explicit examples of non-polynomial integrals of geodesic flows.

8. Geodesic equivalence

Some results from this section are relatively new, so we give them with proofs. We show that geodesic equivalence of metrics leads to integrability of geodesic flows, moreover, the integrals are quadratic in momenta. These allows us to apply the results of Sections 2,3,4,5 to the theory of geodeiscally equivalent metrics. The idea of such investigation was suggested by P. J. Topalov [60], [61] and it is based on the following remark.

Let v and $\bar v$ be Hamiltonian systems on symplectic manifolds (M^{2n},ω) and $(\bar M^{2n},\bar\omega)$ with Hamiltonians H and $\bar H$, respectively. Consider the isoenergy surfaces

$$Q \stackrel{\rm def}{=} \left\{x\in M^{2n} : H(x)=h\right\}, \quad \bar Q \stackrel{\rm def}{=} \left\{x\in \bar M^{2n} : \bar H(x)=\bar h\right\},$$

where h and $\bar h$ are regular values of the functions H, $\bar H$, respectively.

In the paper [60] it was shown, that an orbital diffeomorphism allows one to construct n integrals of the geodesic flow of the system v. This result could be considered as a particular result of a theory, developed in [11], [60]. More precisely, in [11] it was shown, that the existence of a vector field on Q which commutes with the Hamiltonian vector field v allows one to construct a (multi-valued in general) integrals of the Hamiltonian system. In the paper [60] the result of [11] was generalized to tensor fields. It was shown, that if a Hamiltonian flow preserves a tensor field, then there exist (also multi-valued) integrals of the Hamiltonian system.

Now, the orbital diffeomorphism allows one to construct an invariant tensor field. Take the restriction $\bar\omega_{|\bar Q}$ of the symplectic form $\bar\omega$ to the isoenergy surface $\bar Q$, and consider the form $\phi^*\bar\omega_{|\bar Q}$ on Q. This form is preserved by the Hamiltonian flow v, see Lemma 1 in Section 8.2.

For the invariant form $\phi^*\bar\omega_{|\bar Q}$ the integrals are not multi-valued. The explicit formulae for them are given in Theorem 24.

There are not very many examples of orbital diffeomorphisms of mechanical systems, see [19], and all of them are a priory integrable.

A classical example of orbital equivalence of Hamiltonian systems is the geodesic equivalence of metrics. Let $g=(g_{ij})$ and $\bar g=(\bar g_{ij})$ be smooth metrics on the same manifold M^n.

DEFINITION 3. *The metrics g and $\bar{g}$ are* geodesically equivalent*, if they have the same geodesics (considered as unparameterized curves).*

For geodesically equivalent metrics, an orbital diffeomorphism Φ is given by $\Phi(x,\xi) = (x, \frac{\|\xi\|_g}{\|\xi\|_{\bar{g}}}\xi)$. Here $(x,\xi) \in \mathcal{T}M^n$, x is a point of M^n and $\xi \in \mathcal{T}_x M^n$.

The theory of geodesically equivalent metrics is rather classical material. The setting of it is due to E. Beltrami [5], [6]. He has proved, that the metric on a surface, geodesically equivalent to the metric of constant curvature, is also a metric of constant curvature. In 1869 U. Dini [18] formulated the problem of local description of geodesically equivalent metrics, and solved it for dimension two. In 1896 T. Levi-Civita [33] got a local description of geodesically equivalent metrics on manifolds of arbitrary dimension. A significant contribution to the theory of geodesically equivalent metrics was made by A. V. Aminova, N. S. Sinukov, P. Venzi, J. Mikesh, A. V. Pogorelov, see [38] for references.

Let us recall a few examples of geodesically equivalent metrics. There always exists a trivial example, which is not interesting for our consideration: an arbitrary metric g is geodesically equivalent to the metric Cg, where C is a constant.

The first non-trivial in this sense example is the following. Take the torus $T^n = S^1 \times S^1 \times ... \times S^1$. Consider the coordinate system $x^1, ..., x^n$ on T^n, assuming x^k is the cyclic coordinate on the circle number k. Now consider the metrics $ds_1^2 = \sum_{i=1}^n (dx^i)^2$ and $ds_2^2 = \sum_{i,j=1}^n a_{ij}(dx^i)(dx^j)$, where (a_{ij}) is a positive definite symmetric matrix. The metrics ds_1^2 and ds_2^2 are evidently geodesically equivalent, and they are, generally speaking, not proportional.

The second non-trivial example is due to E. Beltrami. Consider the standard sphere $\sum_{i=1}^{n+1}(x^i)^2 = 1$, where $x^1, ..., x^{n+1}$ are standard coordinates in the Euclidean space R^{n+1}. Take a non-degenerate linear transformation $L : R^{n+1} \to R^{n+1}$ of the space R^{n+1}, and consider the corresponding projective transformation l of the sphere. Let x be a point of the sphere. Consider the ray $[0, x)$, where 0 is the zero point of R^{n+1}. Evidently the image $L([0, x))$ is also a ray with the origin in zero. Let the intersection of the ray $L([0, x))$ and the sphere be y. Then, by definition, put $l(x) = y$.

It is easy to see, that the mapping l preserves the geodesics of the sphere. Actually, the geodesics of the sphere are intersections of the planes, which pass through the zero point, with the sphere. The linear mapping L takes planes to planes. Therefore the projective mapping l takes geodesics to geodesics. Then the standard metric $g_{standard}$ of the sphere and the metric $l^* g_{standard}$ are geodesically equivalent.

We would like to point out the following common property of these two examples: the geodesic flows of both metrics are completely Liouville integrable, in the sense that there exist n integrals in involution. We claim that this property is common for all geodesically equivalent metrics in general position.

Denote by G the linear operator $g^{-1}\bar{g} = (g^{i\alpha}\bar{g}_{\alpha j})$. Consider the characteristic polynomial $det(G - \mu E) = c_0\mu^n + c_1\mu^{n-1} + ... + c^n$. The coefficients $c_1, .., c_n$ are smooth functions on the manifold M^n, and $c_0 \equiv (-1)^n$. Consider functions I_k :

$TM^n \to R$, $k = 0, ..., n-1$, given by formulae $I_k(x,\xi) = \left(\frac{det(g)}{det(\bar g)}\right)^{\frac{k+2}{n+1}} \bar g(S_k\xi,\xi)$, where S_k is the linear operator given by $S_k \stackrel{\rm def}{=} \sum_{i=0}^{k} c_i G^{k-i}$, and $\bar g(\nu,\xi)$ denotes the dot product of the tangent vectors ν, ξ in the metrics $\bar g$.

Theorem 21. *If the metrics g and $\bar g$ on M^n are geodesically equivalent, then the functions I_k are integrals of the geodesic flow of the metric g and pairwise commute.*

The metrics $g, \bar g$ are strictly non-proportional, if the characteristic polynomial $det(G - \mu E)$ has no multiply roots. It can be shown, that if the metrics $g, \bar g$ are geodesically equivalent and strictly non-proportional at the point x, then in a neighborhood of the point the integrals I_k are functionally independent almost everywhere.

Theorem 21 allows one to apply the theory of quadratically integrable geodesic flows on surfaces to geodesically equivalent metrics. First of all, for dimension two the converse of Theorem 21 is also true.

COROLLARY 1. Metrics g and $\bar g$ on a surface M^2 are geodesically equivalent, if and only if the function $\left(\frac{det(g)}{det(\bar g)}\right)^{\frac{2}{3}} \bar g(\xi,\xi)$ is an integral of the geodesic flow of the metric g.

Combining Corollary 1 with Theorem 1 and with results of Sections 2, 3, 4, we get

COROLLARY 2. Let metrics $g, \bar g$ on a closed surface of negative Euler characteristic be geodesically equivalent. Then $g = C\bar g$, where C is a constant.

