Three-dimensional manifolds having metrics with the same geodesics.

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Abstract

We prove that if two Riemannian metrics have the same geodesics on a closed three-dimensional manifold which is homeomorphic neither to a lens space nor to a Seifert manifold with zero Euler number, then the metrics are proportional.
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1 Introduction

1.1 Main definitions and result

Definition 1. *Let g be a Riemannian metric on a manifold M^n, $n \geq 2$. A Riemannian metric $\bar{g}$ on M^n is called* **geodesically equivalent** *to g, if any geodesic of $\bar{g}$, considered as unparameterized curve, is a geodesic of g.*

Trivial examples of geodesically equivalent metrics can be obtained by considering *proportional* metrics g and $C \cdot g$, where C is a positive constant.

Definition 2. *The metrics are* **non-trivially geodesically equivalent***, if they are geodesically equivalent and not proportional*

Theorem 1. *Let the Riemannian metrics g and $\bar{g}$ be non-trivially geodesically equivalent on a closed connected three-dimensional manifold M^3. Then the manifold is homeomorphic either to a lens space or to a Seifert manifold with zero Euler number.*

Theorem 1 has been announced in [23].

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1.2 History

Beltrami [1] was the first to observe that two different metrics on surfaces can have the same geodesics. A multidimensional generalization of his example is as follows: g is the restriction of the Euclidean metrics $dx_1^2 + dx_2^2 + ... + dx_{n+1}^2$ to the sphere

$$S^n = \{(x_1, ..., x_{n+1}) \in R^{n+1} : \ x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1\},$$

$\bar{g}$ is the pull-back l^*g, where the mapping $l : S^n \to S^n$ is given by $l : v \mapsto \frac{A(v)}{\|A(v)\|}$ (where A is an arbitrary nondegenerate linear transformation of R^{n+1}). Since l takes great circles of the sphere to great circles, g and $\bar{g}$ are geodesically equivalent. Clearly, if A is not proportional to an orthogonal transformation, g and $\bar{g}$ are non-proportional.

At the end of his paper [1], Beltrami formulated a problem of describing all geodesically equivalent metrics on the surfaces. His motivation came from a certain problem of cartography, which requires the global setting.

Locally, in a neighborhood of almost every point, a complete description of geodesically equivalent metrics has been given by Dini [5] for surfaces and Levi-Civita [15] for manifolds of arbitrary dimension. We will formulate Levi-Civita's Theorem in Section 3. One of the results of Levi-Civita states that, locally, the set of metrics admitting (nontrivial) geodesic equivalence is quite rich: in the most general case, it is controlled by $n = dim(M)$ functions of one variable. For dimensions two and three, every closed manifold has a metric such that in a neighborhood of every point the metric admits a non-proportional geodesically equivalent metric.

A basic philosophical idea of the time of Weyl, E. Cartan and Eisenhart (see, for example, [34]) assumes that the Universe is explained by its infinitesimal structure. Geodesically equivalent metrics provide one of such structures: two metrics are geodesically equivalent if and only if

$$2(n+1)\bar{g}_{ij,k} = 2\bar{g}_{ij}\Theta_{,k} + \bar{g}_{ik}\Theta_{,j} + \bar{g}_{kj}\Theta_{,i}. \tag{1}$$

Here $T_{,l}$ is the covariant derivative of the tensor T with respect to g and Θ denotes the function $\ln\left(\frac{det(\bar{g})}{det(g)}\right)$.

Weyl studied geodesically equivalent metrics on the tensor level and found a few tensor reformulations of geodesic equivalence. One of his most remarkable results is the construction of the projective Weyl tensor [35]: if two metrics are geodesically equivalent, then their projective Weyl tensors coincide. E. Cartan [4] studied geodesic equivalence on the level of affine connections. He introduced the so-called projective connection, which allows reconstruction of geodesics as unparameterized curves. In his book [6], Eisenhart systematically applied both methods and obtained a series of local results.

However, our knowledge of the global (when the manifold is closed or complete) behavior of geodesically equivalent metrics is not satisfactory. All known global results require strong additional geometrical assumptions. Roughly speaking, one takes some geometric condition written in the tensor form (i.e., the metric is assumed to be Kählerian or semisymmetric or generally semisymmetric or T-generalized semisymmetric or Einsteinian or of constant curvature or almost Hermitian or Ricci-flat or recurrent or admitting a concircular vector field or admitting a torse-forming vector field), combines it with equation (1) or with any other tensor reformulation of geodesic equivalence and deduces some new tensor object with global geometric properties, see the survey paper [26].

1.3 Integrable systems and a proof of a weaker version of Theorem 1 modulo the Poincare conjecture

New methods for global investigation of geodesically equivalent metrics have been suggested in [17, 33]. The main observation of [17, 33] is that the existence of $\bar g$ geodesically equivalent to g allows one to construct commuting integrals for the geodesic flow of g, see Theorem 2 below.

Let $g = (g_{ij})$ and $\bar g = (\bar g_{ij})$ be Riemannian metrics on a manifold M^n. Consider the (1,1)-tensor L given by the formula

$$L^i_j \stackrel{\mathrm{def}}{=} \left(\frac{\det(\bar g)}{\det(g)}\right)^{\frac{1}{n+1}} \bar g^{i\alpha} g_{\alpha j}. \tag{2}$$

Then, L determines the family S_t, $t \in R$, of $(1,1)$-tensors

$$S_t \stackrel{\mathrm{def}}{=} \det(L - t\ \mathrm{Id})\,(L - t\ \mathrm{Id})^{-1}. \tag{3}$$

Remark 1. *Although $(L - t\ \mathrm{Id})^{-1}$ is not defined for t lying in the spectrum of L, the tensor S_t is well-defined for any t. Moreover, S_t is a polynomial in t of degree $n-1$ with coefficients being (1,1)-tensors.*

We will identify the tangent and cotangent bundles of M^n by g. This identification allows us to transfer the natural Poisson structure from T^*M^n to TM^n.

Theorem 2 ([17, 33]). *If g, $\bar g$ are geodesically equivalent, then, for any $t_1, t_2 \in R$, the functions*

$$I_{t_i} : TM^n \to R, \quad I_{t_i}(\xi) \stackrel{\mathrm{def}}{=} g(S_{t_i}(\xi), \xi) \tag{4}$$

are commuting integrals for the geodesic flow of g.

Definition 3. *The metrics g and $\bar g$ are called* **strictly-non-proportional** *at a point $x \in M^n$, if the number of different eigenvalues at the point x of g with respect to $\bar g$ is equal to the dimension of the manifold.*

Theorem 2 can be used most efficiently when the metrics are strictly-non-proportional at least at one point of the manifold. Then, the geodesic flow of g is Liouville-integrable, and we can apply the well-developed machinery of integrable systems. The following theorem has been obtained in [21, 24] by combining ideas of [31] with technique from [12]:

Theorem 3 ([21, 24]). *Let M^n, $n \geq 2$, be a connected closed manifold and g, $\bar{g}$ be geodesically equivalent Riemannian metrics on M^n. Suppose there exists a point of the manifold where the number of different eigenvalues of g with respect to $\bar{g}$ equals n. Then the following holds:*

1. *The first Betti number $b_1(M^n)$ is not greater than n.*
2. *The fundamental group of the manifold is virtually Abelian.*
3. *If, in addition, there exists a point where the number of different eigenvalues of g with respect to $\bar{g}$ is less than n, then $b_1(M^n) < n$.*
4. *If $b_1(M^n) = n$, then M^n is homeomorphic to the torus T^n.*

Theorem 3 already gives us the complete list of closed two-dimensional manifolds admitting non-trivial geodesic equivalence.

Indeed, for dimension two, any non-trivially geodesically equivalent metrics are strictly non-proportional at least at one point. Then, by the first statement of Theorem 3, a surface of genus greater than one can not admit nontrivial geodesic equivalence. (A self-contained proof of the last statement can be found in [19] and is based on the theory of quadratically integrable geodesic flows developed in [11, 14].)

It is easy to construct examples of non-trivially geodesically equivalent metrics on the closed surfaces of genus one or zero: Beltrami's construction gives us such examples for the sphere and for the projective space. Since the geodesics of the flat metric are straight lines, any two flat metrics on the torus (or on the Klein bottle) are geodesically related (that is, there exists a diffeomorphism that takes the geodesics of the first metric to the geodesics of the second.)

For dimension three, under the additional assumption that the metrics are strictly non-proportional at least at one point, the following weaker version of Theorem 1 follows from Theorem 3:

Corollary 1 ([22]). *Let the Riemannian metrics g and $\bar{g}$ on a closed connected 3-manifold M^3 be geodesically equivalent. Let them be strictly non-proportional at least at one point. Then, modulo the Poincaré conjecture, the manifold can be covered either by the sphere S^3 or by the torus T^3 or by the product $S^1 \times S^2$.*

Proof: By the second statement of Theorem 3, there exists a finite cover whose fundamental group is isomorphic to $\underbrace{Z \times Z \times ... \times Z}_{k}$. By [7], the fundamental group of a compact 3-manifold has no subgroup isomorphic to $Z \times Z \times Z \times Z$, so k is as most 3.

By a result of Reidemeister (which has been formulated and proven in [8] and [32]), it follows that $Z \times Z$ can not be the fundamental group of a closed 3-manifold. Thus, k is either 3 or 1 or 0. If $k = 3$, then, by the fourth statement of Theorem 3, M^3 is covered by the torus T^3. If $k = 0$, the fundamental group of M^3 is finite so that, modulo the Poincaré conjecture, the manifold is covered by the sphere. Suppose $k = 1$. Then, there exists a finite cover

$\tilde{M}^3$ whose fundamental group is isomorphic to Z. Then, by [36] and modulo the Poincaré conjecture, $\tilde{M}^3$ is reducible. That is, there exists a submanifold $S^2 \subset \tilde{M}^3$ homeomorphic to the 2-sphere such that the set $\tilde{M}^3 \setminus S^2$ is either connected or each of its connected components is not homeomorphic to the 3-ball. The second possibility can not occur in our case: otherwise, since the sphere is simply-connected, the fundamental group of $\tilde{M}^3$ is isomorphic to the free product of the fundamental groups of these components. Hence, at least one of these components is simply-connected and therefore (modulo the Poincaré conjecture) is homeomorphic to the 3-ball. Thus there exists a submanifold S^2 homeomorphic to the 2-sphere which does not divide $\tilde{M}^3$ into two parts.

Consider the generator of the fundamental group Z. It clearly intersects S^2 and the index of the intersection is ± 1.

