Projectively equivalent metrics on the torus

Vladimir S. Matveev*

Keywords: Projectively Equivalent Metrics, Geodesically Equivalent Metrics, Integrable Systems, Levi-Civita Coordinates, Quantum Integrability, Separation of Variables

MSC2000: 37J35, 53A20, 70H06, 53D25, 53A07, 53B10

Abstract

Let Riemannian metrics g and $\bar{g}$ on a closed connected manifold M^n have the same geodesics, and suppose the eigenvalues of one metric with respect to the other are different at least at one point. We show that then the first Betti number $b_1(M^n)$ is not greater than n, and that if there exists a point where the eigenvalues of one metric with respect to the other are not all different, then the first Betti number $b_1(M^n)$ is less than n. In particular, if M^n is covered by the torus T^n, then the eigenvalues of one metric with respect to the other are different at each point. This allows us to classify such metrics on the torus and to separate variables in the equation on the eigenvalues of the Laplacian of g.

1 Introduction

1.1 Metrics with the same geodesics

Definition 1. Two metrics g and $\bar{g}$ on M^n are called *projectively equivalent*, if they have the same geodesics considered as unparameterized curves. The metrics g and $\bar{g}$ are said to be *strictly non-proportional* at $x \in M^n$, if the eigenvalues of g with respect to $\bar{g}$ are all different at x.

Projectively equivalent metrics is a very classical material. In 1865, Beltrami [1] found the first examples of projectively equivalent metrics and formulated a problem of finding all pairs of projectively equivalent metrics. Locally, near the points where the eigenvalues of one metric with respect to the other do not bifurcate, this problem has been solved by Dini [3] for surfaces and Levi-Civita [6] for manifolds of arbitrary dimension. Later, projectively equivalent metrics were considered by Weyl, Eisenhart, E. Cartan, Thomas, Lichnerowicz, Venzi, Voss, Pogorelov, Mikes, Aminova, Sinjukov, Solodovnikov. They found a lot

*Mathematisches Institut, Universität Freiburg, 79104 Germany
matveev@arcade.mathematik.uni-freiburg.de

of beautiful tensor properties of projectively equivalent metric, see the review paper [16] for details.

However, the global behavior of projectively equivalent metrics is not understood completely. Most known global results on projectively equivalent metrics require additional strong geometrical assumptions. For example, for Einstein or (hyper)Kahlerian metrics beautiful results were obtained by Lichnerowicz [7], Venzi [19], Mikes [16] and Hasegawa and Fujimura [4].

1.2 Results

Theorem 1. *Suppose M^n is a connected closed manifold. Let Riemannian metrics g and $\bar{g}$ on M^n be projectively equivalent and strictly non-proportional at least at one point. Then, the following holds:*

1. *The first Betti number $b_1(M^n)$ is not greater than n.*
2. *The fundamental group of M^n is virtually commutative (i.e. it has a commutative subgroup of finite index)*
3. *If in addition there exists a point where the metrics are not strictly non-proportional, then the first Betti number $b_1(M^n)$ is less than n.*

A weaker version of Theorem 1 was announced in [13].

The first Betti number $b_1(T^n)$ of the $n-$torus is precisely n.

Corollary 1. *Let Riemannian metrics g and $\bar{g}$ on T^n be projectively equivalent and strictly non-proportional at least at one point. Then, they are strictly non-proportional at every point.*

As it has been shown in [14], the converse of Corollary 1 is also true:

Theorem 2 ([14]). *Let M^n be closed connected. Let g, $\bar{g}$ on M^n be projectively equivalent. Suppose they are strictly non-proportional at every point of the manifold. Then, the manifold can be covered by the torus T^n.*

In Section 5 we use Corollary 1 to to describe (Theorem 7) and, in a certain sense, to classify (Theorem 8) all projectively equivalent Riemannian metrics on the torus, which are strictly non-proportional at least at one point. It is the first classification result on projectively equivalent metrics on closed $n-$dimensional manifold. Recall that, for surfaces, in view of two-dimensional version of Theorem 3 proved in [10], the classification of projectively equivalent metrics follows immediately from the classification of quadratically-integrable geodesic flows obtained in [2, 5].

In Section 6, we will use the description from Theorem 7 to show that the variables in the equations $\Delta_g(\phi) = \mu\phi$ on the eigenfunctions of the Laplacian Δ_g of the metric g can be separated: the equation splits into maximum n linear ordinary differential equations on the circle.

In Section 7, we will generalize Corollary 1 for the three-dimensional case: we will show that the number of different eigenvalues of one metric with respect to the other is constant for projectively equivalent metrics on the three-torus, see Theorem 10.

1.3 Methods and ideas of proofs

The new technique that allows us to prove Theorem 1 came from theory of integrable geodesic flows. The connection between projectively equivalent metrics and integrable geodesic flows is established by the following theorem.

Consider the (1,1)-tensor field L given by the formula

$$L^i_j \stackrel{\mathrm{def}}{=} \left(\frac{\det(\bar g)}{\det(g)}\right)^{\frac{1}{n+1}} \bar g^{i\alpha} g_{\alpha j} \tag{1}$$

Theorem 3 ([8, 18]). *Let Riemannian metrics g, $\bar g$ be projectively equivalent. For each $t \in \mathbb{R}$, consider the $(1,1)$- tensor field*

$$S_t \stackrel{\mathrm{def}}{=} \det(L - t\ \mathrm{Id})\,(L - t\ \mathrm{Id})^{-1}. \tag{2}$$

Let us identify the tangent and cotangent bundles of M^n by g. Consider the standard Poisson structure on T^*M^n. Then, for every t_1, t_2, the functions

$$I_{t_i} : TM^n \to \mathbb{R}, \quad I_{t_i}(\xi) \stackrel{\mathrm{def}}{=} g(S_{t_i}(\xi), \xi) \tag{3}$$

are commuting integrals for the geodesic flow of g.