COROLLARY 3. Let metrics $g, \bar g$ on the torus T^2 be geodesically equivalent. If they are proportional at a point $x \in T^2$, then $g = C\bar g$, where C is a positive constant.

COROLLARY 4. Let metrics g, $\bar g$ on the sphere S^2 be geodesically equivalent. Then there are three possibilities.

1. The metrics are proportional at exactly two points.
2. The metrics are proportional at exactly four points.
3. The metrics are completely proportional, i.e. $g = C\bar g$, where C is a positive constant.

In the first case the metrics admit a Killing vector field.

Moreover, in Sections 2,3,4 a complete description of quadratically integrable geodesic flows on closed surfaces was obtained. In view of Corollary 1, it gives us a complete description of geodesically equivalent metrics on closed surfaces.

What is the dimension of the space of the metrics, geodesically equivalent to a given one? This question was actively discussed, see [38] for references. Even locally, there exist metrics that have no non-trivially geodesically equivalent one. Even locally, the dimension of space of metrics, geodesically equivalent to a given one, does not exceed $\frac{(n+1)(n+2)}{2}$ and is equal to $\frac{(n+1)(n+2)}{2}$ only for the metrics of constant curvature.

If the metrics $g, \bar{g}$ are geodesically equivalent, then there exists a one-parameter family of metrics, geodesically equivalent to g. Note, that the integrals I_k from Theorem 21 are quadratic in velocities. Then there exists symmetric bilinear forms $\tilde{I}_k$, such that for any $k \in \{0, .., n-1\}$ $\tilde{I}_k(\xi, \xi) = I_k(\xi)$.

Take a real number α and consider the form

$$f_\alpha \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} (-\alpha)^i \tilde{I}_i.$$

The form is positive definite for positive α, and therefore it can be positive definite for small negative α. Consider the metric

$$g_\alpha \stackrel{\text{def}}{=} \left(\frac{det(g)}{det(f_\alpha)}\right)^{\frac{2}{n-1}} f_\alpha.$$

Theorem 22. *If the metrics g, $\bar{g}$ are geodesically equivalent, then for any α, such that the metric g_α is positive definite, the metric g_α is geodesically equivalent to the metric g.*

Let M^n be closed, and let the metrics $g, \bar{g}$ be strictly non-proportional at almost everywhere dense set of points. Then the geodesic flow of the metric g is completely integrable, and almost all orbits lie at the corresponding Liouville tori. Suppose, that the geodesic flow is non-resonant. Then the Liouville foliation is uniquely definite, and any integral of the geodesic flow commutes with the integrals $I_0, ..., I_{n-1}$. Assume in addition, that there are sufficiently many caustics of the Liouville tori of the geodesic flow: almost each point of the surface is an intersection of n caustics. Then the Levi-Civita coordinates are uniquely definite, and the dimension of the space of the metrics (modulo multiplication by a constant), geodesically equivalent to the metric g, is equal one.

The geodesic flow of the metric of an ellipsoid satisfies all these conditions. First of all, it admits a non-trivially geodesically equivalent metric

Consider the ellipsoid
$\sum_{i=1}^n \frac{(x^i)^2}{a_i} = 1$, where $a_i > 0$, $i = 1, ..., n$.

Theorem 23. *The restriction of the metric $\sum_{i=1}^n (dx^i)^2$ to the ellipsoid $\sum_{i=1}^n \frac{(x^i)^2}{a_i} = 1$ is geodesically equivalent to the restriction of the metric*

$$\frac{1}{\sum_{i=1}^n \left(\frac{x^i}{a_i}\right)^2} \left(\sum_{i=1}^n \frac{(dx^i)^2}{a_i}\right)$$

to the ellipsoid.

A very beautiful construction that allows one to find the metric that is geodesically equivalent to the metric of ellipsoid is due to S. Tabachnikov [54].

If we apply Theorem 21 to the metrics from Theorem 23, then the integrals $I_0, ..., I_{n-1}$ are linear combinations of the integrals from [42]. In [42] it was shown, that the geodesic flow of the metric on the ellipsoid is non-resonant, and almost each point is the point of intersection of n caustics.

In Subsection 8.2 we prove Theorem 24 that gives an explicit formula for a one-parameter family of first integrals, if there exist an orbital diffeomorphism between two Hamiltonian systems. In Subsection 8.3, for readers convenience, we formulate Levi-Civita's results about the local form of geodesically equivalent metrics. In Subsection 8.4, 8.5 we apply Theorem 24 to geodesically equivalent metrics. As the result we get the formulae for the integrals I_k. In Subsection 8.5 we prove that the integrals I_k are in involution. In Subsections 8.6, 8.7 we prove Theorems 22, 23.

8.1. Orbital diffeomorphism and integrals.

Let v and $\bar v$ be Hamiltonian systems on symplectic manifolds (M,ω) and $(\bar M,\bar\omega)$ with Hamiltonians H and $\bar H$ respectively. Consider the isoenergy surfaces

$$Q \stackrel{\rm def}{=} \{x\in M : H(x)=h\}, \quad \bar Q \stackrel{\rm def}{=} \left\{x\in \bar M : \bar H(x)=\bar h\right\},$$

where h and $\bar h$ are regular values of the functions $H, \bar H$ respectively. Let $U(Q)\subset M$ and $U(\bar Q)\subset \bar M$ be neighborhoods of the isoenergy surfaces Q and $\bar Q$.

DEFINITION 4. *A diffeomorphism $\Phi : U(Q)\longrightarrow U(\bar Q)$, $\Phi(Q)=\bar Q$, is said to be* orbital *on Q, if the restriction $\Phi|_Q$ takes the orbits of the system v to the orbits of the system $\bar v$.*

Denote the restriction $\Phi|_Q$ by ϕ. Since ϕ takes the orbits of v to the orbits of $\bar v$, it takes the vector field v to the vector field that is proportional to $\bar v$. Denote by $a_1 : Q\to R$ the coefficient of proportionality, i.e. $\phi_*(v)=a_1\bar v$. Since Φ takes Q to $\bar Q$, it takes the differential $\mathrm{d}H$ to a form that is proportional to $d\bar H$. Denote by $a_2 : Q\to R$ the coefficient of proportionality, i.e. $\phi_*\mathrm{d}H = a_2\mathrm{d}\bar H$. By a we denote the product a_1a_2. We denote the Pfaffian of a skew-symmetric matrix X by $\mathrm{Pf}(X)$.

Theorem 24. *Let a diffeomorphism $\Phi : U(Q)\to U(\bar Q)$, $\Phi(Q)=\bar Q$, be orbital on Q. Then, for each value of the parameter t, the polynomial*

$$\mathcal{P}^{n-1}(t) \stackrel{\rm def}{=} \frac{\mathrm{Pf}\,(\Phi^*\bar\omega - t\omega)}{\mathrm{Pf}\,(\omega)\,(t-a)} \qquad (2)$$

is an integral of the system v on Q. In particular, all the coefficients of the polynomial $\mathcal{P}^{n-1}(t)$ are integrals.

PROOF. Denote by σ, $\bar\sigma$ the restrictions of the forms $\omega,\bar\omega$ to Q, $\bar Q$, respectively. Consider the form $\phi^*\bar\sigma$ on Q.

Lemma 1 (Topalov, [60]). *The flow v preserves the form $\phi^*\bar\sigma$.*

PROOF OF LEMMA 1. The Lie derivative L_v of the form $\phi^*\bar\sigma$ along the vector field v satisfies
$$L_v\phi^*\bar\sigma = \mathrm{d}\left[\imath_v\phi^*\bar\sigma\right] + \imath_v\mathrm{d}\left[\phi^*\bar\sigma\right].$$
On the right side both terms vanish. More precisely, for an arbitrary vector $u \in \mathcal{T}_xQ$ at an arbitrary point $x \in Q$ we have
$$\begin{aligned} \imath_v\phi^*\bar\sigma(u) &= \bar\sigma(\phi_*(v), \phi_*(u)) &= \\ &= \bar\sigma(a_1\bar v, \phi_*(u)) &= \\ &= -a_1\mathrm{d}\bar H(\phi_*(u)) &= 0. \end{aligned}$$
Since the form $\bar\omega$ is closed, the form $\bar\sigma$ is also closed and $\mathrm{d}\left[\phi^*\bar\sigma\right] = \phi^*(\mathrm{d}\bar\sigma) = 0$, q. e. d.