Then, the fundamental group of $\tilde{M}^3 \setminus S^2$ is trivial. Hence, $\tilde{M}^3 \setminus S^2$ is homeomorphic to the direct product of the 2-sphere and the interval. Thus $\tilde{M}^3$ is homeomorphic to the direct product of the 2-sphere and the circle. Corollary 1 is proved.

1.4 Examples of geodesically equivalent metrics on lens spaces and on Seifert manifolds with zero Euler number

Beltrami's example allows us to construct non-trivially geodesically equivalent Riemannian metrics on lens spaces. Consider the real four-dimensional space R^4 with the complex coordinates $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$ and with the metric $ds^2 = dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2$. Consider the sphere

$$S^3 = \{(z_1, z_2) \in R^4 : \ |z_1|^2 + |z_2|^2 = 1\} \subset R^4.$$

It is known that every lens space is homeomorphic to the quotient space of S^3 modulo the action of the group Z_p generated by

$$(z_1, z_2) \mapsto (e^{2\pi \frac{q}{p} i} z_1, e^{2\pi \frac{1}{p} i} z_2)$$

for the appropriate mutually-prime p and q. Since the action preserves the restriction of the metric ds^2 to the sphere, the metric generates a metrics on the lens space. We call this metric *the round* metric of the lens space.

Consider the linear transformation A of R^4 given by $A : (z_1, z_2) \mapsto (a_1 z_1, a_2 z_2)$, where $a_1 \neq a_2$ are positive constants. The transformation is not orthogonal and commute with the action of the group Z_q. Then the metric $l^* g$ constructed in Paragraph 1.2 is invariant modulo the action of the group Z_q and, hence, generates a metric on the quotient space non-trivially geodesically equivalent to the round one.

In order to construct geodesically equivalent Riemannian metrics on a closed Seifert manifold with zero Euler number, we use the following observation: for every Riemannian manifolds (M_1, g_1) and (M_2, g_2), the metrics $g_1 + g_2$ and $g_1 + 2g_2$ are geodesically equivalent on the direct product $M_1 \times M_2$. (They even have the same affine connections!)

It is known [29] that every closed Seifert manifold with zero Euler number can be constructed as follows. Consider a closed surface F^2, a finite group H acting on F^2, the group Z_p of $\frac{2\pi}{p}$-rotations of the circle $S^1 = (R \bmod 2\pi))$ and a subgroup G of the group $H \times Z_p$ such that its natural action on the product $F^2 \times S^1$ is free. Then, the quotient space $(F^2 \times S^1)_{/G}$ is a closed Seifert manifold with zero Euler number. Each closed Seifert manifold with zero Euler number can be obtained using this construction [29].

Since H is finite and, therefore, compact, there exists a Riemannian metric g_2 on F^2 invariant with respect to the action of H. Denote by g_1 the metric $d\phi^2$ on S^1. The Riemannian metrics $g_1 + g_2$ and $g_1 + 2g_2$ are invariant modulo the action of G and, therefore, generate two non-trivially geodesically equivalent metrics on the quotient space. Thus, every Seifert manifold with zero Euler number admits non-trivial examples of geodesically equivalent metrics.

1.5 Multidimensional case

Theorem 1 gives us a complete list of closed 3-manifolds admitting non-trivial geodesic equivalence. The following observation shows that, for arbitrary dimension, such list is expected to be sufficiently more complicated.

By local-product structure on M^n we mean two transversal foliation B_r and B_{n-r} and a Riemannian metric g such that they locally look as they come from the direct product of two Riemannian manifolds: every point $p \in M^n$ has a neighborhood $U(p)$ with coordinates

$$(\bar{x}, \bar{y}) = \big((x_1, x_2, ..., x_r), (y_{r+1}, y_{r+2}, ..., y_n)\big)$$

such that the x-coordinates are constant on every fiber of the foliation $B_{n-r} \cap U(p)$, the y-coordinates are constant on every fiber of the foliation $B_r \cap U(p)$, and the metric g has the block-diagonal form

$$ds^2 = \sum_{i,j=1}^{r} G_{ij}(\bar{x}) dx_i dx_j + \sum_{i,j=r+1}^{n} H_{ij}(\bar{y}) dy_i dy_j, \tag{5}$$

where the first block depends on the first r coordinates and the second block depends on the remaining $n-r$ coordinates.

If a manifold admits a local-product structure, it automatically admits non-trivial geodesic equivalence: the metric $g = g_{|B_r} + g_{|B_{n-r}}$ is geodesically equivalent to $\bar{g} = g_{|B_r} + 2 \cdot g_{|B_{n-r}}$.

Although the local-product structures were very popular objects of investigation in 50th (see, for example, [13]), and the local behavior of local-product structures is quite well understood (see, for example, [27]), there is no topological classification (and even no non-trivial topological restrictions) for closed manifolds of dimension greater than three admitting local-product structures. For dimension three, a closed manifold admitting a local-product structure is homeomorphic to a Seifert manifold with zero Euler number, see Corollary B in [10] and [30], or Section 4.4.

For every dimension $n \geq 2$, there exist closed manifolds admitting no non-trivial geodesic equivalences:

Theorem 4 ([25]). *Let M^n, $n \geq 2$, be a connected closed manifold. Suppose it admits a Riemannian metric of negative sectional curvature. Then two Riemannian metrics on the manifold are geodesically equivalent if and only if they are proportional.*

For dimension three Theorem 4 follows from Theorem 1, since neither the sphere nor the product of the surface and the circle admit negatively curved metric [28]. A self-contained proof of Theorem 4 for dimension three can be found in [21].

Under the additional assumption that the metrics are strictly non-proportional at least at one point, the following statement is true:

Theorem 5. *Let M^n, $n \geq 2$, be a connected closed manifold. Suppose there exist two Riemannian metrics g and $\bar{g}$ on M^n such that they are geodesically equivalent and are strictly non-proportional at least at one point. Then there exists a finite cover of the manifold homeomorphic to a product of spheres.*

This theorem is a joint result with P. Topalov. Its proof is quite lengthy and will appear elsewhere, in a joint paper with P. Topalov.

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2 The eigenvalues of L are globally ordered

Within this section, we assume that the Riemannian metrics g and $\bar{g}$ on M^n are geodesically equivalent. We denote by

$$\lambda_1(x) \leq \lambda_2(x) \leq ... \leq \lambda_n(x)$$

the eigenvalues of L at $x \in M^n$.

Our main goal is to prove the following theorem which plays the key role in the proof of Theorem 1:

Theorem 6. *Let (M^n, g) be a connected Riemannian manifold. Suppose every two points of the manifold can be connected by a geodesic. Let Riemannian metric $\bar{g}$ on M^n be geodesically equivalent to g.*

Then, for every $i \in \{1, ..., n-1\}$, for every $x, y \in M^n$, the following holds:

1. $\lambda_i(x) \le \lambda_{i+1}(y)$.

2. If $\lambda_i(x) < \lambda_{i+1}(x)$, *then* $\lambda_i(z) < \lambda_{i+1}(z)$ *for almost every point* $z \in M^n$.

3. If $\lambda_i(x) = \lambda_{i+1}(y)$, *then there exists* $z \in M^n$ *such that* $\lambda_i(z) = \lambda_{i+1}(z)$.

Theorem 6 has been announced in [20, 23].

Corollary 2 ([18]). *Let* (M^n, g) *be a connected Riemannian manifold. Let* $\bar g$ *be geodesically equivalent to* g. *At every point* $x \in M^n$, *denote by* $N(x)$ *the number of different eigenvalues of* L *at the point* x.

Then, for almost every point y *of the manifold,*

$$N(y) = \max_{x \in M^n} (N(x)).$$

By definition, the tensor L is self-adjoint with respect to g. Then, for every $x \in M^n$, there exists a basis in T_xM^n such that g is given by the matrix $\mathrm{diag}(1, 1, ..., 1)$ and L is given by the matrix $\mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$. In this basis, the tensor (3) reads:

$$\begin{aligned} S_t &= \det(L - t\mathrm{Id})(L - t\mathrm{Id})^{(-1)} \\ &= \mathrm{diag}(\Pi_1(t), \Pi_2(t), ..., \Pi_n(t)), \end{aligned}$$

where the polynomials $\Pi_i(t)$ are given by the formula

$$\Pi_i(t) \stackrel{\mathrm{def}}{=} (\lambda_1 - t)(\lambda_2 - t)...(\lambda_{i-1} - t)(\lambda_{i+1} - t)...(\lambda_{n-1} - t)(\lambda_n - t). \tag{6}$$

Then, for every fixed $\xi = (\xi_1, \xi_2, ..., \xi_n) \in T_xM^n$, the function (4) is the following polynomial in t:

$$I_t = \xi_1^2 \Pi_1(t) + \xi_2^2 \Pi_2(t) + ... + \xi_n^2 \Pi_n(t). \tag{7}$$

Consider the roots of this polynomial. From the proof of Lemma 1, it will be clear that they are real. We denote them by

$$t_1(x, \xi) \le t_2(x, \xi) \le ... \le t_{n-1}(x, \xi).$$

Lemma 1. *Let* $\lambda_i(x)$ *and* $t_i(x, \xi)$ *be as above. The following holds:*

1. For every $\xi \in T_xM^n$,

$$\lambda_i(x) \le t_i(x, \xi) \le \lambda_{i+1}(x).$$

In particular, if $\lambda_i(x) = \lambda_{i+1}(x)$ *then* $t_i(x, \xi) = \lambda_i(x) = \lambda_{i+1}(x)$.

2. If $\lambda_i(x) < \lambda_{i+1}(x)$ *then for every constant* τ *the Lebesgue measure of the set*

$$V_\tau \subset T_xM^n, \quad V_\tau \stackrel{\mathrm{def}}{=} \{\xi \in T_xM^n : t_i(x, \xi) = \tau\},$$

is zero.

Proof of Lemma 1: Evidently, the coefficients of the polynomial I_t depend continuously on the eigenvalues λ_i and on the components ξ_i. Then, it is sufficient to prove the first statement of the lemma assuming that the eigenvalues λ_i are all different and that all ξ_i are non-zero. For every $\alpha \neq i$, we evidently have $\Pi_\alpha(\lambda_i) \equiv 0$. Then,

$$I_{\lambda_i} = \sum_{\alpha=1}^{n} \Pi_\alpha(\lambda_i)\xi_\alpha^2 = \Pi_i(\lambda_i)\xi_i^2. \tag{8}$$

Hence I_{λ_i} and $I_{\lambda_{i+1}}$ have different signs and therefore the open interval $]\lambda_i, \lambda_{i+1}[$ contains a root of the polynomial I_t. The degree of the polynomial I_t is equal $n-1$; we have $n-1$ disjoint intervals; each of these intervals contains at least one root so that all roots are real and the ith root lies between λ_i and λ_{i+1}. The first statement of the lemma is proved.

