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Lorenz Geometry, Granada

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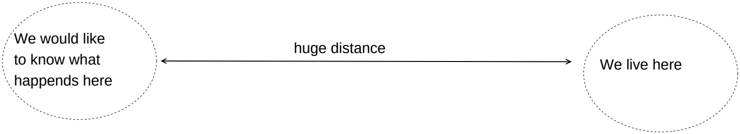

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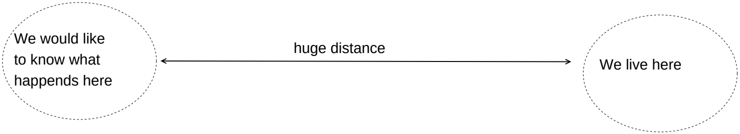

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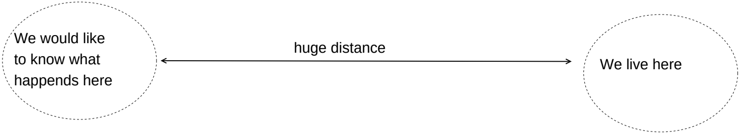

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Then, if the relativistic effects are not negligible (that happens for example is the objects in this part of space time are sufficiently fast or if this region of the universe is big enough),
we obtain as a rule the world lines of the objects as unparameterized curves.

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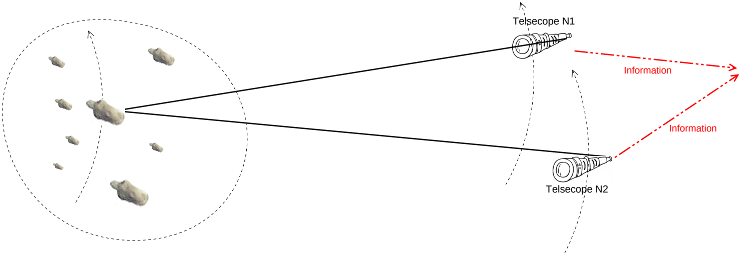
One can obtain unparameterized geodesics by observation:



This place has
4=2+2 coordinates of
any visiable object

We take 2 freely falling observers that measure two angular coordinates of the visible objects and send this information to one place. This place will have 4 functions angle(t) for every visible object which are in the generic case 4 coordinates of the object.

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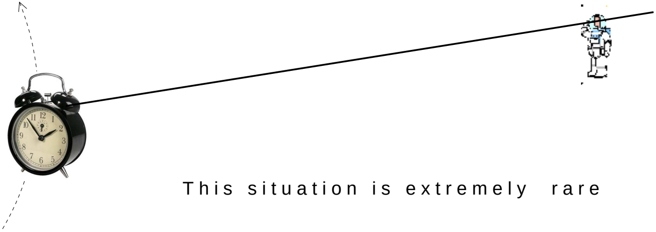

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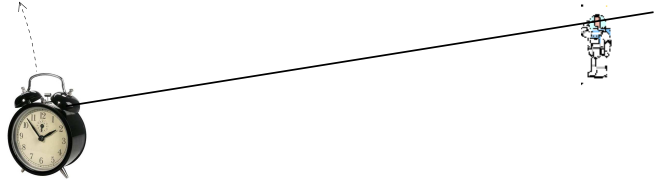

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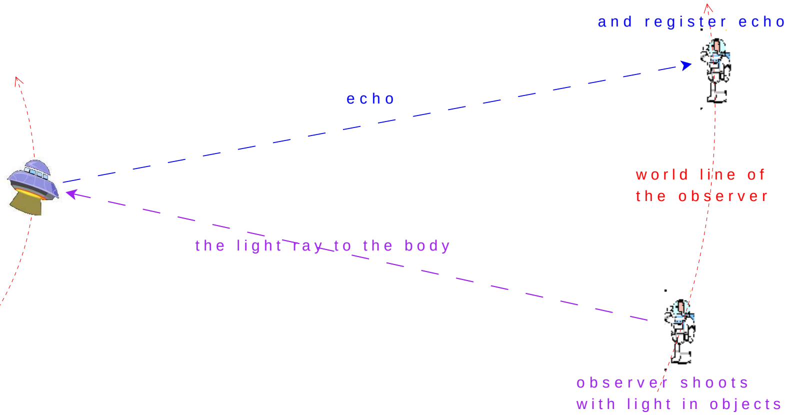

This approach is related to the so called "radar coordinates". It is a hot topic since 1950th (THE LASER ASTROMETRIC TEST OF RELATIVITY — Space Interferometry Mission), but is applicable in a small neighborhood of solar system only.