Remark 1. *Although $(L - t\ \mathrm{Id})^{-1}$ is not defined for t lying in the spectrum of L, the tensor field S_t, and, therefore, the function I_t, is well-defined for all t. Moreover, as it will be clear from Section 2, S_t is a polynomial (in t) of degree $n-1$ with coefficients being (1,1)-tensor fields.*

In Section 2, we will show (Corollary 2) that if the metrics are strictly non-proportional at one point of a connected complete manifold, then it is so at almost every point.

If the metric are real-analytic, the first two statements of Theorem 1 already follows from [17]:

Theorem 4 (Taimanov, [17]). *If a real-analytic closed manifold M^n with a real-analytic metric satisfies at least one of the conditions:*

a) $\pi_1(M^n)$ is not virtually commutative

b) $b_1(M^n) > n$,

then the geodesic flow on M^n is not analytically integrable.

The idea using in the proof of the second statement of Theorem 1 is borrowed from [17].

We will show that each element of the fundamental group $\pi_1(M^n)$ can be realized on one of a finite number of subsets of M^n; each of these subsets has virtually commutative fundamental group and the first Betti number less than n; then, the fundamental group of M^n is virtually commutative, and the first homology group $H_1(M^n;\mathbb{Z})$ is the unity of a finite number of commutative subgroups of rank less than n; then, the rank of $H_1(M^n;\mathbb{Z})$ (which is precisely the first Betti number of M^n) must be less than n.

The subsets are given in the terms of the eigenvalues of the tensor (1); in Section 2 we show that they are globally ordered (Theorem 5) which, together with classical Levi-Civita's theorem (Theorem 6), guarantees that the subsets are well-definite.

1.4 Acknowledgements

The author is grateful to V. Bangert, A. Bolsinov and P. Topalov for useful discussions and to I. Taimanov for explaining his theory of geodesic simplicity. The author thanks the European Post-Doctoral Institute for a partial financial support and the University of Warwick and the Isaac Newton Institute for Mathematical Sciences for their hospitality. My research at INIMS has been supported by EPSRC grant GRK99015.

2 The eigenvalues of L

Let g and $\bar{g}$ be projectively equivalent Riemannian metrics on complete (with respect to g) connected M^n. Consider the tensor L given by (1). The main goal of this section is to prove the following theorem: for each point $x \in M^n$, denote by

$$\lambda_1(x) \leq \lambda_2(x) \leq ... \leq \lambda_n(x)$$

the eigenvalues of L at the point x.

Theorem 5. *For every $i \in \{1, ..., n-1\}$, for every $x, y \in M^n$:*

1. $\lambda_i(x) \leq \lambda_{i+1}(y)$.
2. *If $\lambda_i(x) < \lambda_{i+1}(x)$, then $\lambda_i(z) < \lambda_{i+1}(z)$ for almost every point $z \in M^n$.*

This theorem has been announced in [14]; for three-dimensional manifolds, the theorem has been proven in [15].

Corollary 2. *If the eigenvalues of L are all different at one point of M^n, then they are all different at almost each point of M^n.*

Corollary 2 was announced in [9] and proved by a different method in [11].
Proof of Theorem 5: By definition, the tensor L is self-adjoint. Then, for every $x \in M^n$, there exists a basis in T_xM^n such that the metric g is given by the matrix $\mathrm{diag}(1, 1, ..., 1)$ and the tensor L is given by the matrix $\mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$. Then, the tensor (2) reads:

$$\begin{aligned} S_t &= \det(L - t\mathrm{Id})(L - t\mathrm{Id})^{(-1)} \\ &= \mathrm{diag}(P_1(t), P_2(t), ..., P_n(t)), \end{aligned}$$

where the polynomials $P_i(t)$ are given by the formula

$$P_i(t) \stackrel{\text{def}}{=} (\lambda_1 - t)(\lambda_2 - t)...(\lambda_{i-1} - t)(\lambda_{i+1} - t)...(\lambda_{n-1} - t)(\lambda_n - t).$$

Then, for every fixed $\xi = (\xi_1, \xi_2, ..., \xi_n) \in T_x M^n$, the function (3) is the following polynomial in t:

$$I_t = \xi_1^2 P_1(t) + \xi_2^2 P_2(t) + ... + \xi_n^2 P_n(t). \tag{4}$$

Consider the roots of this polynomial. From the proof of Lemma 1, it will be clear that they are real. We denote them by

$$t_1(x, \xi) \leq t_2(x, \xi) \leq ... \leq t_{n-1}(x, \xi).$$

Lemma 1. *The following holds:*

1. *For every $\xi \in T_x M^n$,*

$$\lambda_i(x) \leq t_i(x, \xi) \leq \lambda_{i+1}(x).$$

   In particular, if $\lambda_i(x) = \lambda_{i+1}(x)$, then $t_i(x, \xi) = \lambda_i(x) = \lambda_{i+1}(x)$.

2. *If $\lambda_i(x) < \lambda_{i+1}(x)$, then for each constant τ the Lebesgue measure of the set*

$$V_\tau \subset T_x M^n, \quad V_\tau \stackrel{\text{def}}{=} \{\xi \in T_x M^n : t_i(x, \xi) = \tau\},$$

   is zero.

Proof of Lemma 1: Evidently, the coefficients of the polynomial I_t depend continuously on the eigenvalues λ_i and on the components ξ_i. Then, it is sufficient to prove the first statement of the lemma assuming that the eigenvalues λ_i are all different and that ξ_i are non-zero. For all $\alpha \neq i$, we evidently have $P_\alpha(\lambda_i) \equiv 0$. Then,

$$I_{\lambda_i} = \sum_{\alpha=1}^{n} \xi_\alpha^2 P_\alpha(\lambda_i) = \xi_i^2 P_i(\lambda_i).$$

Hence I_{λ_i} and $I_{\lambda_{i+1}}$ have different signs and, therefore, the open interval $]\lambda_i, \lambda_{i+1}[$ contains a root of the polynomial I_t. The degree of the polynomial I_t equals $n-1$; we have $n-1$ disjoint intervals; each of these intervals contains at least one root so that all roots are real and the ith root lies between λ_i and λ_{i+1}. The first statement of the lemma is proved.