Let us prove the second statement of Lemma 1. We assume $\lambda_i < \lambda_{i+1}$. Suppose first $\lambda_i < \tau < \lambda_{i+1}$. Then, the set $V_\tau = \{\xi \in T_xM^n : t_i(x,\xi) = \tau\}$ consists of the points ξ where the function $I_\tau(x,\xi) \stackrel{\text{def}}{=} (I_t(x,\xi))_{|t=\tau}$ is zero; then it is a nontrivial quadric in $T_xM^n \equiv R^n$ and its measure is zero.

Let τ be one of the endpoints of the interval $[\lambda_i, \lambda_{i+1}]$. Without loss of generality, we can suppose $\tau = \lambda_i$. Let k be the multiplicity of the eigenvalue λ_i. Then, every coefficient $\Pi_\alpha(t)$ of the quadratic form (7) has a factor $(\lambda_i - t)^{k-1}$. Therefore,

$$\tilde{I}_t \stackrel{\text{def}}{=} \frac{I_t}{(\lambda_i - t)^{k-1}}$$

is a polynomial in t and $\tilde{I}_\tau$ is a nontrivial quadratic form. Evidently, for every point $\xi \in V_\tau$, we have $\tilde{I}_\tau(\xi) = 0$ so that the set V_τ is a subset of a nontrivial quadric in T_xM^n and its measure is zero. Lemma 1 is proved.

Proof of Theorem 6: The first statement of Theorem 6 follows immediately from the first statement of Lemma 1: Let us join the points $x, y \in M^n$ by a geodesic $\gamma : R \to M^n$, $\gamma(0) = x,\ \gamma(1) = y$. Consider the one-parametric family of integrals $I_t(x,\xi)$ and the roots

$$t_1(x,\xi) \leq t_2(x,\xi) \leq ... \leq t_{n-1}(x,\xi).$$

By Theorem 2, each root t_i is constant on each orbit $(\gamma, \dot\gamma)$ of the geodesic flow of g so that

$$t_i(\gamma(0), \dot\gamma(0)) = t_i(\gamma(1), \dot\gamma(1)).$$

Using Lemma 1, we obtain

$$\lambda_i(\gamma(0)) \leq t_i(\gamma(0), \dot\gamma(0)), \quad \text{and} \quad t_i(\gamma(1), \dot\gamma(1)) \leq \lambda_{i+1}(\gamma(1)).$$

Therefore $\lambda_i(\gamma(0)) \leq \lambda_{i+1}(\gamma(1))$ and the first statement of Theorem 6 is proved.

Let us prove the second statement of Theorem 6. Suppose $\lambda_i(y) = \lambda_{i+1}(y)$ for every point y of some subset $V \subset M^n$. Then, by the first statement of

Theorem 6, the value of λ_i is a constant (independent of $y \in V$). Indeed, for every $y_0, y_1 \in V$, we have

$$\lambda_i(y_0) \leq \lambda_{i+1}(y_1) \text{ and } \lambda_i(y_1) \leq \lambda_{i+1}(y_0).$$

Denote this constant by C. Take a point $x \in M^n$ and let us prove that $\lambda_i(x) = \lambda_{i+1}(x) = C$. Let us join the point x with every point of V by all possible geodesics. Consider the set $V_C \subset T_xM^n$ of the initial velocity vectors (at the point x) of these geodesics.

By the first statement of Lemma 1, for every geodesic γ passing through a point of V, the value $t_i(\gamma, \dot\gamma)$ is equal to C. By the second statement of Lemma 1, the measure of the set V_C is zero. Since the set V lies in the image of the exponential mapping of the set V_C, the measure of the set V is also zero. The second statement of Theorem 6 is proved.

In order to prove the third statement of Theorem 6, we need

Lemma 2. *Suppose g and $\bar g$ are geodesically equivalent Riemannian metrics on M^n. Let $\gamma : [0,1] \to M^n$ be a geodesic (for g). Suppose there exists $i \in \{1, ..., n-1\}$ such that*

$$\lambda_i(\gamma(0)) = \lambda_{i+1}(\gamma(1)) \stackrel{\text{def}}{=} \lambda < \lambda_{i+1}(\gamma(0)).$$

Then, for every $\tau \in [0,1]$, the following holds:

1. *$\lambda_i(\gamma(\tau)) = \lambda$ or $\lambda_{i+1}(\gamma(\tau)) = \lambda$,*
2. *$I_\lambda(\gamma(\tau), \dot\gamma(\tau)) = 0$.*

Proof of Lemma 2: By Lemma 1 and Theorem 2, we have

$$\lambda = \lambda_i(\gamma(0)) \leq t_i(\gamma(0), \dot\gamma(0)) = t_i(\gamma(1), \dot\gamma(1)) \leq \lambda_{i+1}(\gamma(1)) = \lambda$$

so that λ is a root of the polynomial $I_t(\gamma(0), \dot\gamma(0))$ and $I_\lambda(\gamma(\tau), \dot\gamma(\tau)) = 0$.

Let us prove that $\lambda_i(\gamma(\tau)) = \lambda$ or $\lambda_{i+1}(\gamma(\tau)) = \lambda$. If λ is a multiple root of the polynomial $I_t(\gamma(0), \dot\gamma(0))$, the statement obviously follows from Lemma 1 and Theorem 2. If $\lambda_{i-1}(\gamma(0)) = \lambda_i(\gamma(0))$, then λ is a multiple root of the polynomial $I_t(\gamma(0), \dot\gamma(0))$.

Below we assume that λ is a simple root of the polynomial $I_t(\gamma, \dot\gamma)$. Let us prove that the differential dI_λ vanishes at every point $(\gamma(\tau), \dot\gamma(\tau))$. Evidently, the differential of every integral is preserved by the geodesic flow so that it is sufficient to prove that the differential vanishes at the point $(\gamma(0), \dot\gamma(0))$.

By assumptions, the eigenvalue λ_i has multiplicity one in a small neighborhood of $\gamma(0)$. Then it is a smooth function on this neighborhood, and the function $I_{\lambda_i(z)}(z, \nu)$ is also smooth on the tangent bundle to this neighborhood. Let us consider the function $I_\lambda(z, \nu) - I_{\lambda_i(z)}(z, \nu)$. Its differential vanishes at the point $(\gamma(0), \dot\gamma(0))$. More precisely, by assumptions, λ is a simple root of the polynomial $I_t(\gamma(0), \dot\gamma(0))$ so that in a neighborhood of the point $\big(\lambda, (\gamma(0), \dot\gamma(0))\big) \in R \times TM^n$ the function $I_t(z, \nu)$ is a monotone function in t. But λ_i is no greater than λ. Then the difference $I_\lambda(z,\nu) - I_{\lambda_i(z)}(z,\nu)$ is either always non-positive or always non-negative in a small neighborhood of $(\gamma(0), \dot\gamma(0))$. By assumptions, $\lambda_i(\gamma(0)) = \lambda$. Then, $I_\lambda(\gamma(0),\dot\gamma(0)) - I_{\lambda_i(\gamma(0))}(\gamma(0),\dot\gamma(0)) = 0$. Hence, the function $I_\lambda(z,\nu) - I_{\lambda_i(z)}(z,\nu)$ has a local extremum at the point $(\gamma(0),\dot\gamma(0))$. Then, its differential vanishes at the point.

The differential of the function $I_{\lambda_i(z)}(z,\nu)$ also vanishes at the point $(\gamma(0),\dot\gamma(0))$. More precisely, in the appropriate coordinates the function I_{λ_i} is given by (8). Then, in view of the first statement of Theorem 6, it is either always non-positive or always non-negative. But $I_{\lambda_i(\gamma(0))}(\gamma(0),\dot\gamma(0))$ is equal to $I_\lambda(\gamma(0),\dot\gamma(0))$ and is zero. Then, the point $(\gamma(0),\dot\gamma(0))$ is an extremum of the function I_{λ_i} and therefore the differential of I_{λ_i} vanishes at the point $(\gamma(0),\dot\gamma(0))$. Thus the differential of the function I_λ vanishes at the point $(\gamma(0),\dot\gamma(0))$.

Let us prove that at every point $\gamma(\tau)$ of the geodesic, λ is an eigenvalue of L. Consider a coordinate system such that g and L at the point $\gamma(\tau)$ are given by the diagonal matrices $\mathrm{diag}(1,1,..,1)$ and $\mathrm{diag}(\lambda_1,\lambda_2,..,\lambda_n)$, respectively. Then, the restriction of the function I_λ to the tangent space $T_{\gamma}(\tau)M^n$ is given by

$$\sum_{\alpha=1}^{n} \xi_\alpha^2 \Pi_\alpha(\lambda)$$

The partial derivatives $\frac{\partial I_\lambda}{\partial \xi_\alpha}$ are

$$\frac{\partial I_\lambda}{\partial \xi_\alpha} = 2\Pi_\alpha(\lambda)\xi_\alpha.$$

Then, since at least one of the components ξ_α is not zero, at least one $\Pi_\alpha(\lambda)$ vanishes. Hence, λ is equal to one of the numbers $\lambda_1, ..., \lambda_n$. By the first statement of Theorem 6, it must be equal to λ_i or λ_{i+1}. Lemma 2 is proved.

Let us prove the third statement of Theorem 6. Let $\lambda_i(\gamma(0)) = \lambda_{i+1}(\gamma(1)) = \lambda$ for some $i \in \{1, ..., n-1\}$ and for some constant λ. If $\lambda_i(\gamma(0)) = \lambda_{i+1}(\gamma(0))$, it is nothing to prove. Below we assume $\lambda_i(\gamma(0)) < \lambda_{i+1}(\gamma(0))$. Then, by Lemma 2, at every point $\gamma(\tau)$ of the geodesic, $\lambda_i(\gamma(\tau)) = \lambda$ or $\lambda_{i+1}(\gamma(\tau)) = \lambda$. Then, every point of the segment $[0,1]$ lies in one of the following sets:

$$\begin{aligned}\Gamma_0 &= \{\tau \in [0,1] : \lambda_i(\gamma(\tau)) = \lambda\},\\ \Gamma_1 &= \{\tau \in [0,1] : \lambda_{i+1}(\gamma(\tau)) = \lambda\}.\end{aligned}$$

The subsets Γ_0, Γ_1 are evidently closed and non-empty. Then they intersect; at each point τ of the intersection we have $\lambda_i(\gamma(\tau)) = \lambda_{i+1}(\gamma(\tau)) = \lambda$. Theorem 6 is proved.