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The mathematical setting: *We are given a family of smooth curves* $\gamma(t;\alpha)$ *in* $U \subseteq \mathbb{R}^4$*; we assume that the family is sufficiently big in the sense that* $\forall x_0 \in U$

$$\Omega_{x_0} := \{\xi \in T_{x_0}U \mid \ \exists \alpha \ \textit{and} \ \exists t_0 \ \textit{with} \ \tfrac{d}{dt}\gamma(t;\alpha)_{|t=t_0} \ \textit{is proportional to} \ \xi\}$$

contains an open subset of $T_{x_0}U$.

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Jürgen Ehlers 1972, who said that "We reject clocks as basic tools for setting up the space-time geometry and propose ... freely falling particles instead. We wish to show how the full space-time geometry can be synthesized ... . Not only the measurement of length but also that of time then appears as a derived operation."

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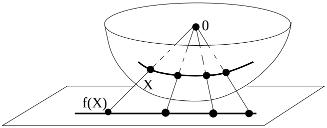

Lagrange example 1789 Radial projection $f : S^2 \to \mathbb{R}^2$ takes geodesics of the sphere to geodesics of the plane, because geodesics on sphere/plane are intersection of plains containing 0 with the sphere/plane.

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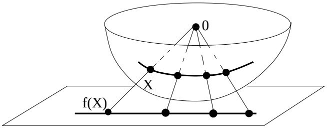

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The example of Lagrange survives for all signatures and for all dimensions.

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Fact (Levi-Civita 1896) The metrics of Dini can be generalized for every dimension: The metric

$$-(T(t)-X_1(x_1))(T(t)-X_2(x_2))(T(t)-X_3(x_3))dt^2+(T(t)-X_1(x_1))(X_1(x_1)-X_2(x_2))(X_1(x_1)-X_3(x_3))dx_1^2+(T(t)-X_2(x_2))(X_1(x_1)-X_2(x_2))(X_2(x_2)-X_3(x_3))dx_2^2+(T(t)-X_3(x_3))(X_1(x_1)-X_3(x_3))(X_2(x_2)-X_3(x_3))dx_3^3$$

is geodesically equivalent to the metric

$$-\frac{(T(t)-X_1(x_1))(T(t)-X_2(x_2))(T(t)-X_3(x_3))}{T(t)^2X_1(x_1)X_2(x_2)X_3(x_3)}dt^2+\frac{(T(t)-X_1(x_1))(X_1(x_1)-X_2(x_2))(X_1(x_1)-X_3(x_3))}{T(t)X_1(x_1)^2X_2(x_2)X_3(x_3)}dx_1^2+\frac{(T(t)-X_2(x_2))(X_1(x_1)-X_2(x_2))(X_2(x_2)-X_3(x_3))}{T(t)X_1(x_1)X_2(x_2)^2X_3(x_3)}dx_2^2+\frac{(T(t)-X_3(x_3))(X_1(x_1)-X_3(x_3))(X_2(x_2)-X_3(x_3))}{T(t)X_1(x_1)X_2(x_2)X_3(x_3)^2}dx_3^3$$

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Example. The so-called Friedman-Lemaitre-Robertson-Walker metric

$$g = -dt^2 + R(t)^2 \frac{dx^2 + dy^2 + dz^2}{1 + \frac{\kappa}{4}(x^2 + y^2 + z^2)} \ ; \quad \kappa = +1; 0; -1,$$

is not geodesically rigid. Indeed, $\forall c$ the metric

$$\bar g = \frac{-1}{(R(t)^2 + c)^2} dt^2 + \frac{R(t)^2}{c(R(t)^2 + c)} \frac{dx^2 + dy^2 + dz^2}{1 + \frac{\kappa}{4}(x^2 + y^2 + z^2)}$$

is geodesically equivalent to g (essentially Levi-Civita 1896; repeated by many relativists (Nurowski, Gibbons et al, Hall) later).