Let us prove the second statement of Lemma 1. Suppose $\lambda_i < \lambda_{i+1}$. Let first $\lambda_i < \tau < \lambda_{i+1}$. Then, the set

$$V_\tau \stackrel{\text{def}}{=} \{\xi \in T_x M^n : t_i(x, \xi) = \tau\},$$

consists of the points ξ where the function $I_\tau(x, \xi) \stackrel{\text{def}}{=} (I_t(x, \xi))_{|t=\tau}$ is zero; then, it is a quadric in $T_x M^n \equiv \mathbb{R}^n$ and its measure is zero.

Let τ be one of the endpoints of the interval $[\lambda_i, \lambda_{i+1}]$. Without loss of generality, we can suppose $\tau = \lambda_i$. Let k be the multiplicity of the eigenvalue λ_i.

Then, every coefficient $P_\alpha(t)$ of the quadratic form (4) has a factor $(\lambda_i - t)^{k-1}$. Therefore, for every fixed $\xi \in T_x M^n$, the function

$$\tilde{I}_t \stackrel{\text{def}}{=} \frac{I_t}{(\lambda_i - t)^{k-1}}$$

is well-definite and is a polynomial in t so that $\tilde{I}_\tau$ is a nontrivial quadratic form. Evidently, for each point $\xi \in V_\tau$, we have $\tilde{I}_\tau(\xi) = 0$ so that the set V_τ is a subset of a quadric in $T_x M^n$ and its measure is zero. **Lemma 1 is proved.**

The first statement of Theorem 5 follows immediately from the first statement of Lemma 1: let us join the points $x, y \in M^n$ by a geodesic

$$\gamma : \mathbb{R} \to M^n, \quad \gamma(0) = x, \ \gamma(1) = y.$$

Consider the one-parametric family of integrals $I_t(x, \xi)$ and the roots

$$t_1(x, \xi) \le t_2(x, \xi) \le ... \le t_{n-1}(x, \xi).$$

By Corollary 3, each root t_i is constant on each orbit $(\gamma, \dot{\gamma})$ of the geodesic flow of g so that

$$t_i(\gamma(0), \dot{\gamma}(0)) = t_i(\gamma(1), \dot{\gamma}(1)).$$

Using Lemma 1, we obtain

$$\lambda_i(\gamma(0)) \le t_i(\gamma(0), \dot{\gamma}(0)) \quad \text{and} \quad t_i(\gamma(1), \dot{\gamma}(1)) \le \lambda_{i+1}(\gamma(1)).$$

Therefore, $\lambda_i(\gamma(0)) \le \lambda_{i+1}(\gamma(1))$. **The first statement of Theorem 5 is proved.**

Let us prove the second statement of Theorem 5. Suppose $\lambda_i(x) < \lambda_{i+1}(x)$. Suppose $\lambda_i(y) = \lambda_{i+1}(y)$ for every point y of some subset V. Then, by the first statement of Theorem 5, the value of λ_i is a constant (independent of $y \in V$). Denote this constant by C. Let us prove that $\lambda_i(x) = \lambda_{i+1}(x) = C$. Let us join the point x with every point of V by all possible geodesics. Consider the set $V_C \subset T_x M^n$ of the initial velocity vectors (at the point x) of these geodesics.

By the first statement of Lemma 1, for each geodesic γ_1 passing through every point of V, the value $t_i(\gamma_1, \dot{\gamma}_1)$ is equal to C. Then, by the second statement of Lemma 1, the measure of the set V_C is zero and, therefore, the measure of the set V is also zero. **Theorem 5 is proved.**

3 Levi-Civita's theorem and projectively equivalent metrics that are strictly non-proportional at each point

A local description of projectively equivalent Riemannian metrics near the points where the eigenvalues of the tensor L given by (1) do not bifurcate has been obtained by Levi-Civita [6]. Here we formulate Levi-Civita's theorem assuming that the eigenvalues of L at the point are different; then, they automatically do not bifurcate in a neighborhood of the point.

Theorem 6 (Levi-Civita [6]). *Consider two Riemannian metrics on an open subset $U^n \subset M^n$. Consider the tensor L given by (1). Suppose the eigenvalues of L are all different at each point $x \in U^n$.*

Then, the metrics are projectively equivalent on U^n if and only if for each point $x \in U^n$ there exist coordinates $x_1, x_2, ..., x_n$ in some neighborhood of the point x such that in these coordinates the metrics have the following model form:

$$\begin{aligned} ds^2_{g_{model}} &= \Pi_1 d(x_1)^2 + \Pi_2 d(x_2)^2 + \cdots + \Pi_n d(x_n)^2, & (5)\\ ds^2_{\bar g_{model}} &= \rho_1\Pi_1 d(x_1)^2 + \rho_2\Pi_2 d(x_2)^2 + \cdots + \rho_n\Pi_n d(x_n)^2, & (6)\end{aligned}$$

where the functions Π_i and ρ_i are given by

$$\begin{aligned} \Pi_i &\stackrel{\text{def}}{=} (\lambda_i - \lambda_1)(\lambda_i - \lambda_2)\cdots(\lambda_i - \lambda_{i-1})(\lambda_{i+1} - \lambda_i)\cdots(\lambda_n - \lambda_i), & (7)\\ \rho_i &\stackrel{\text{def}}{=} \frac{1}{\lambda_1\lambda_2...\lambda_{n-1}}\frac{1}{\lambda_i}.\end{aligned}$$

where, for each i, the function λ_i is a smooth function of the variable x_i.

The notations of the eigenvalues of (1) are compatible with the notations in formulae (5,6,7): in the coordinates $x_1, ..., x_n$, the tensor (1) is given by the diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$.