3 Levi-Civita's Theorem and vanishing of Nijenhuis torsion for L

Let g, $\bar g$ be Riemannian metrics. Consider the tensor L given by (2). At every point $x \in M^n$, consider the different eigenvalues $\phi_1(x) < \phi_2(x) < ... < \phi_m(x)$

of L. Let $k_i(x)$ be the multiplicity of the eigenvalue $\phi_i(x)$ so that $k_1(x)+...+k_m(x)=n$. Consider the ordered set $K(x)\stackrel{\text{def}}{=}\{k_1(x),k_2(x),...,k_m(x)\}$.

Definition 4. *A point $x\in M^n$ is called* **stable** *(with respect to g, $\bar g$), if it has a neighborhood $U(x)$ such that $K(x)=K(y)$ for every $y\in U(x)$.*

Theorem 7 (Levi-Civita's Theorem, [15], 1896). *Let g, $\bar g$ be Riemannian metrics on M^n. Let a point $x\in M^n$ be stable; let $K(x)$ be equal to $\{k_1,k_2,...,k_m\}$. The metrics are geodesically equivalent in some sufficiently small neighborhood $U(x)$ of the point x, if and only if there exists a coordinate system $\bar x=(\bar x_1,...,\bar x_m)$ (in $U(x)$), where $\bar x_i=(x_i^1,...,x_i^{k_i})$, $(1\le i\le m)$, such that the quadratic forms of g and $\bar g$ have the following form:*

$$\begin{aligned}
g(\dot{\bar x},\dot{\bar x}) &= P_1(\bar x)A_1(\bar x_1,\dot{\bar x}_1)+\ P_2(\bar x)A_2(\bar x_2,\dot{\bar x}_2)+\cdots+\\
&+\ \ P_m(\bar x)A_m(\bar x_m,\dot{\bar x}_m),\\
\bar g(\dot{\bar x},\dot{\bar x}) &= \rho_1P_1(\bar x)A_1(\bar x_1,\dot{\bar x}_1)+\rho_2P_2(\bar x)A_2(\bar x_2,\dot{\bar x}_2)+\cdots+\\
&+\ \rho_mP_m(\bar x)A_m(\bar x_m,\dot{\bar x}_m),
\end{aligned}$$

where $A_i(\bar x_i,\dot{\bar x}_i)$ are positive-definite quadratic forms in the velocities $\dot{\bar x}_i$ with coefficients depending on $\bar x_i$,

$$\begin{aligned}
P_i &\stackrel{\text{def}}{=} (\phi_i-\phi_1)\cdots(\phi_i-\phi_{i-1})(\phi_{i+1}-\phi_i)\cdots(\phi_m-\phi_i),\\
\rho_i &= \frac{1}{\phi_1...\phi_m}\frac{1}{\phi_i}
\end{aligned}$$

and $0<\phi_1<\phi_2<...<\phi_m$ are smooth functions such that

$$\phi_i=\begin{cases}\phi_i(\bar x_i), & \text{if}\quad k_i=1\\ \text{constant}, & \text{otherwise.}\end{cases}$$

Remark 2. *In the Levi-Civita coordinates from Theorem 7, L is given by the diagonal matrix*

$$\operatorname{diag}(\underbrace{\phi_1,...,\phi_1}_{k_1},....,\underbrace{\phi_m,...,\phi_m}_{k_m}),\tag{9}$$

so that the notation "ϕ"for the different eigenvalues of L used before Levi-Civita's Theorem is compatible with the notations inside the theorem.

Corollary 3. *Suppose the Riemannian metrics $g,\bar g$ are geodesically equivalent. Then the Nijenhuis torsion of L vanishes.*

Proof: Nijenhuis torsion is a tensor, so it is sufficient to check its vanishing at almost every point. It follows from Theorem 6 that almost every point of M^n is stable. In the Levi-Civita coordinates from Theorem 7, L is given by (9). Since the eigenvalue ϕ_i can depend on $\bar x_i$ only, the Nijenhuis torsion of L vanishes [9]. Corollary 3 is proved.

Remark 3. *A self-contained proof of Corollary 3 which does not require Levi-Civita's Theorem can be found in [3].*

Corollary 4. *Let Riemannian metrics g and $\bar g$ be geodesically equivalent on a connected manifold M^n. Suppose there exists $r \in \{1, ..., n-1\}$ such that $\lambda_r(x) < \lambda_{r+1}(y)$ for and $x, y \in M^n$. At each point $x \in M^n$, consider the subspace $V_r \in T_xM^n$ spanned by the eigenspaces of L corresponding to the eigenvalues $\lambda_1(x), ..., \lambda_r(x)$. Then, the following holds:*

1. *The distribution V_r is integrable.*
2. *For every integral submanifold of the distribution, the restrictions of g and $\bar g$ to it are geodesically equivalent. We denote by $\eta_1 \le ... \le \eta_r$ the eigenvalues of the tensor (2) for the restrictions of the metrics to the integral submanifold.*
3. *For every $i \in \{1, ..., r\}$,*

$$\eta_i = (\lambda_{r+1}\lambda_{r+2}...\lambda_n)^{\frac{1}{r+1}}\lambda_i.$$

Proof: First of all, since $\lambda_r < \lambda_{r+1}$ at every point, the distribution V_r is well-defined and smooth. Since the Nijenhuis torsion of L vanishes, the distribution V_r is integrable [9]. Let us show that the restriction of the metrics to every integral submanifold of the distribution are geodesically equivalent.

Geodesic equivalence is equivalent to the tensor condition (1). Hence, it is sufficient to verify it at almost every point. It follows from Theorem 6 that almost each point of M^n is stable. Take a stable point p and consider the Levi-Civita coordinates $\bar x = (\bar x_1, ..., \bar x_m)$ in a neighborhood of p. There exists q such that $k_1 + k_2 + ... + k_q = r$. In the the Levi-Civita coordinates, the plaques (= the connected components of the intersections of the integral submanifolds with the neighborhood) are the sets

$$\{\bar x_{q+1} = const_{q+1}, ..., \bar x_m = const_m\}.$$

Denote by P the plaque passing through the point p. The coordinates $\bar x_1, ..., \bar x_q$ give us a coordinate system on it. In this coordinates, the restrictions of g and $\bar g$ to every integral submanifold are given by

$$g_{|P}(\dot x, \dot x) = \sum_{i=1}^{q} \tilde P_i \tilde A_i(x_i, \dot x_i), \tag{10}$$

$$\bar g_{|P}(\dot x, \dot x) = \sum_{i=1}^{q} \tilde\rho_i \tilde P_i \tilde A_i(x_i, \dot x_i), \tag{11}$$

where

$$\tilde P_i \stackrel{\mathrm{def}}{=} (\phi_i - \phi_1)...(\phi_i - \phi_{i-1})(\phi_{i+1} - \phi_1)...(\phi_q - \phi_i),$$

$$\tilde A_i(x_i, \dot x_i) \stackrel{\mathrm{def}}{=} (\phi_{q+1} - \phi_i)(\phi_{q+2} - \phi_i)...(\phi_m - \phi_i)A_i(x_i, \dot x_i),$$

$$\tilde{\rho}_i \stackrel{\rm def}{=} \left(\frac{1}{\phi_{q+1}...\phi_m}\right)\left(\frac{1}{\phi_1...\phi_q}\frac{1}{\phi_i}\right).$$

Since the functions $\phi_{q+1}, ..., \phi_m$ are constant on the integral submanifolds, the metrics (10,11) have the form from Levi-Civita's Theorem and, therefore, are geodesically equivalent.

The third statement is an easy exercise in linear algebra. Corollary 4 is proved.

4 Proof of Theorem 1

We assume that the manifold M^3 is closed and that the Riemannian metrics g and $\bar{g}$ on M^3 are non-trivially geodesically equivalent. Our goal is to prove that M^3 is homeomorphic to a lens space or to a Seifert manifold with zero Euler number.

As in the previous sections, we consider L given by (2) and its eigenvalues $\lambda_1 \le \lambda_2 \le \lambda_3$. It follows from Theorem 6 that only the following four cases are possible.

1. There exist numbers $\tau_1 < \tau_2$ and points $x_0, y_0 \in M^3$ such that $\lambda_1(y_0) = \tau_1 = \lambda_2(y_0)$ and $\lambda_2(x_0) = \tau_2 = \lambda_3(x_0)$.

2. The eigenvalue λ_2 is constant, and there exist points $x_1, x_2, x_3 \in M^3$ such that $\lambda_1(x_1) = \lambda_2$, $\lambda_1(x_2) < \lambda_2 < \lambda_3(x_2)$ and $\lambda_3(x_3) = \lambda_2$.

3. At each point $x \in M^3$, at least two eigenvalues are equal and there exists a point where all of them are equal.

4. There exists a constant τ such that either $\lambda_1(x) < \tau < \lambda_2(x)$ for every $x \in M^3$, or $\lambda_2(x) < \tau < \lambda_3(x)$ for every $x \in M^3$.

We will prove that in the first case the manifold is homeomorphic to a lens space; in the second and in the third cases, the manifold is homeomorphic to the sphere or to the projective space; and in the fourth case, the manifold is homeomorphic to a Seifert manifold with zero Euler number.

Remark 4. *In particular, the following statements are true:*

1. *If a lens space is homeomorphic neither to the sphere nor to the projective space, then two non-proportional geodesically equivalent metrics on the lens space are strictly-non-proportional at least at one point.*

2. *Let F^2 be a closed surface of genus greater than one. Then two geodesically equivalent metrics on the product $F^2 \times S^1$ are nowhere strictly-non-proportional.*

Since every manifold covered by S^3 is orientable, in the first three cases we can (and we will) assume that M^3 is orientable.

We also would like to note that under the additional assumption that all eigenvalues of L are not constant, Case 1 follows from [12]. More precisely, if the metrics are strictly non-proportional at least at one point, then, for different t_1, t_2, t_3, the integrals $I_{t_1}, I_{t_2}, I_{t_3}$ are functionally independent almost everywhere. By definition, they are quadratic in velocities and simultaneously diagonalizable at every point of the manifold. Then, g is *Liouville* in the sense of [12], page 1. One can prove that g is *proper* (in the sense of [12], page 2) if and only if every eigenvalue of L is not constant, and that the *rank* (in the sense of [12], page 3) of g is equal to one. Then, by Theorem 3.3.1 from [12], the manifold M^3 is homeomorphic either to S^3 or to RP^3.