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- We give an answer (joint with Bolsinov) in dimension 4 and for Lorenz signature of the metric.
- Actually, we can generalize the answer for all dimensions n and for all signatures.

The goal of my talk: to say something about all this (sub)problems

- **Problem 1. How to reconstruct a metric by its unparameterized geodesics?**
  - **Subproblem 1.1.** Given a big family of curves $\gamma(t; a)$, how to understand whether these curves are reparameterised geodesics of a certain affine connection? How to reconstruct this connection effectively?
  - **Subproblem 1.2.** Given an affine connection $\Gamma = \Gamma^i_{jk}$, how to understand whether there exists a metric g in the projective class of Γ? How to reconstruct this metric effectively?
- **Problem 2.** In what situations is the reconstruction of a metric by the unparameterised geodesics unique (up to the multiplication of the metric by a constant)?
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$$\frac{d^2\gamma^a}{dt^2} + \Gamma^a_{bc}\frac{d\gamma^b}{dt}\frac{d\gamma^c}{dt} = f\left(\frac{d\gamma}{dt}\right)\frac{d\gamma^a}{dt}. \qquad (*)$$

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Take $x_0 \in U$. For every $\gamma(t; \alpha)$ with $\gamma(t_0; \alpha) = x_0$ we view the equations $(*)$ as a system of equations on the entries of $\Gamma(x_0)$ and on the function $f_{|\Omega_{x_0}}$; the coefficients in this system come from known data $\left(\frac{d\gamma(t;\alpha)}{dt}\right)_{|t=t_0}, \left(\frac{d^2\gamma(t;\alpha)}{dt^2}\right)_{|t=t_0}$. Since we have infinitely many curves γ passing through x_0 , we have an infinite system of equations.

Theorem (informal version). *At every point, there exists only one, up to a certain gauge freedom, solution $\left(\Gamma(x_0)^i_{jk}, f_{|\Omega_{x_0}}\right)$.*

Gauge freedom (if there exists one solution, there exist many).

Repeat: $\frac{d^2\gamma^a}{dt^2} + \Gamma^a_{bc}\frac{d\gamma^b}{dt}\frac{d\gamma^c}{dt} = f\left(\frac{d\gamma}{dt}\right)\frac{d\gamma^a}{dt}.$ $(*)$

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We consider two connections Γ and $\bar{\Gamma}$ related by Levi-Civita's formula

$$\Gamma^a_{bc} = \bar{\Gamma}^a_{bc} - \delta^a_b\phi_c - \delta^a_c\phi_b, \tag{1}$$

where $\phi = \phi_i$ is a one form.

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$$(\delta^a_b\phi_c + \delta^a_c\phi_b)\frac{d\gamma^b}{dt}\frac{d\gamma^c}{dt} = 2\left(\frac{d\gamma^b}{dt}\phi_b\right)\frac{d\gamma^a}{dt},$$

we obtain that the same curve γ satisfies the equation (17) with respect to the connection $\bar{\Gamma}$ and the function

$$\bar{f}(v) := f(v) + 2\left(v^b\phi_b\right). \tag{2}$$

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Thus, if (Γ, f) is a solution of $(*)$ (for all γ), then for every $1-$form ϕ the pair $\left(\bar{\Gamma}, \bar{f}\right)$ given by (1,2) is also a solution.

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Thus, Γ and $\bar{\Gamma}$ related by (1) have the same geodesics.

This is the only gauge freedom

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We subtract one equation from the other to obtain

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where $\tilde{\Gamma} = \bar{\Gamma} - \Gamma$, $\phi = f - \bar{f}$.
Since the left hand sind is quadratic in v, ϕ is linear (i.e., $\phi(v) = \phi_i v^i$).
Thus, $\bar{\Gamma}^a_{bc} - \Gamma^a_{bc} = \delta^a_b \phi_c + \delta^a_c \phi_b$ as we claimed.