It is obvious that the kernels of the forms σ and $\phi^*\bar\sigma$ coincide (in the space $\mathcal{T}_xQ$ at each point $x \in Q$) with the linear span of the vector v. Therefore these forms induce two non-degenerate tensor fields on the quotient bundle $\mathcal{T}Q/\langle v\rangle$. We shall denote the corresponding forms on $\mathcal{T}Q/\langle v\rangle$ also by the letters $\sigma, \bar\sigma$.

Lemma 2. *The characteristic polynomial of the operator $(\sigma)^{-1}(\phi^*\bar\sigma)$ on $\mathcal{T}Q/\langle v\rangle$ is preserved by the flow v.*

PROOF of Lemma 2. Since the flow v preserves the Hamiltonian H and the form ω, the flow v preserves the form σ. Since the flow v preserves both forms, it preserves the characteristic polynomial of the operator $(\sigma)^{-1}(\phi^*\bar\sigma)$, q. e. d.

Since both forms are skew-symmetric, each root of the characteristic polynomial of the operator $(\sigma)^{-1}(\Phi^*\bar\sigma)$ has an even multiplicity. Then the characteristic polynomial is the square of a polynomial $\delta^{n-1}(t)$ of degree $n-1$. Hence the polynomial $\delta^{n-1}(t)$ is also preserved by the flow v. It is obvious that
$$\delta^{n-1}(t) = (-1)^{n-1}\frac{\mathrm{Pf}\,(\phi^*\bar\sigma - t\sigma)}{\mathrm{Pf}\,(\sigma)}. \tag{3}$$

The last step of the proof is to verify that
$$(t-a)\delta^{n-1} = \frac{\mathrm{Pf}\,(\Phi^*\bar\omega - t\omega)}{\mathrm{Pf}\,(\omega)} \stackrel{\mathrm{def}}{=} \Delta^n.$$
Take an arbitrary point $x \in Q$. Consider the form $\Phi^*\bar\omega - a\omega$ on $\mathcal{T}_xM$. The form $\imath_v(\Phi^*\bar\omega - a\omega)$ equals zero. More precisely, for any vector $u \in \mathcal{T}_xM$ we have
$$\begin{aligned} \imath_v(\Phi^*\bar\omega - a\omega) &= \bar\omega(\Phi_*(v), \Phi_*(u)) - a\omega(v,u) = \\ &= \bar\omega(a_1 v, \Phi_*(u)) - a\omega(v,u) = \\ &= -a_1\mathrm{d}\bar H(\Phi_*(u)) + a dH = \\ &= -a\mathrm{d}H + adH = 0. \end{aligned}$$
There exists a vector $A \in \mathcal{T}_xM$ such that $\omega(A, v) \neq 0$ and the restriction of the form $\imath_A(\Phi^*\bar\omega - a\omega)$ to the space $\mathcal{T}_xM$ equals zero. More precisely, since the forms

$\Phi^*\bar\omega$, ω are skew-symmetric, then the kernel $K_{\Phi^*\bar\omega-a\omega}$ of the form $\Phi^*\bar\omega-a\omega$ has an even dimension, and the kernel of the restriction of the form $\Phi^*\bar\omega-a\omega$ to $\mathcal{T}_xQ$ has an odd dimension. Thus the intersection $K_{\Phi^*\bar\omega-a\omega}\cap(\mathcal{T}_xM\setminus\mathcal{T}_xQ)$ is not empty. For each vector A from the intersection we obviously have $\omega(A,v)\neq 0$ and $\imath_A(\Phi^*\bar\omega-a\omega)=0$. Without loss of generality we can assume $\omega(A,v)=1$.

Consider a basis $(v,e_1,...,e_{2n-2})$ for the space $\mathcal{T}_xQ$. The set $(A,v,e_1,...,e_{2n-2})$ is a basis for the space $\mathcal{T}_xM$. In this basis we have

$$\begin{aligned}\det(\Phi^*\bar\omega-t\omega) &= \det\left|\begin{array}{cc|c} 0 & a-t & (*) \\ -(a-t) & 0 & 0\cdots 0 \\ \hline -(*) & 0 & (\Phi^*\bar\omega-t\omega)_{\langle e_1,\ldots,e_{2n-2}\rangle}\end{array}\right| \\ &= (a-t)^2\det((\Phi^*\bar\omega-t\omega)_{\langle e_1,\ldots,e_{2n-2}\rangle}) \\ &= (a-t)^2\det(\phi^*\bar\sigma-t\sigma),\end{aligned}$$

where $(\Phi^*\bar\omega-t\omega)_{\langle e_1,\ldots,e_{2n-2}\rangle}$ is the matrix of the form $\Phi^*\bar\omega-t\omega$ in the basis $(e_1,...e_{2n-2})$. Finally, $\delta^{n-1}=\mathcal{P}^{n-1}$, q. e. d .

8.2. Levi-Civita theorems.

Let g and $\bar g$ be geodesically equivalent metrics on a manifold M^n. Denote by $\rho^1,...,\rho^m$ $(1\le m\le n)$ the common eigenvalues of the metrics g and $\bar g$. Suppose the functions $\rho^1,...,\rho^m$ are differ at every point of an open domain $\mathcal{D}\subset M^n$. In the paper [33], Levi-Civita proved that for every point $P\in\mathcal{D}$ there is an open neighborhood $\mathcal{U}(P)\subset\mathcal{D}$ and a coordinate system $\bar x=(\bar x_1,...,\bar x_m)$ (in $\mathcal{U}(P)$), where $\bar x_i=(x_i^1,...,x_i^{k_i})$, $(1\le i\le m)$, such that the quadratic forms of the metrics g and $\bar g$ have the following form:

$$\begin{aligned} g(\dot{\bar x},\dot{\bar x}) &= \Pi_1(\bar x)A_1(\bar x_1,\dot{\bar x}_1)+\ \Pi_2(\bar x)A_2(\bar x_2,\dot{\bar x}_2)+\cdots+ \\ &+\ \Pi_m(\bar x)A_m(\bar x_m,\dot{\bar x}_m), \qquad (4)\\ \bar g(\dot{\bar x},\dot{\bar x}) &= \rho^1\Pi_1(\bar x)A_1(\bar x_1,\dot{\bar x}_1)+\rho^2\Pi_2(\bar x)A_2(\bar x_2,\dot{\bar x}_2)+\cdots+ \\ &+\ \rho^m\Pi_m(\bar x)A_m(\bar x_m,\dot{\bar x}_m), \qquad (5)\end{aligned}$$

where $A_i(\bar x_i,\dot{\bar x}_i)$ are positive-definite quadratic forms in the velocities $\dot{\bar x}_i$ with coefficients depending on $\bar x_i$,

$$\Pi_i \stackrel{\text{def}}{=} (\phi_i-\phi_1)\cdots(\phi_i-\phi_{i-1})(\phi_{i+1}-\phi_i)\cdots(\phi_m-\phi_i) \qquad (6)$$

and $\phi_1,\phi_2,...,\phi_m$, $0<\phi_1<\phi_2<...<\phi_m$, are smooth functions such that

$$\phi_i=\begin{cases}\phi_i(\bar x_i), & \text{if}\quad k_i=1\\ \text{constant}, & \text{else.}\end{cases}$$