Clearly, if one of the eigenvalues of L is constant (we denote this constant by λ), then it follows from (7) that the function I_λ is equal to $\pm F^2$, where the function $F : TM^3 \to R$ is linear in velocities. Then, F is an integral of the geodesic flow. If the metrics are strictly non-proportional at least at one point of the manifold, the function F is not identically zero. Then, there exists a Killing vector field for g. Thus,

Remark 5. *If a lens space is homeomorphic neither to the sphere nor to the projective space, then if a metric on the lens space admits a non-trivial geodesic equivalence then it admits a (non-trivial) Killing vector field.*

4.1 Proof for Case 1

We assume that M^3 is orientable and that there exist numbers $\tau_1 < \tau_2$ and points $x_0, y_0 \in M^3$ such that $\lambda_1(y_0) = \tau_1 = \lambda_2(y_0)$ and $\lambda_2(x_0) = \tau_2 = \lambda_3(x_0)$.

We will show that M^3 can be cut into tree pieces M_1, M_2 and M_3 such that M_1 and M_3 are homeomorphic to the solid torus and M_2 is homeomorphic to the product of the interval and the torus. One connected component of the boundary of M_2 is glued to M_1 and the other connected component of the boundary of M_2 is glued to M_3. The meridians of the solid tori M_1 and M_3 are not homotopic in M_2. Then, M^3 is homeomorphic to a lens space.

There exists a connected component M of the set

$$\{x \in M^3 : \tau_1 < \lambda_2(x) < \tau_2\}$$

such that its closure $\bar{M}$ has points where $\lambda_2 = \tau_1$ and points where $\lambda_2 = \tau_2$.

Let us show that the closure $\bar{M}$ has a point X_1 where $\lambda_1 = \lambda_2 = \tau_1$.

Take $y_1 \in \bar{M}$ such that $\lambda_2(y_1) = \tau_1$. Suppose $\lambda_1(y_1) < \tau_1$. Let us join the point y_1 and y_0 by a geodesic segment γ. It follows from Lemma 2 that at each point of the segment we have $I_{\tau_1} = 0$ and $\lambda_1 = \tau_1$ or $\lambda_2 = \tau_1$. Without loss of generality, we can assume that y_0 is the only point of the geodesic segment where $\lambda_1 = \tau_1$. Then, at every point of the segment, $\lambda_2 = \tau_1$, and every point of the segment except the endpoint y_0 is stable. By Levi-Civita's Theorem, in a neighborhood of a stable point, there exist coordinates (x_1, x_2, x_3) such that the metrics are given by:

$$\begin{aligned} ds_g^2 &= (\lambda_1(x_1)-\lambda_2(x_2))(\lambda_1(x_1)-\lambda_3(x_3))dx_1^2 \\ &+ (\lambda_2(x_2)-\lambda_1(x_1))(\lambda_3(x_3)-\lambda_2(x_2))dx_2^2 \\ &+ (\lambda_3(x_3)-\lambda_1(x_1))(\lambda_3(x_3)-\lambda_2(x_2))dx_3^2 \end{aligned} \tag{12}$$

$$\begin{aligned} ds_{\bar g}^2 &= \frac{(\lambda_1(x_1)-\lambda_2(x_2))(\lambda_1(x_1)-\lambda_3(x_3))}{\lambda_1(x_1)^2\lambda_2(x_2)\lambda_3(x_3)}dx_1^2 \\ &+ \frac{(\lambda_2(x_2)-\lambda_1(x_1))(\lambda_3(x_3)-\lambda_2(x_2))}{\lambda_1(x_1)\lambda_2(x_2)^2\lambda_3(x_3)}dx_2^2 \\ &+ \frac{(\lambda_3(x_3)-\lambda_1(x_1))(\lambda_3(x_3)-\lambda_2(x_2))}{\lambda_1(x_1)\lambda_2(x_2)\lambda_3(x_3)^2}dx_3^2, \end{aligned} \tag{13}$$

and the integral I_{τ_1} is given by

$$\begin{aligned} I_{\tau_1} &= \big((\lambda_1(x_1)-\lambda_2(x_2))(\lambda_1(x_1)-\lambda_3(x_3))\big)\big((\lambda_2(x_2)-\tau_1)(\lambda_3(x_3)-\tau_1)\big)\xi_1^2 \\ &+ \big((\lambda_2(x_2)-\lambda_1(x_1))(\lambda_3(x_3)-\lambda_2(x_2))\big)\big((\lambda_1(x_1)-\tau_1)(\lambda_3(x_3)-\tau_1)\big)\xi_2^2 \\ &+ \big((\lambda_3(x_3)-\lambda_1(x_1))(\lambda_3(x_3)-\lambda_2(x_2))\big)\big((\lambda_1(x_1)-\tau_1)(\lambda_2(x_2)-\tau_1)\big)\xi_3^2. \end{aligned}$$

Substituting $\lambda_2 = \tau_1$, we see that the condition $I_{\tau_1} = 0$ implies that the velocity vector of the geodesic segment γ is orthogonal to $\frac{\partial}{\partial x_2}$. Then, the geodesic segment lies on the plane (x_1, x_3). Since λ_2 is independent of the coordinates x_1, x_3, the plane (x_1, x_3) lies in $\bar M$. Then, the geodesic segment lies in $\bar M$, and, therefore, the point y_0 lies in $\bar M$.

Similarly, $\bar M$ has a point Y_1 where $\lambda_3 = \lambda_2 = \tau_2$.

Take constants μ_1, μ_2 such that $\tau_1 < \mu_1 < \mu_2 < \tau_2$. Consider the connected component M_1 of the set

$$\{x \in M^3 :\ \lambda_2(x) \le \mu_1\}$$

containing the point X_1. By Sard's Theorem, we can choose the constants μ_1, μ_2 such that, at every point of the boundary of M_1, the differential $d\lambda_2$ does not vanish. Let us show that M_1 is homeomorphic to the solid torus.

At each point of M_1, let us consider a vector v_3 satisfying

$$\begin{cases} L(v_3) = \lambda_3 v_3 \\ g(v_3, v_3) = (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) \end{cases} \tag{14}$$

At each point of M_1, we have precisely two possibilities to choose v_3. Then, we can globally define a vector field v_3 satisfying conditions (14) on M_1 or on the double cover of M_1. By assumptions, M_1 is orientable. Hence, if its double cover is homeomorphic to the solid torus, then M_1 is itself homeomorphic to the solid torus. Therefore, we can suppose that the vector field v_3 satisfying conditions (14) is defined already on M_1. It follows from Lemma 3 that λ_2 is constant along the integral curves of v_3. Hence, M_1 is invariant with respect to the flow of v_3.

By definition and by Theorem 6, $\lambda_2 < \lambda_3$ at every point of M_1. Consider the distribution V_2 spanned by the eigenspaces of L corresponding to λ_1, λ_2. By Corollary 4, the distribution is integrable and, therefore, defines a two-dimensional foliation.

In the Levi-Civita coordinates, the plaques of the foliation are the coordinate planes (x_1, x_2). In the Levi-Civita coordinates, the metrics are given by (12,13), and the vector field equals $\pm\frac{\partial}{\partial x_3}$. Then, its flow takes leafs of the foliation to leafs. Since M_1 is connected, the flow acts transitively on the space of leafs. Then, all leafs are diffeomorphic, and all leafs have the same number of intersections with the integral curve of v_3 passing through X_1.

Consider the leaf P of the foliation passing through the point X_1. Let us show that the leaf is homeomorphic to the 2-disk, and that it has a finite number of intersections with the integral curve of v_3 passing through X_1.

We denote by h and $\bar{h}$ the restriction of g and $\bar{g}$ to P. By Corollary 4, the metrics $h, \bar{h}$ are geodesically equivalent. As in Corollary 4, we denote by $\eta_1 \le \eta_2$ the eigenvalues of the tensor (2) for the metrics $h, \bar{h}$.

Recall that, by Corollary 3, the eigenvalue λ_3 is constant on P, and the eigenvalues λ_1, λ_2 are constant along the integral curves of v_3. By Corollary 4, for every point y,

$$\eta_1(y) = (\lambda_3)^{\frac{1}{3}}\lambda_1(y) \le (\lambda_3)^{\frac{1}{3}}\lambda_2(y) = \eta_2(y).$$

Then, every point of the boundary is stable, and for every point X of the intersection of the integral curve of v_3 passing through X_1 with P, we have $\eta_1(X) = \eta_2(X) = (\lambda_3)^{\frac{1}{3}}\tau_1$.

Since in the Levi-Civita coordinates, the boundary of the leaf is given by the equation $x_2 = const$, it is a smooth curve. Then, it is sufficient to show that the universal cover of P has finitely many points where $\eta_1 = \eta_2 = (\lambda_3)^{\frac{1}{3}}\tau_1$.

Suppose the universal cover of P has infinitely many points where $\eta_1 = \eta_2 = (\lambda_3)^{\frac{1}{3}}\tau_1$.

Let us show that P is geodesically convex (in the weak sense) with respect to h. It is sufficient to show that every geodesic passing through at least one point of the interior can not be tangent to the boundary.

Since every point of the boundary is stable, by Levi-Civita's Theorem, there exist coordinates where the metrics and the one-parametric family of integrals (4) for the geodesic flow of h are given by

$$\begin{array}{rcl} ds_h^2 & = & (\eta_2(x_2) - \eta_1(x_1))(dx_1^2 + dx_2^2), \\ ds_{\bar{h}}^2 & = & \left(\dfrac{1}{\eta_1(x_1)} - \dfrac{1}{\eta_2(x_2)}\right)\left(\dfrac{dx_1^2}{\eta_1(x_1)} + \dfrac{dx_2^2}{\eta_2(x_2)}\right), \\ I_t & = & (\eta_2(x_2) - \eta_1(x_1))\big((\eta_2(x_2) - t)\xi_1^2 + (\eta_1(x_1) - t)\xi_2^2\big). \end{array}$$

Suppose the initial velocity vector $\dot{\gamma}(0)$ of a geodesic γ is tangent to the boundary of P. Then, its second component vanishes. Hence, $t_1(\gamma, \dot{\gamma}) = \eta_2 = (\lambda_3)^{\frac{1}{3}}\mu_1$. Then, by Lemma 1, the geodesic γ has no points where $\lambda_2 > \mu_1$. Thus, P is geodesically convex. Hence, the universal cover of P is geodesically convex as well. Then, there exist three geodesics $\gamma_1, \gamma_2, \gamma_3$ (on the universal cover) such that:

1. They intersect transversally at a point y where $\eta_1 < \eta_2$.
2. Every of the geodesics passes through at least one point where $\eta_1 = \eta_2 = (\lambda_3)^{\frac{1}{3}}\tau_1$.