Proof of Theorem 2. We assume that M^n is closed and connected, that Riemannian metrics g and $\bar g$ on M^n are projectively equivalent and that they are strictly non-proportional at each point of M^n. Then, the eigenvalues $\lambda_1 < \lambda_2 < ... < \lambda_n$ of the tensor L given by (1) are all different at every point of M^n. Hence, they are everywhere defined smooth functions on M^n. Therefore, the functions Π_i given by (7) are also smooth on M^n. For each i, at each point of M^n, consider the vector v_i satisfying the conditions

$$\left\{\begin{array}{rcl} Lv_i &=& \lambda_i v_i\\ g(v_i, v_i) &=& \Pi_i.\end{array}\right. \qquad (8)$$

The only freedom we have is the sign of the vector. Then, we can globally define the vector fields v_i, $i = 1, 2, ..., n$ satisfying (8) on some finite cover $\tilde M^n$ of M^n. In Levi-Civita's coordinates $x_1, ..., x_n$ from Theorem 6, the vector fields v_i are equal to $\pm\frac{\partial}{\partial x_i}$; then, they commute; by definition they never vanish. Then, $\tilde M^n$ must be homeomorphic to the torus T^n. **Theorem 2 is proved.**

4 Proof of Theorem 1

We assume that M^n is closed connected, that Riemannian metrics g and $\bar g$ on M^n are projectively equivalent and that they are strictly non-proportional at least at one point.

If the metrics are strictly non-proportional at each point, then by Theorem 2 our manifold M^n is covered by the $n-$torus, whose fundamental group is commutative, and whose first Betti number is precisely n. Therefore, we need to prove the second and the third statements of Theorem 1 under assumption that there exists a point where the metrics are not strictly non-proportional.

Consider the tensor L given by (1) and its eigenvalues

$$\lambda_1(x) \le \lambda_2(x) \le ... \le \lambda_n(x).$$

By Theorem 5, there exist numbers $\tau_1, \tau_2, ..., \tau_{n-1} \in \mathbb{R}$ such that, for all $i \in \{1, 2, ..., n-1\}$ and for every point x of the manifold,

$$\lambda_i(x) \le \tau_i \le \lambda_{i+1}(x).$$

For all $1 \le i \le n-1$, consider the following subsets of M^n:

$$V_i^- \stackrel{\mathrm{def}}{=} \{x \in M^n : \quad \lambda_i(x) < \tau_i\};$$

$$V_i^+ \stackrel{\mathrm{def}}{=} \{x \in M^n : \quad \lambda_{i+1}(x) > \tau_i\}.$$

Some of the sets V_i^+, V_i^- can be empty. For example, if $\lambda_1(x)$ is constant and $\tau_1 = \lambda_1$, then the set V_1^- is empty. By definition, the eigenvalue λ_i is a globally defined continuous function on M^n. Therefore, the sets V_i^+ and V_i^- are open. Below $V_i^\pm$ will denote either V_i^+ or V_i^-. Consider the sets

$$V_1^\pm \cap V_2^\pm \cap ... \cap V_{n-1}^\pm.$$

There are finitely many (not greater than 2^{n-1}) of such sets; each of them is open.

Let us take a point $x \in M^n$ where

$$\lambda_1(x) < \lambda_2(x) < ... < \lambda_n(x).$$

For each set $V_1^\pm \cap V_2^\pm \cap ... \cap V_{n-1}^\pm$ containing the point x, we denote by

$$(V_1^\pm \cap V_2^\pm \cap ... \cap V_{n-1}^\pm)_x$$

its connected component containing the point x. Let us show that the first Betti number of each of the sets $(V_1^\pm \cap V_2^\pm \cap ... \cap V_{n-1}^\pm)_x$ is less than n, and that its fundamental group $\pi_1((V_1^\pm \cap V_2^\pm \cap ... \cap V_{n-1}^\pm)_x)$ is virtually commutative.

Let us fix one of the sets $(V_1^\pm \cap V_2^\pm \cap ... \cap V_{n-1}^\pm)_x$ and denote it by V; V is not empty. At each point of V, the eigenvalues of L are all different. Then, they are smooth functions on V. Therefore, the functions Π_i given by (7) are also smooth on V.

For all i, at each point of V, consider the vector v_i satisfying conditions (8). The only freedom we have is the sign of the vector. Then, we can globally define the vector fields v_i, $i = 1, 2, ..., n$, satisfying conditions (8) on the universal cover $\tilde{V}$ of V.

We see that in Levi-Civita's coordinates $x_1, x_2, ..., x_n$ from Theorem 6, the vector fields v_i are equal to $\pm\frac{\partial}{\partial x_i}$. Then, the vector fields v_i commute and for all $j \ne i$ the eigenvalue λ_j is constant on the integral curves of v_i.

We can globally define Levi-Civita's coordinates on the universal cover $\tilde{V}$: Choose an origin P_0. Join each point P with P_0 by a curve. Then, the coordinate x_i is equal to the action of the vector field v_i along the curve. Evidently, the definition is independent on the curve.

By definition of the set V, if the universal cover $\tilde{V}$ contains the points with the coordinates $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$ and $(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$, then it contains the whole parallelepiped

$$\{(x_1, ..., x_n) : \ \min(\bar{x}_i, \hat{x}_i) \leq x_i \leq \max(\bar{x}_i, \hat{x}_i), \quad i = 1, 2, .., n\}.$$

Then, the coordinates uniquely define the point, and the universal cover $\tilde{V}$ is homeomorphic to the band

$$\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : \ \alpha_1 < x_1 < \beta_1, \alpha_2 < x_2 < \beta_2, ..., \alpha_n < x_n < \beta_n\},$$

where $\alpha_i, \beta_i \in \mathbb{R}_\infty$ (so that they can be either real numbers or $\pm\infty$.).

The fundamental group $\pi_1(V)$ naturally acts on $\tilde{V}$. The action preserves the metric g and the tensor L. Therefore, if γ is an element of the fundamental group, then for each vector field v_i either $\gamma(v_i) = v_i$ or $\gamma(v_i) = -v_i$. Consider the subgroup $H \subset \pi_1(V)$ of the elements that preserve the directions of the vector fields. The subgroup H has finite (at most 2^n) index in the fundamental group $\pi_1(V)$.

Evidently, in coordinates $x_1, x_2, ..., x_n$, the group H acts by parallel translations. Then, the group H is commutative, free and has finite index in $\pi_1(V)$. Our next goal is to show that the rank of H is less than n.