Algorithm how to reconstruct the pair (Γ, f) up to the gauge freedom

Repeat: $$\frac{d^2\gamma^a}{dt^2} + \Gamma^a_{bc}\frac{d\gamma^b}{dt}\frac{d\gamma^c}{dt} = f\left(\frac{d\gamma}{dt}\right)\frac{d\gamma^a}{dt}. \qquad (*)$$

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\begin{array}{rcl}
f\left(\frac{d\gamma}{dt}\right) & = & \left(\frac{d^2\gamma^1}{d^2t} + \Gamma^1_{ab}\frac{d\gamma^a}{dt}\frac{d\gamma^b}{dt}\right)/\frac{d\gamma^1}{dt} \\
\frac{d\gamma^2}{dt}\Gamma^1_{ab}\frac{d\gamma^a}{dt}\frac{d\gamma^b}{dt} - \frac{d\gamma^1}{dt}\Gamma^2_{ab}\frac{d\gamma^a}{dt}\frac{d\gamma^b}{dt} & = & \frac{d^2\gamma^2}{d^2t}\frac{d\gamma^1}{dt} - \frac{d\gamma^2}{dt}\frac{d^2\gamma^1}{d^2t} \\
 & \vdots & \\
\frac{d\gamma^n}{dt}\Gamma^1_{ab}\frac{d\gamma^a}{dt}\frac{d\gamma^b}{dt} - \frac{d\gamma^1}{dt}\Gamma^n_{ab}\frac{d\gamma^a}{dt}\frac{d\gamma^b}{dt} & = & \frac{d^2\gamma^n}{d^2t}\frac{d\gamma^1}{dt} - \frac{d\gamma^n}{dt}\frac{d^2\gamma^1}{d^2t}.
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&\vdots \\
\frac{d\gamma^n}{dt}\Gamma^1_{ab}\frac{d\gamma^a}{dt}\frac{d\gamma^b}{dt} - \frac{d\gamma^1}{dt}\Gamma^n_{ab}\frac{d\gamma^a}{dt}\frac{d\gamma^b}{dt} &= \frac{d^2\gamma^n}{d^2t}\frac{d\gamma^1}{dt} - \frac{d\gamma^n}{dt}\frac{d^2\gamma^1}{d^2t}.
\end{aligned}
\tag{4}
$$

The first equation of (4) is equivalent to the equation $(*)$ for $a = 1$ solved with respect to $f\left(\frac{d\gamma}{dt}\right)$. We obtain the second, third, etc, equations of (4) by substituting the first equation of (4) in the equations $(*)$ with $a = 2, 3, \mathrm{etc.}$

Note that the subsystem of (4) containing the the second, third, etc. equations of (4) does not contain the function f and is therefore a linear system on Γ^i_{jk}.

$$
\begin{aligned}
\frac{d\gamma^2}{dt}\,\Gamma^1_{ab}\frac{d\gamma^a}{dt}\frac{d\gamma^b}{dt} - \frac{d\gamma^1}{dt}\,\Gamma^2_{ab}\frac{d\gamma^a}{dt}\frac{d\gamma^b}{dt} &= \frac{d^2\gamma^2}{d^2t}\,\frac{d\gamma^1}{dt} - \frac{d\gamma^2}{dt}\,\frac{d^2\gamma^1}{d^2t} \\
&\vdots \\
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Then, for every 'geodesic' $\gamma(t_0, \alpha)$ gives us $n-1$ linear (inhomogeneous) equations on the components $\Gamma(x_0)^i_{jk}$.

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Then, for every 'geodesic' $\gamma(t_0,\alpha)$ gives us $n-1$ linear (inhomogeneous) equations on the components $\Gamma(x_0)^i_{jk}$. We take a sufficiently big number N (if $n=4$, it is sufficient to take $N=12$) and substitute N generic geodesics $\gamma(t;\alpha)$ passing through x_0 in this subsystem.

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At every point x_0, we obtain an inhomogeneous linear system of equations on $\frac{n^2(n+1)}{2}$ unknowns $\Gamma(x_0)^i_{jk}$.

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In the case the solution of this system does not exist (at least at one point x_0), there exists no connection whose (reparameterized) geodesics are $\gamma(t; \alpha)$.

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In the case the solution of this system does not exist (at least at one point x_0), there exists no connection whose (reparameterized) geodesics are $\gamma(t;\alpha)$. In the case it exists, the solution of the system above gives this connection uniquely up to the gauge freedom described above.

Subproblem 1.2: Given an affine connection $\Gamma = \Gamma^i_{jk}$, how to understand whether there exists a metric g in the projective class of Γ?