It is easy to see that the functions ρ^i as functions of ϕ_i and the function ϕ_i as functions of ρ^i are given by

$$\begin{aligned}\rho^i &= \frac{1}{\phi_1...\phi_m}\frac{1}{\phi_i}\\ \phi_i &= \frac{1}{\rho_i}(\rho_1\rho_2...\rho_m)^{\frac{1}{m+1}}\end{aligned}$$

DEFINITION 5. *Let metrics g and $\bar{g}$ be given by formulae (4) and (5) in a coordinate chart $\mathcal{U}$. Then we say that the metrics g and $\bar{g}$ have* Levi-Civita local form (of type m)*, and the coordinate chart $\mathcal{U}$ is* Levi-Civita *coordinate chart (with respect to the metrics).*

Levi-Civita proved that the metrics g and $\bar{g}$ given by formulae (4) and (5) are geodesically equivalent. If we replace ϕ_i by $\phi_i + c$, $i = 1, ..., m$, where c is a (positive for simplicity) constant, in (4) and (5), we obtain the following one-parameter family of metrics, geodesically equivalent to g:

$$g_c(\dot{\bar{x}}, \dot{\bar{x}}) = \frac{1}{(\phi_1 + c)\cdots(\phi_m + c)}\left\{\frac{1}{\phi_1 + c}\Pi_1 A_1 + \cdots + \frac{1}{\phi_m + c}\Pi_m A_m\right\}. \quad (7)$$

The next theorem is essentially due to Painlevé, see [33].

Theorem 25. *If the metrics g and $\bar{g}$ are geodesically equivalent, then the function*

$$I_0 \stackrel{\text{def}}{=} \left(\frac{\det(g)}{\det(\bar{g})}\right)^{\frac{2}{n+1}} \bar{g}(\dot{\bar{x}}, \dot{\bar{x}}), \quad (8)$$

is an integral of the geodesic flow of the metric g.

Substituting g_c instead of $\bar{g}$ in (8), we obtain the following one-parameter family of integrals

$$\begin{aligned} I_c &\stackrel{\text{def}}{=} \left(\frac{\det(g)}{\det(g_c)}\right)^{\frac{2}{n+1}} g_c(\dot{\bar{x}}, \dot{\bar{x}}) = \\ &= C[(\phi_1 + c)\cdots(\phi_m + c)]\left\{\frac{1}{\phi_1 + c}\Pi_1 A_1 + \cdots + \frac{1}{\phi_m + c}\Pi_m A_m\right\} \\ &= C\{L_1 c^{m-1} + L_2 c^{m-2} + \cdots + L_m\}, \end{aligned}$$

where

$$\begin{aligned} L_1 &= \Pi_1 A_1 + \cdots + \Pi_m A_m \text{ — the twice energy integral,} \\ L_2 &= \sigma_1(\phi_2, ..., \phi_m)\Pi_1 A_1 + \cdots + \sigma_1(\phi_1, ..., \phi_{m-1})\Pi_m A_m, \\ L_3 &= \sigma_2(\phi_2, ..., \phi_m)\Pi_1 A_1 + \cdots + \sigma_2(\phi_1, ..., \phi_{m-1})\Pi_m A_m, \\ &\vdots \\ L_m &= (\phi_2 ... \phi_m)\Pi_1 A_1 + \cdots + (\phi_1 ... \phi_{m-1})\Pi_m A_m, \end{aligned}$$

σ_k denotes the elementary symmetric polynomial of degree k, and $C \stackrel{\text{def}}{=} \left[(\phi_1 + c)^{k_1 - 1}\cdots(\phi_m + c)^{k_m - 1}\right]^{\frac{2}{n+1}}$ is a constant. Therefore the functions L_k, $k = 1, ..., m$, are integrals of the geodesic flows of the metric g. We call these integrals *Levi-Civita integrals*.

From the results of [46] it follows that Levi-Civita integrals are in involution. More precisely, let $D = (d^i_j)$ be an $m \times m$ matrix. Suppose that for any i, j the element d^i_j depends only on the variables $\bar{x}_j$. Denote by Δ the determinant of the matrix D and by Δ^i_j the minor of the element d^i_j. In the paper [46] it was shown that, for arbitrary functions $A_i(\bar{x}_i, \dot{\bar{x}}_i)$, quadratic in velocities $\dot{\bar{x}}_i$, the Lagrangian system with Lagrangian

$$T_1 = \Delta\left(\frac{A_1(\bar{x}_1, \dot{\bar{x}}_1)}{\Delta^1_1} + \frac{A_2(\bar{x}_2, \dot{\bar{x}}_2)}{\Delta^1_2} + ... + \frac{A_m(\bar{x}_m, \dot{\bar{x}}_m)}{\Delta^1_m}\right)$$

admits $(m-1)$ integrals

$$T_i = \Delta\left(A_1(\bar{x}_1, \dot{\bar{x}}_1)\frac{\Delta^i_1}{(\Delta^1_1)^2} + A_2(\bar{x}_2, \dot{\bar{x}}_2)\frac{\Delta^i_2}{(\Delta^1_2)^2} + ... + A_m(\bar{x}_m, \dot{\bar{x}}_m)\frac{\Delta^i_m}{(\Delta^1_m)^2}\right),$$

where $i = 2, ..., m$, and if we identify the tangent and cotangent bundles the Lagrangian T_1 and consider the standard symplectic form on the cotangent bundle, then the integrals are in involution.

If we take $d^i_j = (\phi_j)^{m-i}$, then Δ and Δ^i_j are given by

$$\Delta^i_j = (-1)^{m-1}\sigma^{i-1}(\phi_1, \phi_2, ..., \phi_{j-1}, \phi_{j+1}, ..., \phi_m)\prod_{\alpha>\beta\geq 1, \alpha\neq j, \beta\neq j}(\phi_\alpha - \phi_\beta),$$

$$\Delta = (-1)^m \prod_{\alpha>\beta\geq 1}(\phi_\alpha - \phi_\beta).$$

Therefore,

$$\frac{\Delta\Delta^i_j}{(\Delta^1_j)^2} = \sigma^{i-1}(\phi_1, \phi_2, ..., \phi_{j-1}, \phi_{j+1}, ..., \phi_m)\Pi_j,$$

so $T_i = -L_i$ and thus the integrals L_i are in involution, q. e. d.

8.3. Geodesic equivalence and corresponding integrals.

Let metrics g and $\bar{g}$ on a manifold M (of dimension n) be geodesically equivalent.

By definition, put

$$U^r_g M \stackrel{\text{def}}{=} \{(x, \xi) \in \mathcal{T}M : \quad ||\xi||_g = r\},$$

where $x \in M$, $\xi \in \mathcal{T}_x M$ and $||\xi||_g \stackrel{\text{def}}{=} \sqrt{g(\xi, \xi)} = \sqrt{g_{ij}\xi^i\xi^j}$ is the norm of the vector ξ in the metric g.

By the geodesic flow of the metric g we mean the Lagrangian system of differential equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$ on $\mathcal{T}M$ with Lagrangian $L \stackrel{\text{def}}{=} \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j$. Because of the Legendre transformation, the geodesic flow could be considered as a Hamiltonian system on $\mathcal{T}M$ (as a symplectic form we take $\omega_g \stackrel{\text{def}}{=} \mathrm{d}[g_{ij}\xi^j dx^i]$) with the Hamiltonian $H_g \stackrel{\text{def}}{=} \frac{1}{2}g_{ij}\xi^i\xi^j$.