Evidently, if either α_i or β_i is finite, then the group H preserves the coordinate number i. Consider the numbers $i_1, i_2, ..., i_k$ such that $\alpha_{i_j} = -\infty$ and $\beta_{i_j} = \infty$.

The group H is a discrete subgroup of the group $(\mathbb{R}^k, +)$; then, its rank can be maximum k. If $k < n$, then the Betti number of V is not greater than k and is automatically less than n. Let k be equal to n so that the group $(\mathbb{R}^n, +)$ freely acts on $\tilde{V}$ by parallel translations. Then, the the group H is a discrete subgroup of the group $(\mathbb{R}^n, +)$. If the rank of H is n, then the quotient space $\mathbb{R}^n/_H$ is homeomorphic to the torus and, therefore, is compact; this quotient space naturally covers V so that V is also compact. Since V is open by definition, it coincides with the whole manifold M^n so that at each point of M^n the eigenvalues of L are all different, which contradicts the assumptions.

Thus, $\pi_1((V_1^\pm \cap V_2^\pm \cap ... \cap V_{n-1}^\pm)_x)$ has a subgroup of finite index isomorph to $\mathbb{Z}^k$, where $k < n$.

Let us now show that each element of the fundamental group $\pi_1(M^n)$ can be realized on one of the sets $(V_1^\pm \cap V_2^\pm \cap ... \cap V_{n-1}^\pm)_x$. By Hopf-Rinow theorem, every element of the fundamental group can be realized by a geodesic loop γ, $\gamma(0) = \gamma(1) = x$. We consider this geodesic loop as a curve $(\gamma, \dot{\gamma})$ on TM^n. The values of the roots t_i of the polynomial $I_t(\gamma, \dot{\gamma})$ are constant on the curve $(\gamma, \dot{\gamma})$. If $t_i(\gamma, \dot{\gamma}) \neq \tau_i$ for each i, then, by Lemma 1, the geodesic loop γ already lies in one of the sets $(V_1^\pm \cap V_2^\pm \cap ... \cap V_{n-1}^\pm)_x$. Suppose $t_i(\gamma, \dot{\gamma}) = \tau_i$ for some numbers i. Let us slightly perturb the initial velocity vector $\dot{\gamma}(0)$ and consider a geodesic γ_ϵ such that $\gamma_\epsilon(0)=\gamma(0)$, $|\dot\gamma_\epsilon(0)-\dot\gamma(0)|=\epsilon<<1$ and $t_i(\gamma_\epsilon,\dot\gamma_\epsilon)\neq\tau_i$ for all i. The geodesic γ lies then in one of the sets $(V_1^\pm\cap V_2^\pm\cap...\cap V_{n-1}^\pm)_x$. If ϵ is small, then the geodesic segment $\gamma_\epsilon(T)$, $0\le T\le 1$, lies in a thin regular neighborhood of the geodesic loop γ and the point $\gamma_\epsilon(1)$ lies in a small disk neighborhood of x and we can connect the points $\gamma_\epsilon(1)$ and x by a segment in this disk. Then, the curve which is made from this segment and from the geodesic segment $\gamma_\epsilon(T)$, $0\le T\le 1$, represents the same homotopy class as the geodesic loop γ so that each element of the fundamental group can be realized on one of the sets $(V_1^\pm\cap V_2^\pm\cap...\cap V_{n-1}^\pm)_x$.

Finally, the fundamental group $\pi_1(M^n)$ is a unity of finite number of virtually commutative subgroups. Then, one of these subgroups has a finite index in $\pi_1(M^n)$, see Lemma 4 in [17]. Thus, $\pi_1(M^n)$ is virtually commutative.

The image under the natural homomorphism $\pi_1(M^n)\to H_1(M^n;\mathbb{Z})$ of this subgroup must have a finite index in $H_1(M^n;\mathbb{Z})$. Since the rank k of the subgroup is less than n, the rank of $H_1(M^n;\mathbb{Z})$ (which is precisely $b_1(M_n)$) is less than n as well. **Theorem 1 is proved.**

5 Projectively equivalent metrics on the torus

Consider $\mathbb{R}^n$ with the standard coordinates $x_1,x_2,...,x_n$. Consider an $n-$lattice G on $\mathbb{R}^n$. By an $n-$lattice we mean the set of the vectors $k_1v_1+k_2v_2+...+k_nv_n$, where $v_1,...,v_n\in\mathbb{R}^n$ are linearly independent vectors and $k_1,k_2,...,k_n\in\mathbb{Z}$.

Let λ_i, $i=1,...,n$, be smooth functions on $\mathbb{R}^n$ satisfying the following three conditions:

(i) For every i, the function λ_i depends on the the variable x_i only.

(ii) $0<\lambda_i(x)<\lambda_j(y)$ for all $i<j$ and for all x,y.

(iii) The functions λ_i are invariant modulo the lattice so that for every vector $v=(v^1,v^2,...,v^n)\in G$, and for all $x_i\in\mathbb{R}$

$$\lambda_i(x_i+v^i)=\lambda(x_i).$$

Consider the Riemannian metrics g_{model} and $\bar g_{model}$ on $\mathbb{R}^n$ given by the formulae (5,6). By Levi-Civita's theorem, the metrics are projectively equivalent; by definitions, they are invariant modulo the lattice so they generate two projectively equivalent metrics on the torus $\mathbb{R}^n/_G$. We will call such metrics *model* metrics corresponding to the lattice G and to the functions λ_i.