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Theorem (Eastwood-Matveev 2006) g lies in a projective class of a connection Γ^i_{jk} if and only if $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(n+1)}$ is a solution of

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Here $\sigma^{ab} := g^{ab} \cdot \det(g)^{1/(n+1)}$ should be understood as an element of $S^2M \otimes (\Lambda_n)^{2/(n+1)}M$. In particular,

$$\nabla_a \sigma^{bc} = \underbrace{\frac{\partial}{\partial x^a}\sigma^{bc} + \Gamma^b_{ad}\sigma^{dc} + \Gamma^c_{da}\sigma^{bd}}_{\text{Usual covariant derivative}} - \underbrace{\frac{2}{n+1}\Gamma^d_{da}\,\sigma^{bc}}_{\text{addition coming from volume form}}$$

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The system (5) is an overdetermined linear system of PDE of finite type the first order on the unknown functions σ^{bc}; in theory, there exists an algorithmic method to understand the existence of a solution. The method is highly computational and hardly applicable in this case.

If the searched metric is Ricci-flat, there exists a trick that simplifies the algorithm.

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Weyl has shown that the projective Weyl tensor does not depend of the choice of connection within the projective class: if the connections Γ and $\bar\Gamma$ are related by the formula

$$\Gamma^a_{bc} = \bar\Gamma^a_{bc} - \delta^a_b\phi_c - \delta^a_c\phi_b, \tag{1}$$

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then their projective Weyl tensors coincide. Now, from the formula (6), we know that, if the searched $\bar g$ is Ricci-flat, projective Weyl tensor coincides with the Riemann tensor $\bar R^i{}_{jk\ell}$ of $\bar g$. Thus, if we know the projective class of the Ricci-flat metric $\bar g$, we know its Riemann tensor.

Then, the metric $\bar{g}$ must satisfy the following system of equations due to the symmetries of the Riemann tensor:

$$\begin{cases} \bar{g}_{ia} {W^a}_{jkm} + \bar{g}_{ja} {W^a}_{ikm} = 0 \\ \bar{g}_{ia} {W^a}_{jkm} - \bar{g}_{ka} {W^a}_{mij} = 0 \end{cases} \tag{7}$$

The first portion of the equations is due to the symmetry $(\bar{R}_{ijkm} = -\bar{R}_{jikm})$, and the second portion is due to the symmetry $(\bar{R}_{kmij} = \bar{R}_{ijkm})$ of the curvature tensor of $\bar{g}$.

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We see that for every point $x_0 \in U$ the system (7) is a system of linear equations on $\bar{g}(x_0)_{ij}$. The number of equations (around 100) is much bigger than the number of unknowns (which is 10).

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Theorem (Folklore – Petrov, Hall, Rendall, Mcintosh) Let $W^i{}_{jk\ell}$ be a tensor in $\mathbb{R}^4$ such that it is skew-symmetric with respect to k, ℓ and such that its traces $W^a{}_{ak\ell}$ and $W^a{}_{ja\ell}$ vanish. Assume that for all 1-forms $\xi_i \neq 0$ we have $W^a{}_{jk\ell}\xi_a \neq 0$. Then, the equations (7) have no more than one-dimensional space of solutions.

Thus, for generic Γ, we can algorithmically reconstruct the conformal class of the metric $\bar{g}$ by solving the system of linear equations (7). Then, we obtain the ansatz

$$\sigma^{ij} = e^{\lambda} a^{ij}, \text{ where } a^{ij} \text{ is known and comes from } (7).$$

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Subproblem 2.1. What metrics are geodesically rigid?

Theorem (Matveev arXiv:1101.2069) *Almost every 4D metric is geodesically rigid and can be reconstructed by an algorithm similar to the one above (the algorithm requires solution of linear system of equations and integration)*.

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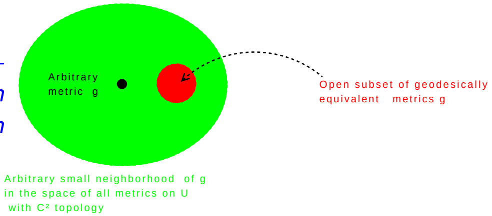
the set of geodesically rigid 4D metrics contains an open everywhere dense (in C^2-topology) subset.