Then, by Lemma 1, $t_1(\gamma_i, \dot\gamma_i) = (\lambda_3)^{\frac{1}{3}}\tau_1$, $i = 1, 2, 3$. Then, the velocity vectors of the geodesics lie on the quadric

$$\{(\xi_1, \xi_2) \in R^2 \equiv T_yP: \ \ (\eta_2 - (\lambda_3)^{\frac{1}{3}}\tau_1)\xi_1^2 + (\eta_1 - (\lambda_3)^{\frac{1}{3}}\tau_1)\xi_2^2\}.$$

Since the quadric is isomorphic to a pair of intersected straight lines, the velocity vectors at the point of intersection of at least two of the geodesics are proportional. The contradiction shows that the fundamental group of P is finite, and that the number of intersections of P with the integral curve of v_3 passing through X_1 is finite. Then, the foliation is indeed a fibration, and the fiber is homeomorphic to the 2-disk. Since the base of the fibration is one-dimensional and closed, it is homeomorphic to the circle. Finally, M_1 is homeomorphic to the solid torus.

Similarly, the connected component M_3 of the set

$$\{x \in M^3: \ \mu_2 \le \lambda_2(x)\} \subset M^3$$

containing the point X_3 is homeomorphic to the solid torus.

Let us denote by M_2 the closure of the set $M \setminus \big(M_1 \cup M_3\big)$.

Clearly, M_2 is a connected compact manifold with the boarder. The boarder has two connected components; one component coincides with the boarder of the solid torus M_1, the other component coincides with the boarder of the solid torus M_3. It follows from the implicit function theorem that the union $M_1 \cup M_2 \cup M_3$ is open and closed in M^3, and, therefore, coincides with M^3.

Let us show that M_2 is homeomorphic to the direct product of the 2-torus and of the interval.

At each point of M_2, the eigenvalues of L are all different. At each point of M_2, let us consider vectors v_1, v_2, v_3 satisfying

$$\left\{\begin{array}{rcl} L(v_i) & = & \lambda_i v_i \\ g(v_1, v_1) & = & (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \\ g(v_2, v_2) & = & (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3) \\ g(v_3, v_3) & = & (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) \end{array}\right. \tag{15}$$

The only freedom we have is the signs of the vector. Hence, at each point we have precisely eight possibilities to choose v_1, v_2, v_3. Then, on some finite cover $\tilde M_2$ of M_2, we can globally define vector fields v_1, v_2, v_3 satisfying conditions (15). More precisely, we can define the vector fields v_1, v_2, v_3 locally and therefore along every curve. Then, for every $x_0 \in M_2$, we have an action of the fundamental group $\pi_1(M_2, x_0)$ on the signs of the vectors at the point x_0. Consider the subgroup $\Gamma \subset \pi_1(M_2, x_0)$ preserving the signs of the vectors and the cover $\tilde{M}_2$ of M_2 corresponding to this subgroup. Evidently, the subgroup is normal and its index in $\pi_1(M_2, x_0)$ is finite (at most eight). On the cover $\tilde{M}_2$, we can globally define the vector fields v_1, v_2, v_3.

In the Levi-Civita coordinates from Theorem 7, the metrics are given by (12,13). By direct calculations, it is easy to verify that the vector fields v_1, v_2, v_3 are equal to $\pm\frac{\partial}{\partial x_1}$, $\pm\frac{\partial}{\partial x_2}$, $\pm\frac{\partial}{\partial x_3}$, respectively. Hence, they commute and the flows of the vector fields v_1 and v_3 are well-defined and preserve the function λ_2. Consider the action of the Abelian group $(R^2, +)$ on $\tilde{M}_2$ generated by the shifts along the integral curves of the vector fields v_1, v_3. The action is locally-free. Then, it defines a two-dimensional foliation of $\tilde{M}_2$.

In the Levi-Civita coordinates, the plaques of the foliation are the coordinate planes x_1, x_3. The shifts along the vector field v_2 evidently takes leafs to leafs and act transitively on the space of leafs. Hence, all leafs are diffeomorphic. Since $d\lambda_2 \neq 0$ at the points of the boarder of M_2, by implicit function theorem, the connected components of the boarder are the leafs of the foliation. Then every leaf of the foliation is homeomorphic to the torus. Since M_1 has a boundary, the integral curves of v_2 are homeomorphic to the interval. Since the torus and the interval are compact, every leaf has only a finite number of intersections with an integral curve of v_2 (in fact, the number of intersections is always one). Then, the foliation is indeed a fibration over the interval. Finally, $\tilde{M}_2$ is homeomorphic to the direct product of the torus and of the interval.

Now let us show that in fact the subset M_2 is itself diffeomorphic to the product $I \times T^2$. Consider the action of the factor group $\pi_1(M_2, x_0)/_\Gamma$ on $\tilde{M}_2$. Since M^3 is orientable, every non-trivial element of $\pi_1(M, x_0)/_\Gamma$ must change the signs of precisely two vector fields. Clearly, the group $\pi_1(M_2, x_0)/_\Gamma$ preserves the sign of v_2: otherwise the boarder of M_2 is connected, which contradicts to the fact that it contains points where $\lambda_2 = \mu_1$ and $\lambda_2 = \mu_2$.

In particular, a connected component of boundary is invariant with respect to the action of the group $\pi_1(M_2, x_0)/_\Gamma$. Then every non-trivial element of the group $\pi_1(M_2, x_0)/_\Gamma$ changes the signs of v_2 and v_3 on a connected components of boundary, and, therefore, has stable points. Thus the factor group $\pi_1(M_2, x_0)/_\Gamma$ is trivial. Hence, M_2 is diffeomorphic to the product $T^2 \times I$. Finally, M^3 is glued from two solid tori and the direct product $T^2 \times I$.

Take a point of $M_1 \cap M_2$ and consider the integral curve of the vector field v_1 passing through the point. The integral curve clearly lies on the intersection of the boarder with the leaf of the foliation V_2 on M_1. Then, it is a meridian of the solid torus M_1. Similarly, an integral curve of the vector field v_3 is a meridian of the solid torus M_3. Since they are not homotopic in M_2, M^3 is homeomorphic to a lens space. Theorem 1 is proved under assumptions of Case 1.

4.2 Proof for Case 2

We assume that the manifold is orientable, that the eigenvalue λ_2 is constant, and that there exist $x_1, x_2, x_3 \in M^3$ such that $\lambda_1(x_1) = \lambda_2$, $\lambda_1(x_2) < \lambda_2 <$

$\lambda_3(x_2)$ and $\lambda_3(x_3) = \lambda_2$. Our goal is to show that the manifold is homeomorphic to the sphere S^3 or to the projective space RP^3.

Consider the following subsets of M^3:

$$\begin{aligned} M &\stackrel{\text{def}}{=} \{x \in M^3 : (\lambda_1(x) - \lambda_2)(\lambda_3(x) - \lambda_2) = 0\}, \\ M_1 &\stackrel{\text{def}}{=} \{x \in M^3 : \lambda_1(x) = \lambda_2, \quad \lambda_3(x) = \lambda_2\}, \\ M_2 &\stackrel{\text{def}}{=} \{x \in M^3 : \lambda_1(x) = \lambda_2, \quad \lambda_3(x) \geq \lambda_2\}, \\ M_3 &\stackrel{\text{def}}{=} \{x \in M^3 : \lambda_1(x) \leq \lambda_2, \quad \lambda_3(x) = \lambda_2\}. \end{aligned}$$

We will show that the set M is a smooth submanifold homeomorphic to the circle. The points of M_1 divide M into intervals. The number of the intervals is four or two. In the first case, the difference $M^3 \setminus M_3$ is diffeomorphic to the product $S^2 \times I$, where I is (an open) interval. Then, M^3 is homeomorphic to S^3. In the second case, the fundamental group of M^3 is isomorphic to Z_2. Then, the universal cover of M^3 is as in the first case. Then, it is homeomorphic to S^3. Finally, M^3 is homeomorphic to RP^3.

The following lemma shows that the set M_1 is not empty:

Lemma 3. *Suppose $\lambda_1(x_1) = \lambda_2 = \lambda_3(x_3) < \lambda_3(x_1)$ for some points $x_1, x_3 \in M^3$. Then, $\lambda_3(x) = \lambda_2$ or $\lambda_1(x) = \lambda_2$ at every point x of every geodesic connecting the points x_1, x_3.*

Proof: Let us denote by $\gamma : R \to M^3$ a geodesic connecting the points x_1 and x_3. We assume that $\gamma(1) = x_1$ and $\gamma(2) = x_3$.

At each tangent space to the manifold we can find coordinates such that g and L are given by the diagonal matrixes $\operatorname{diag}(1,1,1)$ and $\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$, respectively. Consider the function

$$I'_t \stackrel{\text{def}}{=} \left(\frac{d}{dt} I_t\right).$$

For every fixed t, the function I'_t is an integral of the geodesic flow. By (7),

$$\begin{aligned} I_t &= (\lambda_2 - t)(\lambda_3(x) - t)\xi_1^2 + (\lambda_1(x) - t)(\lambda_3(x) - t)\xi_2^2 + (\lambda_2 - t)(\lambda_3(x) - t)\xi_3^2 \quad (16) \\ I'_t &= (\lambda_2 - t + \lambda_3(x) - t)\xi_1^2 + (\lambda_1(x) - t + \lambda_3(x) - t)\xi_2^2 + (\lambda_2 - t + \lambda_3(x) - t)\xi_3^2, \\ I'_{\lambda_2} &= (\lambda_1(x) - \lambda_2)(\xi_2^2 + \xi_3^2) + (\lambda_3(x) - \lambda_2)(\xi_2^2 + \xi_1^2). \quad (17) \end{aligned}$$

We see that the function I'_{λ_2} is non-negative at $(\gamma(1), \dot\gamma(1)) \in TM^3$ and is non-positive at the point $(\gamma(2), \dot\gamma(2)) \in TM^3$. Then, it is zero at each point of the geodesic orbit $(\gamma, \dot\gamma)$. Since $\lambda_1 = \lambda_2 < \lambda_3$ at the point x_1, the components ξ_1 and ξ_3 in (17) vanish so that $\dot\gamma(1)$ is an eigenvector of L with the eigenvalue $\lambda_3(x_1)$.