Since the metrics $g, \bar{g}$ are geodesically equivalent, the mapping $\Phi : \mathcal{T}M \to \mathcal{T}M$, $\Phi(x, \xi) = \left(x, \xi\frac{||\xi||_g}{||\xi||_{\bar{g}}}\right)$, takes the orbits of the geodesic flow of the metric g to the orbits of the geodesic flow of the metric $\bar{g}$. This mapping is a diffeomorphism (for $r \neq 0$), takes $U_g^r M$ to $U_{\bar g}^r M$ and is orbital on $U_g^r M$. Obviously the surfaces U_g^r, $U_{\bar g}^r$ are regular isoenergy surfaces $\{H_g = \frac{r}{2}\}$, $\{H_{\bar g} = \frac{r}{2}\}$.

By Theorem 24, in order to obtain a family of first integrals we have to find the polynomial $\Delta^n(t)$ and divide it by $(t-a)$. In our case $H_g = H_{\bar g} \circ \Phi$. Therefore the function a from Theorem 24 equals to $\frac{||\xi||_{\bar g}}{||\xi||_g}$.
In coordinates we have

$$\omega_g = \mathrm{d}[g_{ij}\xi^j dx^i]$$

and

$$\omega_{\bar g} = \mathrm{d}[\bar g_{ij}\xi^j dx^i].$$

Therefore,

$$\begin{aligned}
\Phi^*\omega_{\bar g} &= \mathrm{d}\left[\frac{||\xi||_g}{||\xi||_{\bar g}}\bar g_{ij}\xi^j dx^i\right] = \\
&= \frac{\partial}{\partial x^k}\left[\frac{||\xi||_g}{||\xi||_{\bar g}}\bar g_{ij}\xi^j\right] dx^k \wedge dx^i - \\
&- \frac{\partial}{\partial \xi^k}\left[\frac{||\xi||_g}{||\xi||_{\bar g}}\bar g_{ij}\xi^j\right] dx^i \wedge d\xi^k.
\end{aligned}$$

It is easy to see that at a point $\xi \in \mathcal{T}_x M$ the quantities

$$A_{ik} \stackrel{\mathrm{def}}{=} -\frac{\partial}{\partial \xi^k}\left[\frac{||\xi||_g}{||\xi||_{\bar g}}\bar g_{ij}\xi^j\right]$$

form an element of $\mathcal{T}_x M \otimes \mathcal{T}_x M$. Without loss of generality we can assume that in the space $\mathcal{T}_x M$ the metrics g and $\bar g$ are given in principal axes. Then

$$\begin{aligned}
A_{ij} &\stackrel{\mathrm{def}}{=} -\rho^i(x)\frac{\partial}{\partial \xi^j}\left(\xi^i \frac{\sqrt{{\xi^1}^2 + \dots + {\xi^n}^2}}{\sqrt{\rho^1{\xi^1}^2 + \dots + \rho^n{\xi^n}^2}}\right) = \\
&= \rho^i\delta^i_j\frac{||\xi||_g}{||\xi||_{\bar g}} - \rho^i\xi^i\left(\frac{\frac{||\xi||_{\bar g}}{||\xi||_g} - \rho^j\frac{||\xi||_g}{||\xi||_{\bar g}}}{||\xi||^2_{\bar g}}\xi^j\right) = \\
&= \mathrm{diag}(\mu_1, \dots, \mu_n) - A \otimes B.
\end{aligned}$$

Here $\rho^i, i = 1, \dots, n$ are common eigenvalues (here we allow ρ^i to be equal to ρ^j for some i,j) of the metrics g and $\bar g$, $\mu_i \stackrel{\mathrm{def}}{=} -\rho^i\frac{||\xi||_g}{||\xi||_{\bar g}}$, $A_i \stackrel{\mathrm{def}}{=} \rho^i\xi^i$ and

$$B_i \stackrel{\mathrm{def}}{=} \frac{\frac{||\xi||_{\bar g}}{||\xi||_g} - \rho^i\frac{||\xi||_g}{||\xi||_{\bar g}}}{||\xi||^2_{\bar g}}\xi^i.$$

We have

$$\begin{aligned}
\det(\Phi^*\omega_{\bar g} - t\omega_g) &= \det\left|\begin{array}{c|c} (*) & (A_{ij} + t\delta_{ij}) \\ \hline -(A_{ij} + t\delta_{ij}) & 0 \end{array}\right| \\
&= \det(A_{ij} + t\delta_{ij})^2.
\end{aligned}$$

Therefore,

$$\Delta^n(t) = \det\left(\operatorname{diag}(t+\mu_1, ..., t+\mu_n) - a \otimes b\right). \tag{9}$$

Lemma 3. *The following relation holds:*

$$\begin{aligned} \Delta^n(t) &= (t+\mu_1)\cdots(t+\mu_n) - (a_1b_1)(t+\mu_2)\cdots(t+\mu_n) - \dots \\ &- (t+\mu_1)\cdots(t+\mu_{n-1})(a_nb_n). \end{aligned} \tag{10}$$

The lemma follows from induction considerations.

To divide the polynomial by $(t-a)$ we shall use the Horner scheme. Suppose that $\Delta^n(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$ and $\delta^{n-1}(t) = t^{n-1} + b_{n-2}t^{n-2} + \cdots + b_0$. Then we have

$$b_{n-1} = a_n = 1, \tag{11}$$

$$b_{n-2} = a_{n-1} + a, \tag{12}$$

$$\dots$$

$$b_k = a_{k+1} + ab_{k+1}, \tag{13}$$

$$\dots$$

$$0 = a_0 + ab_0. \tag{14}$$

It follows from lemma 3 that

$$\begin{aligned} a_0 &= (\mu_1...\mu_n) - (A_1B_1)(\mu_2...\mu_n) - \cdots - (\mu_1...\mu_{n-1})A_nB_n = \\ &= (-1)^n \left(\frac{||\xi||_g}{||\xi||_{\bar g}}\right)^n (\rho^1\cdots\rho^n). \end{aligned}$$

Combining with (14) we get

$$b_0 = -\frac{a_0}{a} = (-1)^{n+1}\left(\frac{||\xi||_g}{||\xi||_{\bar g}}\right)^{n+1}(\rho^1\cdots\rho^n).$$

Since $\frac{1}{2}g_{ij}\xi^i\xi^j$ is an integral of the geodesic flow of the metric g, the function

$$I_0 \stackrel{\text{def}}{=} (\rho^1\cdots\rho^n)^{-\frac{2}{n+1}}\bar g(\xi,\xi) \tag{15}$$

is also an integral of the geodesic flow of the metric g. Using Lemma 3 we have

$$\begin{aligned} a_{n-1} &= (\mu_1 + ... + \mu_n) - (A_1B_1 + ... + A_nB_n) = \\ &= \frac{||\xi||_g}{||\xi||_{\bar g}^3}\Big\{({\rho^1}^2{\xi^1}^2 + ... + {\rho^n}^2{\xi^n}^2) - \\ &- (\rho^1 + ... + \rho^n)(\rho^1{\xi^1}^2 + ... + \rho^n{\xi^n}^2)\Big\} - \frac{||\xi||_{\bar g}}{||\xi||_g}. \end{aligned}$$

Using (12) we get

$$\begin{aligned} b_{n-2} &= a_{n-2} + a = \\ &= \frac{||\xi||_g}{||\xi||_{\bar g}^3}\Big\{({\rho^1}^2{\xi^1}^2 + ... + {\rho^n}^2{\xi^n}^2) - \\ &- (\rho^1 + ... + \rho^n)(\rho^1{\xi^1}^2 + ... + \rho^n{\xi^n}^2)\} \end{aligned}$$

Therefore, the function

$$\begin{aligned} I_1 &\stackrel{\text{def}}{=} (\rho^1\cdots\rho^n)^{-\frac{3}{n+1}}\Big\{({\rho^1}^2{\xi^1}^2+...+{\rho^n}^2{\xi^n}^2)- \\ &- (\rho^1+...+\rho^n)(\rho^1{\xi^1}^2+...+\rho^n{\xi^n}^2)\Big\} \end{aligned}$$

is an integral. (It is easy to see that $\frac{||\xi||_g^2}{||\xi||_{\bar g}^2}=(\rho^1\cdots\rho^n)^{-\frac{2}{n+1}}\frac{||\xi||_g^2}{I_0}$.)