Theorem 7. *Let Riemannian metrics g and $\bar g$ on the torus T^n be projectively equivalent and strictly non-proportional at least at one point. Then, there exist an $n-$lattice G, functions $\lambda_1,...,\lambda_n$ satisfying conditions (i-iii) and a diffeomorphism $\phi:T^n\to\mathbb{R}^n/_G$ such that $g=\phi^*g_{model}$, $\bar g=\phi^*\bar g_{model}$, where g_{model} and $\bar g_{model}$ are model metrics with respect to the lattice G and the functions λ_i.*

Thus, every pair of projectively equivalent Riemannian metrics on the $n-$torus which are strictly non-proportional at least at one point is given by an $n-$lattice G and by functions λ_i satisfying conditions (i-iii). The following theorem answers when two of such sets of data define the same pair of metrics:

Theorem 8. *Let G^1 and G^2 be two $n-$lattices on $\mathbb{R}^n$. Suppose the functions $\lambda_1^1, ..., \lambda_n^1$ satisfy conditions (i-iii) with respect to the lattice G^1; suppose the functions $\lambda_1^2, ..., \lambda_n^2$ satisfy conditions (i-iii) with respect to the lattice G^2. Consider the pairs of model projectively-equivalent metrics $g^1_{model}, \bar g^1_{model}$ (which are model metrics with respect to the lattice G^1 and the functions λ_i^1) and $g^2_{model}, \bar g^2_{model}$ (which are model metrics with respect to the lattice G^2 and the functions λ_i^2).*

Then, there exists a diffeomorphism $\phi : \mathbb{R}^n/_{G^1} \to \mathbb{R}^n/_{G^2}$ such that $g^1_{model} = \phi^* g^2_{model}$ and $\bar g^1_{model} = \phi^* \bar g^2_{model}$, if and only if there exists $\alpha_1, ..., \alpha_n \in \mathbb{R}$ and $\epsilon_1, ..., \epsilon_n \in \{+1, -1\}$ such that the coordinate change

$$x_i \mapsto \epsilon_i x_i + \alpha_i, \quad i = 1, 2, ..., n,$$

takes the lattice G^1 to the lattice G^2 and the functions λ_i^1 to the functions λ_i^2.

Proof of Theorems 7,8: Let g and $\bar g$ be projectively equivalent Riemannian metrics on the torus T^n. Suppose they are strictly non-proportional at least at one point of the torus. Then, by Corollary 1, they are strictly non-proportional at each point of the torus. As in Section 3, for every i, at each point of the torus, consider a vector v_i satisfying conditions (8). The only freedom we have is the sign of the vector. Then, on the universal cover $\mathbb{R}^n$, we can globally define vector fields v_i satisfying (8). As in Section 4, consider the coordinate system on $\mathbb{R}^n$ defined as follows: choose an origin P_0. Join every point P with P_0 by a curve. Then, the coordinate x_i is equal to the action of the vector field v_i along the curve. Evidently, the definition is independent of the curve, and the coordinate system is isomorph to the standard one on $\mathbb{R}^n$.

The fundamental group $\pi_1(T^n)$ acts on the universal cover $\mathbb{R}^n$. The action preserves the metrics $g, \bar g$; therefore, it preserves the coordinate net. Then, the group $\pi_1(T^n)$ acts by translations and compositions of translations and reflections. Since the fundamental group $\pi_1(T^n)$ is commutative and isomorphic to $\mathbb{Z}^n$, compositions of translations and reflections can not occur so that the fundamental group acts by translations. Since the action is co-compact, free and is a discrete subgroup of the group of all translations, the vectors of such translations form an $n-$lattice G. Since the action preserves the metrics, it must preserve the eigenvalues of the tensor L given by 1. Then, the functions $\lambda_1 < ... < \lambda_n$ are invariant modulo the lattice G. By Levi-Civita's theorem, for each i the function λ_i depends on the variable x_i only. The only freedom we have is in choosing the origin P_0 of the coordinate system, which gives us the translation $x_i \mapsto x_i + \alpha_i$, and the freedom in choosing the directions of the vectors v_i, which gives us the changes of the signs of the coordinates. Thus, every pair of projectively equivalent strictly non-proportional Riemannian metrics on the torus is model, and two pairs of model metrics are the same if their data satisfy the conditions in Theorem 8. **Theorems 7, 8 are proved.**

6 Complete separation of variables

The goal of this Section is to separate the variables in the equation on the eigenfunctions of the Laplacian for the model metrics on the torus.

We will use the following "quantum" version of Theorem 3 announced in [9] and proved in [11, 12]:

Theorem 9 ([9, 11, 12]). *Let Riemannian metrics g and $\bar{g}$ be projectively equivalent. Consider the tensors L and S_t given by the formulae (1), (2). Then, for all $t_1, t_2 \in \mathbb{R}$, the operators $\mathcal{I}_{t_i} : C^2(M^n) \to C^0(M^n)$,*

$$\mathcal{I}_{t_i}(f) \stackrel{\text{def}}{=} \operatorname{div}(S_{t_i}(\operatorname{grad}(f))) \tag{9}$$

commute and commute with the Laplacian of the metric g.

Consider $\mathbb{R}^n$ with the standard coordinates $x_1, ..., x_n$. Let G be an $n-$lattice; let functions $\lambda_1, ..., \lambda_n$ satisfy conditions (i-iii). Consider the model metrics (5,6) on the torus $\mathbb{R}^n/_G$.

By Theorem 9, for all $t_1, t_2 \in \mathbb{R}$, the operators $\mathcal{I}_{t_i}$ given by (9) commute and commute with the Laplacian Δ of the metric g_{model}. Actually, the condition that the operators commute already implies that they commute with the Laplacian of g_{model}. Indeed, it is convenient to consider the family of operators $\mathcal{I}_t$ as a polynomial (in t) of degree $n-1$ with coefficients being second-order differential operators; the leading coefficient is precisely the Laplacian of g_{model}.

As it has been shown in [11], since $\mathbb{R}^n/_G$ is closed, the operators are self-adjoint with respect to the standard scalar product on $L^2(\mathbb{R}^n/_G)$. Since they commute, there exists a countable basis

$$\Phi = \{\phi_0, \phi_1, \phi_2, ...,\}$$

of the space $L^2(\mathbb{R}^n/_G)$ such that each of the functions ϕ_m is an eigenfunction of all operators $\mathcal{I}_t$. That is, each function $\phi \in \Phi$ satisfies the equation of the form:

$$\mathcal{I}_t(\phi) = F(t)\phi, \tag{10}$$

where $F(t)$ is a polynomial in t of degree not greater than $n-1$. The leading coefficient of the polynomial is precisely the eigenvalue of the Laplacian corresponding to the eigenfunction ϕ.