Substituting $t = \lambda_2$ in (16), we obtain that the function I_{λ_2} vanishes at $(\gamma(1), \dot\gamma(1))$. Thus λ_2 is a double-root of the polynomial $I_t(\gamma, \dot\gamma)$. It follows from Lemma 1 that the leading coefficient of the polynomial $I'_t(\gamma, \dot\gamma)$ is not zero.

Let us prove that the differential dI'_{λ_2} vanishes at each point of the geodesic orbit $(\gamma, \dot\gamma)$. Since I'_{λ_2} is an integral, it is sufficient to show this at the point $(\gamma(1), \dot\gamma(1))$ only. We have,

$$I'_{\lambda_1(x)} = (\lambda_3(x) - \lambda_1(x))(\xi_1^2 + \xi_2^2) + (\lambda_2 - \lambda_1(x))(\xi_1^2 + \xi_3^2).$$

We see that the function $I'_{\lambda_1(x)}$ is non-negative. At the point $(\gamma(1), \dot\gamma(1))$ it is zero (since $\lambda_1 = \lambda_2$ at $\gamma(1)$ and the components ξ_1 and ξ_2 of $\dot\gamma(1)$ equal zero). Then, the function has a minimum at the point $(\gamma(1), \dot\gamma(1))$, and its differential vanishes.

Let us show that the differential of the function $I'_{\lambda_1(x)} - I'_{\lambda_2}$ also vanishes at the point $(\gamma(1), \dot\gamma(1))$. Indeed, the function I'_t is a linear polynomial in t with non-zero leading coefficient at the point $(\gamma(1), \dot\gamma(1))$. Since $\lambda_1(x) < \lambda_2$, then the function $I'_{\lambda_1(x)} - I'_{\lambda_2}$ is either everywhere positive or everywhere negative. Since it vanishes at $(\gamma(1), \dot\gamma(1))$, the differential of $I'_{\lambda_1(x)} - I'_{\lambda_2}$ vanishes at the point $(\gamma(1), \dot\gamma(1))$.

Thus the differential dI'_{λ_2} is zero at each point of the geodesic orbit $(\gamma, \dot\gamma)$. At each point of γ, the components $\frac{\partial I'_{\lambda_2}}{\partial \xi_i}$ of dI'_{λ_2} are

$$2(\lambda_3 - \lambda_2)\xi_1,\ 2(\lambda_3 - \lambda_2 - \lambda_2 + \lambda_1)\xi_2,\ 2(\lambda_1 - \lambda_2)\xi_3.$$

Since the differential vanishes, all its components are equal to zero. Then, $\lambda_2 = \lambda_3$ or $\lambda_1 = \lambda_2$ or $\xi_2 \neq 0$.

On the other hand, by (16), using that $I_{\lambda_2} = 0$, we see that $(\lambda_2 - \lambda_1(x))(\lambda_2 - \lambda_3(x)) = 0$ or $\xi_2 = 0$.

Finally, every point of γ lies in the set M. Lemma 3 is proved.

By Lemma 3, there exists $x_0 \in M^3$ such that $\lambda_1(x_0) = \lambda_2 = \lambda_3(x_0)$, so that the set M_1 is not empty.

Lemma 4. *The following holds:*

1. *The set M_1 is finite.*
2. *The set M is a submanifold of M^3 homeomorphic to the circle.*
3. *M_1 divides M into a collection of intervals. Every interval is a subset either of M_2 or of M_3. Every point of M_1 is an endpoint of precisely two intervals; one of this intervals is a subset of M_2 and the other is a subset of M_3.*

Proof. Let us prove that M_1 is finite. Suppose M_1 is infinite. Then, there exists a small ball containing at least three points $y_1, y_2, y_3 \in M_1$. Then, for almost every point x of the ball, there exist three geodesics $\gamma_1, \gamma_2, \gamma_3$ such that $\gamma_i(0) = y_i$, $\gamma_i(1) = x$ and the velocity vectors $\dot\gamma_1(1), \dot\gamma_2(1), \dot\gamma_3(1)$ are mutually transversal. Let us show that $x \in M$. Indeed, by Lemma 1, for every $i = 1, 2, 3$

we have that λ_2 is a double root of $I_t(\gamma_i(1), \dot\gamma_i(1))$. Then, the components (ξ_1, ξ_2, ξ_3) of the velocity vectors $\dot\gamma_i(1)$ satisfy

$$\left\{ \begin{array}{rcl} (\lambda_1(x) - \lambda_2)(\lambda_3(x) - \lambda_2)\xi_2^2 & = & 0 \\ (\lambda_1(x) - \lambda_2)(\xi_2^2 + \xi_3^2) + (\lambda_3(x) - \lambda_2)(\xi_2^2 + \xi_1^2) & = & 0. \end{array} \right. \tag{18}$$

If $\lambda_1(x) < \lambda_2 < \lambda_3(x)$, the solutions of (18) are organized into two intersected straight lines so that the velocity vectors $\dot\gamma_i(1)$ are not mutually transversal. Then, almost every point of the ball lies in M which contradicts Theorem 6. Finally, the set M_1 is finite. The first statement of Lemma 4 is proved.

Let us prove the second and the third statement of our lemma. Take a point $x_0 \in M_1$ and consider the ball B_ϵ and the sphere S_ϵ of small radius ϵ with center x_0. Since M_1 is discrete, we can assume that x_0 is the only point of $M_1 \cap B_\epsilon$. Let us show that the intersection $M \cap S_\epsilon$ consists of precisely two points. First of all, it consists of maximum two points. Indeed, for every geodesic γ passing through a point of M, we evidently have $I_{\lambda_2}(\gamma, \dot\gamma) = 0$. Hence, at every point x of the geodesic such that $x \in M^3 \setminus M$, the geodesic is orthogonal to the eigenvector of L corresponding to λ_2. If the intersection $M \cap S_\epsilon$ has more than two points, then almost every point of the ball is an intersection of three geodesics passing through the points of M with linear independent velocity vectors at the point of the intersection. Hence, almost every point of the ball lies in M which is impossible by Theorem 6. Thus $M \cap S_\epsilon$ consists of at most two points.

The intersection $M \cap S_\epsilon$ is not empty. Otherwise, at every point of S_ϵ, the eigenspace of L with the eigenvalue λ_2 is one-dimensional. Since it is orthogonal to the geodesics passing through x_0, it is tangent to the sphere S_ϵ, which contradicts the Euler characteristic of the sphere S_ϵ. Thus the intersection $M \cap S_\epsilon$ is not empty.

Consider the geodesic connecting a point of $M \cap S_\epsilon$ with the point x_0. By Lemma 4, every point of the geodesic lies in M. The second intersection of the geodesic with the sphere gives us the second point of $M \cap S_\epsilon$.

Thus the intersection $M \cap S_\epsilon$ consists of precisely two points, and the intersection $M \cap B_\epsilon$ is the geodesic connecting these two points.

Let us show that one of these points lies in M_2 and the other in M_3. The point x_0 divides the set $M \cap B_\epsilon$ into two intervals I_1 and I_2; we will show that $\lambda_1 = \lambda_2$ at every point of one of the intervals and $\lambda_3 = \lambda_2$ at every point of the other interval.

Take $y \in B_\epsilon$ such that $\lambda_1(y) < \lambda_2 < \lambda_3(y)$. Consider all geodesics in B_ϵ connecting the points of $M \cap B_\epsilon$ with the point y. The velocity vectors of these geodesics at the point y satisfy the first equation of (18). Hence, they lie on the plane orthogonal to the eigenvector of L corresponding to λ_2. The velocity vectors (at the point y) of the geodesics connecting the point x_0 with y are organized in one of the rays of two intersected straight lines (18). The velocity vectors of the geodesics coming from the points of the interval I_1 lie from one side of this ray, the velocity vectors of the geodesics coming from the points of the interval I_2 lie from the other side.

Evidently, the function I'_{λ_2} is positive from one side of the ray and negative from the other. Hence, $\lambda_3 > \lambda_2$ on one of these intervals and $\lambda_1 < \lambda_2$ on the other.

Finally, we have that M is closed, contains points where $\lambda_1(x) \neq \lambda_2$, or $\lambda_3(x) \neq \lambda_2$, every geodesic connecting these points with the point x_0 lies in M and, for every point of M_1, the sphere of a small radius around the point of M_1 has precisely two points of M. Then, M is homeomorphic to the circle. Hence, M_2 and M_3 are organized in a collection of intervals with endpoints in M_1. Every point of M_1 is an endpoint of precisely two intervals; one of the intervals is a subset of M_2 and the other is a subset of M_3. Lemma 4 is proved.

Consider the difference $M^3 \setminus M_3$. Since M has codimension two, $M^3 \setminus M_3$ is connected. By definition, $\lambda_2 < \lambda_3$ at every point of $M^3 \setminus M_3$. At each point of $M^3 \setminus M_3$, there exist precisely two vectors satisfying conditions (14). Then, on $M^3 \setminus M_3$ or on a double-cover of $M^3 \setminus M_3$ we can define a vector field v_3 satisfying (14).

Suppose first the vector field v_3 is defined already on $M^3 \setminus M_3$. Let us show that $M^3 \setminus M_3$ is homeomorphic to the product $S^2 \times I$.

Consider the distribution orthogonal to v_3. By Corollary 4, the distribution is integrable so that it generates a two-dimensional foliation of $M^3 \setminus M_3$. Locally, in the Levi-Civita coordinates, g and $\bar{g}$ have the form (12,13). Then, $v_3 = \pm \frac{\partial}{\partial x_3}$ and the plaques of the foliation are the coordinate planes (x_1, x_2). Then, the shifts along the integral curves of v_3 take leafs of the foliation to leafs, and act transitively on the space of leafs. Then, all leafs are homeomorphic. Let us show that every leaf is homeomorphic to the two-sphere.

By Sard's Theorem, there exists a regular value of λ_3. Consider y_0 from the preimage of the regular value. By definition, $d_{\lambda_3} \neq 0$ at each point of the preimage. Consider the connecting component of the preimage containing the point x_0. By implicit function theorem, it is a two-dimensional manifold.

Now, by Corollary 3, λ_3 is constant along the leaf of the foliation passing through y_0. Since the leaf of the foliation is also a two-dimensional manifold, it coincides with the connecting component of the preimage. Hence, it is closed. Finally, all leafs of the foliation are closed.

By Corollary 4, the restrictions of the metrics to the leaf are non-trivially geodesically equivalent and are proportional at the point of M_2. Then, the leaf is homeomorphic to the sphere or to the projective plane [2]. Since, by assumptions, the manifold is orientable, the projective plane can not occur. Thus the leaf is homeomorphic to the sphere. Finally, every leaf is homeomorphic to the sphere.