Arguing as above, we see that the functions

$$\begin{aligned} I_k &\stackrel{\text{def}}{=} (\rho^1\cdots\rho^n)^{-\frac{k+2}{n+1}}\Big\{({\rho^1}^{k+1}{\xi^1}^2+...+{\rho^n}^{k+1}{\xi^n}^2)- \\ &- (\rho^1+...+\rho^n)({\rho^1}^k{\xi^1}^2+...+{\rho^n}^k{\xi^n}^2)+\cdots \\ &+ (-1)^k\sigma_k(\rho^1,...,\rho^n)(\rho^1{\xi^1}^2+...+\rho^n{\xi^n}^2)\Big\}, \end{aligned}$$

are integrals of the geodesic flow of the metric g, where by σ_k we denote the elementary symmetric polynomial of degree k. It is obvious that $(-1)^k\sigma_k=c_k$ from Theorem 1, and therefore $I_k=\left(\frac{\det(g)}{\det(\bar g)}\right)^{\frac{k+2}{n+1}}\bar g(S_k\xi,\xi)$. Thus I_k, $k=0,...,n-1$, are integrals of the geodesic flow of the metric g, q. e. d.

8.4. Liouville integrability.

The last step of the proof of Theorem 21 is to verify that the integrals $I_0,...,I_{n-1}$ are in involution. We proceed along the following plan. First we show that it is sufficient to prove the involutivity in each Levi-Civita chart. Then we prove that in each Levi-Civita chart the integrals $I_0,...,I_{n-1}$ are linear combinations of Levi-Civita integrals, and therefore commute.

Let $g,\bar g$ be metrics on M. A point $x\in M$ is called *stable*, if in a neighborhood of x the number of different eigenvalues of the metrics $g,\bar g$ does not depend of a point.

Denote by $\mathcal{M}$ the set of stable points of M. The set $\mathcal{M}$ is an open subset of M. Obviously

$$\mathcal{M}=\bigsqcup_{1\le q\le n}\mathcal{M}^q, \tag{16}$$

where $\mathcal{M}^q$ denotes the set of stable points whose number of distinct common eigenvalues equals q. Points $x\in M\setminus\mathcal{M}$ are called *points of bifurcation.*

Lemma 4. *The set $\mathcal{M}$ is everywhere dense in M.*

PROOF of Lemma 4. Denote by $N(x)$ the number of distinct common eigenvalues of the metrics $g,\hat g$ at a point x. Recall that the common eigenvalues of the metrics $g,\hat g$ at a point $x\in M$ are roots of the characteristic polynomial $P_x(t)=\det{(G-tE)}_{|x}$, where $G=\left(g^{i\alpha}\bar g_{\alpha j}\right)$. In particular, all roots of $P_x(t)$ are real.

Let us prove that, for a sufficiently small neighborhood of an arbitrary point $x\in M$, for any y from the neighborhood the number $N(x)$ is no less than $N(y)$.

Take a small $\epsilon > 0$ and an arbitrary root ρ of $P_x(t)$. Let us prove that for a sufficiently small neighborhood $U(x) \subset M$, for any $y \in U(x)$ there is a root ρ_y, $\rho - \epsilon < \rho_y < \rho + \epsilon$, of the polynomial $P_y(t)$. If ϵ is small, then for a sufficiently small neighborhood $U(x)$ of the point x, for any $y \in U(x)$ the numbers $\rho + \epsilon$ and $\rho - \epsilon$ are not roots of $P_y(t)$. Consider the circle $S_\epsilon \stackrel{\text{def}}{=} \{z \in C : \quad |z - \rho| = \epsilon\}$ on the complex plane C. Clearly the number of roots (with multiplicities) of the polynomial P_y inside the circle is equal to

$$\frac{1}{2\pi i}\int_{S_\epsilon} \frac{P'_y(z)}{P_y(z)} dz.$$

Since for any $y \in U(x)$ there are no roots of P_y on the circle S_ϵ, then the function

$$\frac{1}{2\pi i}\int_{S_\epsilon} \frac{P'_y(z)}{P_y(z)} dz$$

continuously depends on $y \in U(x)$, and therefore is a constant. Clearly it is positive. Thus for any $y \in U(x)$ there is at least one root of P_y that lies between $\rho + \epsilon$ and $\rho - \epsilon$. Then for any y from a sufficiently small neighborhood of x we have $N(y) \geq N(x)$.

Now let us prove the lemma. Evidently the set $\mathcal{M}$ is an open subset of M. Then it is sufficient to prove that for any open subset $U \subset M$ there is a stable point $x \in U$. Suppose otherwise, i.e. let all the points of U be points of bifurcation. Take a point $y \in M$ with maximal value of the function N on it. We have that in a neighborhood $U(y)$ of the point y the function N is constant and equals $N(y)$. Then the point y is a stable point, and we get a contradiction, q. e. d.

Now let the metrics $g, \bar{g}$ be geodesically equivalent. Since the set of points of bifurcation is nowhere dense, it is sufficient to prove the involutivity in each Levi-Civita chart. Let the metrics g and $\bar{g}$ be given by

$$\begin{aligned} g(\dot{\bar{x}}, \dot{\bar{x}}) &= \Pi_1(\bar{x})A_1(\bar{x}_1, \dot{\bar{x}}_1) + \Pi_2(\bar{x})A_2(\bar{x}_2, \dot{\bar{x}}_2) + \cdots + \\ &+ \Pi_m(\bar{x})A_m(\bar{x}_m, \dot{\bar{x}}_m), \end{aligned} \tag{17}$$

$$\begin{aligned} \bar{g}(\dot{\bar{x}}, \dot{\bar{x}}) &= \rho^1\Pi_1(\bar{x})A_1(\bar{x}_1, \dot{\bar{x}}_1) + \rho^2\Pi_2(\bar{x})A_2(\bar{x}_2, \dot{\bar{x}}_2) + \cdots \\ &+ \rho^m\Pi_m(\bar{x})A_m(\bar{x}_m, \dot{\bar{x}}_m). \end{aligned} \tag{18}$$

We show that the integrals I_k are linear combinations of the Levi-Civita integrals. We have

$$\bar{G} = \mathrm{diag}(\underbrace{\rho_1, ..., \rho_1}_{k_1}, ..., \underbrace{\rho_m, ..., \rho_m}_{k_m}), \tag{19}$$

where $\rho_k = \frac{1}{(\phi_1 \ldots \phi_m)}\frac{1}{\phi_k}$. It is easy to check that

$$S_k = (-1)^k \mathrm{diag}(\underbrace{\sigma_k^1, ..., \sigma_k^1}_{k_1}, ..., \underbrace{\sigma_k^m, ..., \sigma_k^m}_{k_m}), \tag{20}$$

where

$$\sigma_k^l \stackrel{\text{def}}{=} \sigma_k(\underbrace{\rho_1,...,\rho_1}_{k_1},...,\underbrace{\rho_l,...,\rho_l}_{k_l-1},...,\underbrace{\rho_m,...,\rho_m}_{k_m}). \tag{21}$$