We will first investigate this system on the universal cover $\mathbb{R}^n$.

Every polynomial of degree less than n is uniquely defined by its values at n different points; for each point $x \in \mathbb{R}^n$, as these n different points we take $\lambda_1(x), ..., \lambda_n(x)$. As has been shown in [11] (and can be verified by direct calculations), in the coordinates $x_1, ..., x_n$ the operators $\mathcal{I}_{\lambda_i}$ are given by $\frac{\partial^2}{\partial {x_i}^2}$. Then, (10) is equivalent to the following system of equations:

$$\left\{\begin{array}{rcl} \frac{\partial^2}{\partial {x_1}^2}\phi & = & F(\lambda_1)\phi \\ & \vdots & \\ \frac{\partial^2}{\partial {x_n}^2}\phi & = & F(\lambda_n)\phi. \end{array}\right. \tag{11}$$

Since the function λ_i depends on the variable x_i only, the function $F(\lambda_i)$ depends on the variable x_i only so that the ith equation of (11) depends on the variable x_i only. Then, for a fixed polynomial $F(t)$, every solution of the system (11) is a finite sum of functions of the form $X_1(x_1)X_2(x_2)...X_n(x_n)$, where each function X_i is a solution of the ith equation of (11).

Now let us investigate when a solution of (11) on $\mathbb{R}^n$ generates a solution on the torus.

First of all, if the functions λ_i are not all constant, then the lattice can not be arbitrary:

Consider the numbers $i_1, ..., i_k$ such that the corresponding functions λ_{i_r} are not constant. Later, the notations i_r will be reserved for the non-constant functions λ. Let the lattice G be given by the basic vectors

$$v_1 = (v_1^1, ..., v_1^n), ..., v_n = (v_n^1, ..., v_n^n).$$

Let us show that there exists an $n-$sub-lattice $\tilde{G} \subset G$ whose basis $\tilde{v}_1 = (\tilde{v}_1^1, ..., \tilde{v}_1^n), ..., \tilde{v}_n = (\tilde{v}_n^1, ..., \tilde{v}_n^n)$ satisfies

$$\begin{cases} \tilde{v}_{i_r}^m = P_{i_r} > 0, & \text{if} \quad m = i_r \\ \tilde{v}_{i_r}^m = 0, & \text{if} \quad m \neq i_r \end{cases} \tag{12}$$

for every i_r. Indeed, consider the $(n \times n)$ matrix $\mathfrak{G}$ with entries $\mathfrak{G}_q^p = v_q^p$. In other words, the rows of the matrix $\mathfrak{G}$ are the vectors v_i. By definition, each function λ_{i_r} is periodic (since it is periodic with respect to each $v_m^{i_r}$ and at least one of $v_m^{i_r}$ is not zero) and not constant. Then, it has a minimal period $p_{i_r} > 0$. Then, every period of the function λ_{i_r} is an integer multiple of p_{i_r}. Then, every $v_m^{i_r}$ is an integer multiple of p_{i_r}. Each $n-$sub-lattice of G is generated by its basis which is a system of n linear independent vectors of G. The procedure of choosing such basis is equivalent to the multiplying (from left) the matrix $\mathfrak{G}$ by some integer non-degenerate matrix. It is easy to see that, multiplying by integer elementary matrices, we can organize Gauß diagonalization process for the columns numbers $i_1, ..., i_k$ of the matrix $\mathfrak{G}$ and, therefore, we can make the entries $v_{i_r}^m$ satisfying conditions (12). Thus, there exists a $n-$sub-lattice $\tilde{G}$ of the lattice G given by a basis satisfying (12).

The torus $\mathbb{R}/_{\tilde{G}}$ finitely covers the torus $\mathbb{R}/_G$. We will consider system (11) on the torus $\mathbb{R}/_{\tilde{G}}$. In order to do this, we must consider the solutions of (11) invariant modulo the lattice $\tilde{G}$.

Let $\mathcal{I}_t(\phi) = F(t)\phi$ for a function $\phi : \mathbb{R}^n \to C$ and for a polynomial $F(t)$.

Consider the numbers $j_1, ..., j_{n-k}$ such that the corresponding functions λ_{j_m} are constant. Later, the notation j_m will be reserved for constant functions λ. If ϕ is invariant modulo the lattice $\tilde{G}$, then all $F(\lambda_{j_m})$ are evidently nonpositive. Then, the roots $\sqrt{F(\lambda_{j_m})}$ are pure imaginary or zero. By definition, we will assume $\mathfrak{Im}\sqrt{F(\lambda_{j_m})} \geq 0$.

Every function ϕ satisfying the equations number $j_1, ..., j_{n-k}$ from (11) has the form

$$\sum_{\epsilon_1, ..., \epsilon_{n-k} = \pm 1} C_{\epsilon_1 ... \epsilon_{n-k}}(x_{i_1}, ..., x_{i_k}) \exp\left(\epsilon_1\sqrt{F(\lambda_{j_1})}x_{j_1} + ... + \epsilon_{n-k}\sqrt{F(\lambda_{j_{n-k}})}x_{j_{n-k}}\right), \tag{13}$$

where $C_{\epsilon_1...\epsilon_{n-k}}$ is a complex-valued function of variables $x_{i_1},...,x_{i_k}$. We assume that if $\sqrt{F(\lambda_{j_m})}=0$, then $C_{\epsilon_1...\epsilon_m...\epsilon_{n-k}} = C_{\epsilon_1...-\epsilon_m...\epsilon_{n-k}}$. Then, the function ϕ uniquely defines the decomposition (13). Therefore, ϕ is invariant modulo the vectors $\tilde{v}_{j_1},...,\tilde{v}_{j_{n-k}}$, if and only if each term