By Corollary 4, for every leaf of the foliation, one of the eigenvalues of the tensor (2) constructed for the restrictions of g and $\bar{g}$ to the leaf, is constant. Then, the leaf has precisely two points where the metrics are proportional [2]. By Corollary 4, the metrics are proportional precisely at the points of the intersection of the leaf with M_2. Then, the different points of the intersection belong to different connected component of M_2. Thus the foliation is indeed a fibration. The base of the fibration is homeomorphic to the interval. Thus $M^3 \setminus M_3$ is homeomorphic to the product $S^2 \times I$, and the set M_2 consists of precisely two intervals. Hence, by Lemma 4, the set M_3 consists of precisely two intervals. Since $M^3 = (M^3 \setminus M_3) \cup M_3$, then M^3 is homeomorphic to S^3.

Now suppose the vector field v_3 can not be defined on $M^3 \setminus M_3$. By Lemma 4, the set M_3 is a union of non-intersected smooth intervals. Hence, the fundamental groups of M^3 and of $M^3 \setminus M_3$ coincide. Then, there exists a double cover of M^3 satisfying the assumptions of the first case. Then, the double cover of M^3 is homeomorphic to S^3. Finally, M^3 is homeomorphic to RP^3 [16]. Theorem 1 is proved under assumptions of Case 2.

4.3 Proof for Case 3

By Theorem 6, without loss of generality we can assume that $\lambda_1(y) < \lambda_2(y) = \lambda_3(y) = const \stackrel{\rm def}{=} \lambda$ for almost each point $y \in M^3$, and that there exists $x_0 \in M^3$ such that $\lambda_1(x_0) = \lambda_2(x_0) = \lambda_3(x_0)$.

We will show that all geodesics starting from the point x_0 intersect at some point y_0 such that $\lambda_1(y_0) = \lambda_2(y_0) = \lambda_3(y_0)$, and that they do not have (transversal) intersections before y_0. Therefore, if $x_0 \neq y_0$ then M^3 is homeomorphic to the 3-sphere, and if $x_0 = y_0$ then M^3 is homeomorphic to the projective space RP^3.

At each point $x \in M^3$, we can find a basis of the space T_xM^3 such that in this basis g is given by the diagonal matrix $\mathrm{diag}(1,1,1)$ and L is given by the diagonal matrix $\mathrm{diag}(\lambda_1(x), \lambda_2(x), \lambda_3(x))$. In this basis, the polynomial $I_t(x,\xi)$ given by (4) reads

$$I_t = (\lambda - t)^2\xi_1^2 + (\lambda_1(x) - t)(\lambda - t)(\xi_2^2 + \xi_3^2).$$

Therefore, for every t, the functions

$$\tilde I_t \stackrel{\rm def}{=} \frac{I_t}{(\lambda - t)} = (\lambda - t)\xi_1^2 + (\lambda_1(x) - t)(\xi_2^2 + \xi_3^2)$$

is an integral for the geodesic flow of g. Substitution $t = \lambda$ gives us the integral

$$\tilde I_\lambda = (\lambda_1(x) - \lambda)(\xi_2^2 + \xi_3^2).$$

For every geodesic γ passing through the point y, we clearly have $\tilde I_\lambda(\gamma, \dot\gamma) \equiv 0$. Then, for every $x \in \gamma$ such that $\lambda_1(x) \neq \lambda$, the sum $(\xi_2^2 + \xi_3^2)$ vanishes. Hence, the velocity vector $\dot\gamma(x)$ is an eigenvector of L with the eigenvalue $\lambda_1(x)$. Then, two geodesics passing through y can transversally intersect only at the points where $\lambda_1 = \lambda$. Then, there can be maximum two of them and M^3 is homeomorphic either to S^3 or to RP^3. Theorem 1 is proved under the assumptions of Case 3.

4.4 Proof for Case 4

Suppose there exists $\tau \in R$ such that $\lambda_1(x) < \tau < \lambda_2(x)$ for every $x \in M^3$. Our goal is to show that M^3 is homeomorphic to a Seifert manifold with zero Euler number.

Consider the (1,1)-tensor D given by

$$D \stackrel{\text{def}}{=} \left(L^2 - \big(\text{trace}(L) - \lambda_1(x)\big)L + \frac{\det(L)}{\lambda_1(x)}\text{Id} + L - \lambda_1(x)\text{Id}\right)^{-1}. \tag{19}$$

Let us show that the condition $\lambda_1 < \tau < \lambda_2$ ensures that D is well-defined, is self-adjoint with respect to g, and all its eigenvalues are positive.

At each tangent space T_xM^3, consider coordinates where g and L are given by $\text{diag}(1,1,1)$ and $\text{diag}(\lambda_1, \lambda_2, \lambda_3)$, respectively. We have:

$$\begin{aligned}
L^2 - \big(\text{trace}(L) - \lambda_1\big)L + \frac{\det(L)}{\lambda_1}\text{Id} &= L^2 - (\lambda_2 + \lambda_3)L + \lambda_2\lambda_3\text{Id} = \\
&= (L - \lambda_2\text{Id})(L - \lambda_3\text{Id}) = \\
&= \text{diag}\big((\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1), 0, 0\big), \\
L - \lambda_1(x)\text{Id} &= \text{diag}\big(0, (\lambda_2 - \lambda_1), (\lambda_3 - \lambda_1)\big).
\end{aligned}$$

Thus,

$$D = \text{diag}\big((\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1), (\lambda_2 - \lambda_1), (\lambda_3 - \lambda_1)\big),$$

so that D is self-adjoint and all its eigenvalues are positive.

For every $x \in M^3$, consider $v_1 \in T_xM^3$ satisfying the following conditions:

$$\begin{cases} L(v_1) &= \lambda_1 v_1 \\ g(v_1, v_1) &= (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1). \end{cases} \tag{20}$$

At every point of the manifold, we have precisely two possibilities to choose v_1. Then, at least on the double-cover of the manifold, we can globally define a vector field v_1 satisfying (20). By [30], a closed manifold is homeomorphic to a Seifert manifold with zero Euler number if and only if its double cover is homeomorphic to a Seifert manifold with zero Euler number. Then, we can assume that the vector field v_1 is defined already on M^3.

Let us show that the vector field v_1 is Killing with respect to the metric h, and that the 1-form ω given by

$$\omega : TM \to R, \quad \omega(\xi) \stackrel{\text{def}}{=} h(\xi, v_1)$$

is closed.

It is sufficient to verify these at almost every point. By Corollary 2, almost every point $p \in M^3$ is stable. Suppose first that the number of different eigenvalues of L at the stable point p is three. Then, by Levi-Civita's Theorem, there exists coordinates where g and $\bar g$ are given by (12,13).

Then, $v_1 = \pm\frac{\partial}{\partial x_1}$, and the metric h is given by

$$ds_h^2 = dx_1^2 + (\lambda_3(x_3) - \lambda_2(x_2))dx_2^2 + (\lambda_3(x_3) - \lambda_2(x_2))dx_3^2.$$

We see that all components of h are independent of x_1. Thus the vector field v_1 is Killing with respect to h. We see that the form ω is equal to $\pm dx_1$. Then, it is closed.

Suppose the number of different eigenvalues of L at the stable point p is precisely two. Then, $\lambda_2 = \lambda_3 \stackrel{\text{def}}{=} \lambda$ are constants. By Levi-Civita's Theorem, there exist coordinates where g and $\bar{g}$ are given by

$$\begin{aligned} ds_g^2 &= (\lambda - \lambda_1(x_1))\big(dx_1^2 + B(x_2,x_3)(dx_2^2 + dx_3^2)\big) \\ ds_{\bar{g}}^2 &= \left(\frac{1}{\lambda_1(x_1)} - \frac{1}{\lambda}\right)\left(\frac{dx_1^2}{\lambda_1(x_1)} + \frac{B(x_2,x_3)(dx_2^2+dx_3^2)}{\lambda}\right), \end{aligned}$$

Then, $v_1 = \pm\sqrt{\lambda - \lambda_1(x_1)}\frac{\partial}{\partial x_1}$, and the metric h is given by

$$ds_h^2 \;=\; \frac{dx_1^2}{\lambda_1(x_1) - \lambda} + B(x_2,x_3)(dx_2^2 + dx_3^2).$$

After the substitution $\tilde{x}_1 = \int \frac{dx_1}{\sqrt{\lambda_1(x_1)-\lambda}}$, all components of the metric h are independent of $\tilde{x}_1$. Thus v_1 is a Killing vector field. The form ω is equal to $d\tilde{x}_1$ and is evidently closed.

Now we are able to prove that M^3 is a Seifert manifold with zero Euler number. Let us show that we always can find a Killing (with respect to h) vector field v_1^{new} such that its integral curves are closed.

Consider the group $\mathfrak{G}$ of all isometrics of (M^3, h). It is known (see, for example, [13], Chapter 6, Paragraph 3) that $\mathfrak{G}$ is a compact Lie group. The flow F of the vector field v_1 is a one-parametric subgroup of $\mathfrak{G}$. Its closure is a compact Lie subgroup of $\mathfrak{G}$. It is evidently commutative. Then it is isomorphic (as a Lie group) to the torus. We can clearly choose a one-dimensional subgroup F^{new} of the torus in such a way that it is closed and its generator in the Lie algebra is close enough to the generator of F.

Denote by v_1^{new} the corresponding vector field on the manifold. More precisely, let us parameterize the subgroup F^{new} by $t \in (R \bmod 2\pi)$ such that

$$F_{t_1}^{new} \cdot F_{t_2}^{new} = F_{t_1+t_2}^{new}.$$

By definition, let us put the vector field v_1^{new} (at $x \in M^3$) equal to $\frac{d}{dt}\big(F_t^{new}(x)\big)$. If the generator of the Lie algebra of the subgroup F^{new} is close enough to the generator of the algebra of F, the scalar product $h(v_1^{new}, v_1)$ is never zero. In particular, v_1^{new} never vanishes.

By construction, the vector field is Killing and its integral curves are closed. Hence, the integral curves of v_1^{new} give us a foliation of M^3; every leaf of the foliation is homeomorphic to the circle. Thus M^3 is Seifert.

Finally, since by the construction the integral $\int_C \omega$ along every integral curve C of v_1^{new} is not zero, then the integral curve C represents an element of infinite order of $H_1(M^3, Z)$. Then, the Euler number of M^3 is zero [29, 30]. Theorem 1 is proved under assumptions of Case 4.

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