We have

$$\sigma_k^1 = \frac{1}{(\phi_1...\phi_m)^k}\sigma_k\Big(\underbrace{\frac{1}{\phi_1},...,\frac{1}{\phi_1}}_{k_1-1},...,\underbrace{\frac{1}{\phi_m},...,\frac{1}{\phi_m}}_{k_m}\Big) = \tag{22}$$

$$= \frac{1}{(\phi_1...\phi_m)^k}\sum_{|\alpha|=k}\binom{k_1-1}{\alpha_1}\binom{k_2}{\alpha_2}\cdots\binom{k_m}{\alpha_m}\frac{1}{\phi_1^{\alpha_1}}\frac{1}{\phi_2^{\alpha_2}}\cdots\frac{1}{\phi_m^{\alpha_m}}, \tag{23}$$

$$\tag{24}$$

where $|\alpha| \stackrel{\text{def}}{=} \alpha_1 + \cdots + \alpha_m$ and $\alpha_i \geq 0$. Substituting $\binom{k_l-1}{\alpha_l} + \binom{k_l-1}{\alpha_l-1}$ for $\binom{k_l}{\alpha_l}$ (we assume that $\binom{k}{0} = 1$, $\binom{k}{-1} = 0$, $k \geq 0$) for $2 \leq l \leq m$ we obtain

$$\sigma_k^1 = \frac{1}{(\phi_1...\phi_m)^k}\left(B_k + B_{k-1}\sigma_1\Big(\frac{1}{\phi_2},...,\frac{1}{\phi_m}\Big) + \cdots + \right.$$
$$\left. + \quad B_{k-m+1}\sigma_{m-1}\Big(\frac{1}{\phi_2},...,\frac{1}{\phi_m}\Big)\right),$$

where

$$B_k \stackrel{\text{def}}{=} \sum_{|\alpha|=k}\binom{k_1-1}{\alpha_1}\cdots\binom{k_m-1}{\alpha_m}\frac{1}{\phi_1^{\alpha_1}}\cdots\frac{1}{\phi_m^{\alpha_m}}. \tag{25}$$

Note that

$$\left(\frac{\det(g)}{\det(\bar g)}\right)^{\frac{k+2}{n+1}} = C_k(\phi_1...\phi_m)^{k+2}, \tag{26}$$

where $C_k = [\phi_1^{k_1-1}...\phi_m^{k_m-1}]^{\frac{k+2}{n+1}}$. Therefore,

$$I_k \stackrel{\text{def}}{=} \left(\frac{\det(g)}{\det(\bar g)}\right)^{\frac{k+2}{n+1}}\bar g(S_k\dot{\bar x},\dot{\bar x}) =$$
$$= (-1)^k C_k(\phi_1...\phi_m)^{k+2}\left\{\rho^1\sigma_k^1\Pi_1A_1 + \cdots + \rho^m\sigma_k^m\Pi_mA_m\right\} =$$
$$= (-1)^k C_k(\phi_1...\phi_m)^{k+2}\left\{\frac{1}{\phi_1...\phi_m}\frac{1}{\phi_1}\left\{\frac{1}{(\phi_1...\phi_m)^k}(B_k+\right.\right.$$
$$\left.\left. + \quad \cdots + B_{k-m+1}\sigma_{m-1}\Big(\frac{1}{\phi_2},...,\frac{1}{\phi_m}\Big)\Big)\right\}\Pi_1A_1 + \cdots\right\} =$$
$$= (-1)^k C_k\left\{B_kL_m + B_{k-1}L_{m-1} + \cdots + B_{k-m+1}L_1\right\}, \tag{27}$$

where L_i are Levi-Civita integrals.

Finally, since the integrals $I_0,...,I_{n-1}$ are linear combinations of Levi-Civita integrals with constant coefficients, and since Levi-Civita integrals commute, then the integrals $I_0,...,I_{n-1}$ also commute, q. e. d.

Remark 1. *Let m be the number of distinct common eigenvalues of geodesically equivalent metrics $g, \bar{g}$ at a point x. Then in a neighborhood U of the point x the number of functionally independent almost everywhere Levi-Civita integrals is no less than m. Therefore the dimension of the space generated by the differentials $(dI_0, dI_1, ..., dI_{n-1})$ no less than m in almost all points of $\mathcal{T}U$.*

8.5. A family of geodesically equivalent metrics.

Lemma 5. *Let A be the diagonal $n \times n$ matrix $\mathrm{Diag}(\frac{1}{a_1 a}, \frac{1}{a_2 a}, ..., \frac{1}{a_i a})$, where a is $\prod_{i=1}^n a_i$, and a_i are positive. Let the characteristic polynomial $det(A - \mu E)$ be $c_0\mu^n + c_1\mu^{n-1} + ... + c^n$. Then for any α the matrix*

$$A \sum_{k=0}^{n-1} (-\alpha)^k \left(\frac{1}{det(A)}\right)^{\frac{k+2}{n+1}} \sum_{i=0}^{k} A^{k-i} c_i$$

is equal to $\prod_{i=0} n(a_i + \alpha) Diag(\frac{1}{a_1+\alpha}, \frac{1}{a_2+\alpha}, ..., \frac{1}{a_n+\alpha})$.

PROOF. It is clear that $\left(\frac{1}{det(A)}\right)^{\frac{k+2}{n+1}}$ equals a^{k+2}, and that $c_k = \sigma^k\left(\frac{-1}{aa_1}, \frac{-1}{aa_2}, ..., \frac{-1}{aa_n}\right)$, where σ^k denotes the symmetric polynomial of degree k.Then,

$$A \sum_{k=0}^{n-1} (-\alpha)^k \left(\frac{1}{det(A)}\right)^{\frac{k+2}{n+1}} \sum_{i=0}^{k} A^{k-i} c_i = A \sum_{k=0}^{n-1} (-\alpha)^k a^{k+2} \sum_{i=0}^{k} A^{k-i} \sigma^i \left(\frac{-1}{aa_1}, ..., \frac{-1}{aa_n}\right).$$

The matrix

$$A \sum_{k=0}^{n-1} (-\alpha)^k a^{k+2} \sum_{i=0}^{k} A^{k-i} \sigma^i \left(\frac{-1}{aa_1}, ..., \frac{-1}{aa_n}\right)$$

is evidently diagonal. The element number l on the diagonal is given by

$$\begin{aligned}
\frac{1}{aa_l} \sum_{k=0}^{n-1} (-\alpha)^k a^{k+2} \sum_{i=0}^{k} \left(\frac{1}{aa_l}\right)^{k-i} \sigma^i \left(\frac{-1}{aa_1}, ..., \frac{-1}{aa_n}\right) &= \\
\frac{a}{a_l} \sum_{k=0}^{n-1} (-\alpha)^k \sum_{i=0}^{k} \sigma^i \left(\frac{-1}{a_1}, ..., \frac{-1}{a_n}\right) \left(\frac{1}{a_l}\right)^{k-i} &= \\
\frac{a}{a_l} \sum_{i=0}^{n-1} \sigma^i \left(\frac{-1}{a_1}, ..., \frac{-1}{a_n}\right) \sum_{k=i} n - 1 \frac{(-\alpha)^k}{a_l^{k-i}} &= \\
\frac{a}{a_l} \sum_{i=0}^{n-1} a_l^i \sigma^i \left(\frac{-1}{a_1}, ..., \frac{-1}{a_n}\right) \frac{\left(\frac{-\alpha}{a_l}\right)^n - \left(\frac{-\alpha}{a_l}\right)^i}{-\frac{\alpha}{a_l} - 1} &= \\
-\frac{a}{a_l + \alpha} \sum_{i=0}^{n} a_l^i \sigma^i \left(\frac{-1}{a_1}, ..., \frac{-1}{a_n}\right) \left(\left(\frac{-\alpha}{a_l}\right)^n - \left(\frac{-\alpha}{a_l}\right)^i\right) &=
\end{aligned}$$

$$
\begin{aligned}
-\frac{a(-\alpha)^n}{a_l+\alpha}\sum_{i=0}^{n}\frac{1}{a_l^{n-i}}\sigma^i\left(\frac{-1}{a_1},...,\frac{-1}{a_n}\right)+\frac{a}{a_l+\alpha}\sum_{i=0}^{n}\sigma^i\left(\frac{-1}{a_1},...,\frac{-1}{a_n}\right) &= \\
\frac{-a(-\alpha)^n}{a_l+\alpha}\prod_{i=0}^{n}\left(\frac{1}{a_l}-\frac{1}{a_i}\right)+\frac{a}{a_l+\alpha}\prod_{i=1}^{n}\left(1+\frac{\alpha}{a_i}\right) &= \\
0+\frac{1}{a_l+\alpha}\prod_{i=1}^{n}(a_i+\alpha),
\end{aligned}
$$

q. e. d.