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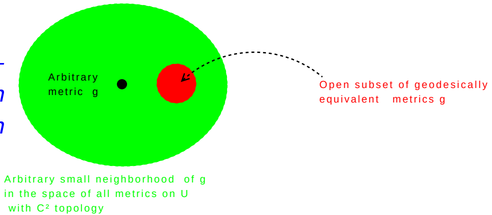

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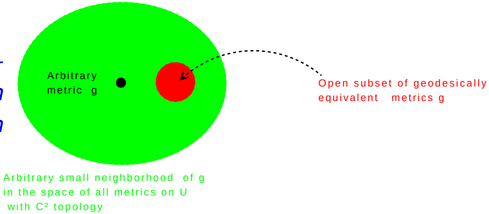

The result survives for all $n \geq 4$. The result survives in D3, if we replace the uniform C^2- topology by the uniform C^3-topology. In D2, the result is again true, if we replace the uniform C^2- topology by the uniform C^8-topology.

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Notation: Given g and $\bar g$ on M, we consider

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Theorem (Gluing Lemma). Then, the metrics

$$g = \begin{pmatrix} h_1\chi_2(L_1) & 0 \\ 0 & h_2\chi_1(L_2) \end{pmatrix}, \quad \bar{g} = \begin{pmatrix} \frac{1}{\chi_2(0)}\bar{h}_1\chi_2(L_1) & 0 \\ 0 & \frac{1}{\chi_1(0)}\bar{h}_2\chi_1(L_2) \end{pmatrix}.$$

on $M_1 \times M_2$, where χ_i is the characteristic polynomial of L_i, are geodesically equivalent.

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Example.

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Example. We we take two 1-dimensional manifolds

$$\left(I_1, h_1 = dx^2, \bar{h}_1 = \frac{1}{X(x)^2}dx^2\right) \text{ and } \left(I_2, h_2 = -dy^2, \bar{h}_2 = -\frac{1}{Y(x)^2}dy^2\right).$$

The corresponding tensors L_1 and L_2 and their characteristic polynomials are

$$L_1 = (X(x))\ ; \quad L_2 = (Y(y))\ ; \quad \chi_1(t) = t - X(x)\ ; \quad \chi_2(t) = t - Y(y).$$

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We see that these metrics are precisely the Dini metrics from the introduction of my talk.

Building blocks and Splitting Lemma

Def. *Building block* is $(U \subset \mathbb{R}^m, h \sim \bar{h})$ such that at every point $L(h, h_1)$ has only one real eigenvalue, or two complex-conjugate eigenvalues.

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The only possible 1-dimensional building block was described above:

$$\left(I_1, h_1 = dx^2, \bar{h}_1 = \frac{1}{X(x)^2}dx^2\right) \text{ and } \left(I_2, h_2 = \pm dy^2, \bar{h}_2 = \pm\frac{1}{Y(x)^2}dy^2\right).$$

Two-dimensional building Block were described in Bolsinov-Matveev-Pucacco 2009 (see also Darboux 1886 and Petrov 1949)

	Complex-Liouville case	Jordan-block case
g	$\Im(h)dxdy$	$(1+xY'(y))\,dxdy$
$\bar g$	$-\left(\frac{\Im(h)}{\Im(h)^2+\Re(h)^2}\right)^2 dx^2$ $+$ $2\frac{\Re(h)\Im(h)}{(\Im(h)^2+\Re(h)^2)^2}dxdy$ $+$ $\left(\frac{\Im(h)}{\Im(h)^2+\Re(h)^2}\right)^2 dy^2$	$\frac{1+xY'(y)}{Y(y)^4}\left(-2Y(y)dxdy\right.$ $\left.+(1+xY'(y))dy^2\right)$

Trivial block: $\bar g = \mathrm{const}\cdot g$.