PROOF of Theorem 22. Because of Lemma 4, it is sufficient to prove the theorem only in Levi-Civita chart. Let the metrics g and $\bar{g}$ be given by

$$
\begin{aligned}
g(\dot{\bar{x}},\dot{\bar{x}}) &= \Pi_1(\bar{x})A_1(\bar{x}_1,\dot{\bar{x}}_1)+\Pi_2(\bar{x})A_2(\bar{x}_2,\dot{\bar{x}}_2)+\cdots+ \\
&+ \Pi_m(\bar{x})A_m(\bar{x}_m,\dot{\bar{x}}_m), \qquad (28)\\
\bar{g}(\dot{\bar{x}},\dot{\bar{x}}) &= \frac{1}{\phi_1(\phi_1^{k_1}\phi_2^{k_2}...\phi_m^{k_m})}\Pi_1(\bar{x})A_1(\bar{x}_1,\dot{\bar{x}}_1)+\frac{1}{\phi_2(\phi_1^{k_1}\phi_2^{k_2}...\phi_m^{k_m})}\Pi_2(\bar{x})A_2(\bar{x}_2,\dot{\bar{x}}_2)+\cdots+ \\
&+ \frac{1}{\phi_1(\phi_m^{k_1}\phi_2^{k_2}...\phi_m^{k_m})}\Pi_m(\bar{x})A_m(\bar{x}_m,\dot{\bar{x}}_m), \qquad (29)
\end{aligned}
$$

For this metrics the operator $\bar{G}\stackrel{\text{def}}{=}(g\bar{g}^{-1})$ is given by the diagonal matrix $\text{Diag}(\underbrace{\phi_1,\phi_1,...,\phi_1}_{k_1},\underbrace{\phi_2,\phi_2,...,\phi_2}_{k_2},...,\underbrace{\phi_m,\phi_m,...,\phi_m}_{k_m})$. It is easy to see that for any $\xi,\nu\in\mathcal{T}_xM^n$ we have $\bar{g}(\xi,\nu)=g(G\xi,\nu)$. Then the formula for $\tilde{I}_k$ is

$$
\tilde{I}_k(\xi,\xi)=\left(\frac{1}{det(G)}\right)^{\frac{k+2}{n+1}}g(G\sum_{i=0}^{k}G^{k-i}c_i\xi,\xi),
$$

and f_α is given by

$$
\begin{aligned}
f_\alpha &= \sum_{k=0}^{n-1}(-\alpha)^k\tilde{I}_k \\
&= \sum_{k=0}^{n-1}\left(\frac{1}{det(G)}\right)^{\frac{k+2}{n+1}}g(G\sum_{i=0}^{k}G^{k-i}c_i\xi,\xi) \\
&= g(G\sum_{k=0}^{n-1}\left(\frac{1}{det(G)}\right)^{\frac{k+2}{n+1}}\sum_{i=0}^{k}G^{k-i}c_i\xi,\xi) \qquad (30)
\end{aligned}
$$

Combining (30) with Lemma 5, we have that the form f_α is given by

$$
\begin{aligned}
f_\alpha(\dot{\bar{x}},\dot{\bar{x}}) &= \frac{(\phi_1+\alpha)^{k_1}(\phi_2+\alpha)^{k_2}...(\phi_m+\alpha)^{k_m}}{\phi_1+\alpha}\Pi_1(\bar{x})A_1(\bar{x}_1,\dot{\bar{x}}_1)+\frac{(\phi_1+\alpha)^{k_1}(\phi_2+\alpha)^{k_2}...(\phi_m+\alpha)^{k_m}}{\phi_2+\alpha}\Pi_2(\bar{x})A_2(\bar{x}_2,\dot{\bar{x}}_2)+\cdots+ \\
&+ \frac{(\phi_1+\alpha)^{k_1}(\phi_2+\alpha)^{k_2}...(\phi_m+\alpha)^{k_m}}{\phi_m+\alpha}\Pi_m(\bar{x})A_m(\bar{x}_m,\dot{\bar{x}}_m).
\end{aligned}
$$

Then the metric

$$g_\alpha \stackrel{\text{def}}{=} \left(\frac{det(g)}{det(f_\alpha)}\right)^{\frac{2}{n-1}} f_\alpha$$

is given by the formula

$$g_\alpha(\dot{\bar{x}},\dot{\bar{x}}) = \frac{1}{(\phi_1+\alpha)^{k_1}\cdots(\phi_m+\alpha)^{k_m}}\left\{\frac{1}{\phi_1+\alpha}\Pi_1 A_1 + \cdots + \frac{1}{\phi_m+\alpha}\Pi_m A_m\right\}, \tag{31}$$

and is evidently geodesically equivalent to g, q. e. d.

8.6. Geodesically equivalent metrics on the ellipsoid.

PROOF OF THEOREM 23. We show that in the elliptic coordinate system the restriction of the metrics

$$ds^2 \stackrel{\text{def}}{=} \sum_{i=1}^n (dx^i)^2 \text{ and } dr^2 \stackrel{\text{def}}{=} \frac{1}{\sum_{i=1}^n \left(\frac{x^i}{a_i}\right)^2}\left(\sum_{i=1}^n \frac{(dx^i)^2}{a_i}\right)$$

to the ellipsoid $\sum_{i=1}^n \frac{(x^i)^2}{a_i} = 1$ have Levi-Civita local form, and therefore are geodesically equivalent. More precisely, consider elliptic coordinates $\nu^1,...,\nu^n$. Without loss of generality we can assume that $a^1 < a^2 < ... < a^n$. Then the relation between the elliptic coordinates $\bar\nu$ and the Cartesian coordinates $\bar x$ is given by

$$x^i = \sqrt{\frac{\prod_{j=1}^n (a^i-\nu^j)}{\prod_{j=1, j\neq i}^n (a^i - a^j)}}. \tag{32}$$

Recall that the elliptic coordinates are non-degenerate almost everywhere, and the set

$$\{\nu^1 = 0, a_1 < \nu^2 < a_2, a_2 < \nu^3 < a_3, ..., a_{n-1} < \nu^n < a^n\}$$

is the part of the ellipsoid $\{x^1 > 0, x^2 > 0, ..., x^n > 0\}$. Since for any i the symmetry $x^i \to -x^i$ takes the ellipsoid to the ellipsoid and preserves the metrics ds^2 and dr^2, it is sufficient to check the statement of the theorem only in the quadrant $\{x^1 > 0, x^2 > 0, ..., x^n > 0\}$.

In the elliptic coordinates the restriction of the metric ds^2 to the ellipsoid has the following form

$$\sum_{i=1}^n \Pi_i A_i (d\nu^i)^2, \tag{33}$$

where $\Pi_i \stackrel{\text{def}}{=} \prod_{j=1, j\neq i}^n (\nu^i - \nu^j)$, and $A_i \stackrel{\text{def}}{=} \frac{\nu^i}{\prod_{j=1}^n (a^j - \nu^i)}$. The restriction of the metric dr^2 to the ellipsoid is

$$(a^1a^2...a^n)\sum_{i=1}^n \rho^i \Pi_i A_i (d\nu^i)^2, \tag{34}$$

where $\rho^i \stackrel{\text{def}}{=} \frac{1}{\nu^i(\nu^1\nu^2...\nu^n)}$. We see that the metrics ds^2, dr^2 have Levi-Civita local form, and therefore are geodesically equivalent, q. e. d.

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