Three-dimensional building Block were described in Petrov 1949 and Eisenhart 1925

$$
\begin{aligned}
g &= \left(4\,x_2\left(\frac{d}{dx_3}\lambda\left(x_3\right)\right)+2\right)dx_1\,dx_3+dx_2{}^2\\
&+\ 2\,x_1\left(\frac{d}{dx_3}\lambda\left(x_3\right)\right)dx_2\,dx_3+x_1{}^2\left(\frac{d}{dx_3}\lambda\left(x_3\right)\right)^2dx_3{}^2,\\
\bar{g} &= \frac{1}{\lambda(x_3)^6}\Big[\left(4\,x_2\lambda\left(x_3\right)^2\left(\frac{d}{dx_3}\lambda\left(x_3\right)\right)+2\,\lambda\left(x_3\right)^2\right)dx_1\,dx_3+\lambda\left(x_3\right)^2dx_2{}^2\\
&-\ \left(4\,x_2\lambda\left(x_3\right)\left(\frac{d}{dx_3}\lambda\left(x_3\right)\right)+2\lambda\left(x_3\right)-2\,x_1\lambda\left(x_3\right)^2\left(\frac{d}{dx_3}\lambda\left(x_3\right)\right)\right)dx_2\,dx_3\\
&+\ \left(4\,x_2{}^2\left(\frac{d}{dx_3}\lambda\left(x_3\right)\right)^2+4\,x_2\left(\frac{d}{dx_3}\lambda\left(x_3\right)\right)-4\,x_1x_2\lambda\left(x_3\right)\left(\frac{d}{dx_3}\lambda\left(x_3\right)\right)^2\right)dx_3{}^2\\
&+\ \left(1+x_1{}^2\lambda\left(x_3\right)^2\left(\frac{d}{dx_3}\lambda\left(x_3\right)\right)^2-2\,x_1\lambda\left(x_3\right)\left(\frac{d}{dx_3}\lambda\left(x_3\right)\right)\right)dx_3{}^2\Big]
\end{aligned}
$$

where λ is a function of x_3, and

$$
\begin{aligned}
g &= 2\,dx_3\,dx_1+h\left(x_2,x_3\right)_{11}dx_2{}^2+2\,h(x_2,x_3)_{12}dx_2\,dx_3+h(x_2,x_3)_{22}dx_3{}^2,\\
\bar{g} &= 2\,\alpha\,dx_3\,dx_1+\alpha\,h\left(x_2,x_3\right)_{11}dx_2{}^2+2\,\alpha\,h\left(x_2,x_3\right)_{12}dx_2\,dx_3+\beta dx_3{}^2+\alpha\,h\left(x_2,x_3\right)_{22}dx_3{}^2,
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where α and β are constants.

Success report

We have described explicitly all building blocks that can be used in constructing 4D metrics of Lorenz signature; Splitting-Gluing Lemmas give us the explicit construction.

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We have described explicitly all building blocks that can be used in constructing 4D metrics of Lorenz signature; Splitting-Gluing Lemmas give us the explicit construction. Let us count the number of cases: we can represent 4 as the sum of natural numbers by 4 different ways:

Dim of blocks	Description of blocks	# of cases
1+1+1+1	All building blocks are one-dimensional, and the metric is essentially the Levi-Civita metric from the introduction	1
1+1+2	The first two building blocks are one-dimensional, the third is two-dimensional	3
2+2	Both building blocks are two-dimensional; at least one of them is trivial (i.e., $\bar{h} = \mathrm{const} \cdot h$)	3
1+3	The first building block is one-dimensional, the second is three-dimensional ('Petrov', 'Eisenhart', or trivial)	3

Summary: I said something about all this (sub)problems

- **Problem 1. How to reconstruct a metric by its unparameterized geodesics?**
  - **Subproblem 1.1.** Given a big family of curves $\gamma(t; a)$, how to understand whether these curves are reparameterised geodesics of a certain affine connection? How to reconstruct this connection effectively? **Solved completely**
  - **Subproblem 1.2.** Given an affine connection $\Gamma = \Gamma^i_{jk}$, how to understand whether there exists a metric g in the projective class of Γ? How to reconstruct this metric effectively? **Suggested an effective way for Ricci-flat metrics**
- **Problem 2.** In what situations is the reconstruction of a metric by the unparameterised geodesics unique (up to the multiplication of the metric by a constant)?
  - **Subproblem 2.1.** What metrics 'interesting' for general relativity are geodesically rigid? **Almost every metric is geodesically rigid**
  - **Subproblem 2.2.** Construct all pairs of nonproportional geodesically equivalent metrics. **Solved completely**

Thank you!!!