$$C_{\epsilon_1...\epsilon_{n-k}}(x_{i_1},...,x_{i_k})\exp\left(\epsilon_1\sqrt{F(\lambda_{j_1})}x_{j_1}+...+\epsilon_{n-k}\sqrt{F(\lambda_{j_{n-k}})}x_{j_{n-k}}\right) \quad (14)$$

is invariant modulo the vectors $\tilde{v}_{j_1},...,\tilde{v}_{j_{n-k}}$; that means that either $C_{\epsilon_1...\epsilon_{n-k}}(x_{i_1},...,x_{i_k}) \equiv 0$ or the vector

$$\frac{1}{2\pi i}\left(\epsilon_1\sqrt{F(\lambda_{j_1})},...,\epsilon_{n-k}\sqrt{F(\lambda_{j_{n-k}})}\right) \in \mathbb{R}^{n-k} \quad (15)$$

lies in the lattice dual to the lattice generated by the vectors

$$(\tilde{v}_{j_1}^{j_1},\tilde{v}_{j_1}^{j_2},...,\tilde{v}_{j_1}^{j_{n-k}}),...,(\tilde{v}_{j_{n-k}}^{j_1},\tilde{v}_{j_{n-k}}^{j_2},...,\tilde{v}_{j_{n-k}}^{j_{n-k}}) \in \mathbb{R}^{n-k}. \quad (16)$$

The function ϕ of the form (13) satisfies the equations number $i_1,...,i_k$ from (10) and is invariant modulo the translations along the vectors $\tilde{v}_{i_1},...,\tilde{v}_{i_{n-k}}$ if and only if each coefficient $C_{\epsilon_1...\epsilon_{n-k}}(x_{i_1},...,x_{i_k})$ satisfies the equations number $i_1,...,i_k$ and is invariant modulo the translations. Then, the coefficient $C_{\epsilon_1...\epsilon_{n-k}}(x_{i_1},...,x_{i_k})$ is a finite sum of functions of the form $X_{i_1}(x_{i_1})X_{i_2}(x_{i_2})...X_{i_k}(x_{i_k})$. Each of the functions $X_{i_r}(x_{i_r})$ is a solution of the equation number i_r from the system (10), and satisfies the following condition reflecting that the term (14) is invariant modulo the translations along $\tilde{v}_{i_r}$:

$$X_{i_r}(x_{i_r}-P_{i_r}) = X_{i_r}(x_{i_r})\exp\left(\epsilon_1\sqrt{F(\lambda_{j_1})}\tilde{v}_{i_r}^{j_1}+...+\epsilon_{n-k}\sqrt{F(\lambda_{j_{n-k}})}\tilde{v}_{i_r}^{j_{n-k}}\right).$$

Then, for every $r=1,...,k$, there exists a function $\tilde{X}_{i_r}(x_{i_r})$, periodic with the period P_{i_r}, such that

$$X_{i_r}(x_{i_r}) = \tilde{X}_{i_r}(x_{i_r})\exp\left(\frac{\epsilon_1\sqrt{F(\lambda_{j_1})}\tilde{v}_{i_r}^{j_1}+...+\epsilon_{n-k}\sqrt{F(\lambda_{j_{n-k}})}\tilde{v}_{i_r}^{j_{n-k}}}{-P_{i_r}}x_{i_r}\right). \quad (17)$$

The function $\tilde{X}_{i_r}$ satisfies the equation

$$\begin{aligned}&\frac{\partial^2}{\partial x_{i_r}^2}\tilde{X}_{i_r}(x_{i_r})\\ &-2\frac{\epsilon_1\sqrt{F(\lambda_{j_1})}\tilde{v}_{i_r}^{j_1}+...+\epsilon_{n-k}\sqrt{F(\lambda_{j_{n-k}})}\tilde{v}_{i_r}^{j_{n-k}}}{P_{i_r}}\frac{\partial}{\partial x_{i_r}}\tilde{X}_{i_r}(x_{i_r})\\ &+\left(\frac{\epsilon_1\sqrt{F(\lambda_{j_1})}\tilde{v}_{i_r}^{j_1}+...+\epsilon_{n-k}\sqrt{F(\lambda_{j_{n-k}})}\tilde{v}_{i_r}^{j_{n-k}}}{P_{i_r}}\right)^2\tilde{X}_{i_r}(x_{i_r}) \quad = F(\lambda_{i_r})\tilde{X}_{i_r}(x_{i_r})\end{aligned} \quad (18)$$

on the circle ($\mathbb{R} \mod P_{i_r}$).

Thus, the partial differential equation $\Delta(\phi)=\mu\phi$ is reduced to the system of equations (18): each solution of the equation $\Delta(\phi)=\mu\phi$ is a finite sum of terms of the form

$$X_1(x_{i_1})...X_k(x_{i_k})\left(\exp\left(\epsilon_1\sqrt{F(\lambda_{j_1})}x_{j_1}+...+\epsilon_{n-k}\sqrt{F(\lambda_{j_{n-k}})}x_{j_{n-k}}\right)\right),$$

such that the vector (15) made from the entries $\epsilon_{n-k}\sqrt{F(\lambda_{j_{n-k}})}$ lies in the lattice dual to the lattice (16) and the functions $X_1, ..., X_k$ are given by (17), where the function $\tilde{X}_r$ is periodic with the period P_{i_r} and satisfies (18).

7 The eigenvalues of L for projectively equivalent metrics on the 3-torus never bifurcate

Theorem 10. *Let g and $\bar{g}$ be projectively equivalent Riemannian metrics on the three-dimensional torus T^3. Then, the number of different eigenvalues of the tensor L given by (1) is constant on the torus.*

Proof: Let Riemannian metrics g and $\bar{g}$ on T^3 be projectively equivalent. By Corollary 1, if they are strictly non-proportional at least at one point, then at each point of T^3 the number of different eigenvalues of L is precisely three and, therefore, is constant. If the metrics are proportional at each point of T^3, then at each point of T^3 the number of different eigenvalues is equal to one and, therefore, is constant. So, the only case we need to consider is when there exists a point $x \in T^3$ such that at these point the number of different eigenvalues is equal to two; our goal is to show that then the number of different eigenvalues at every other point can not be equal to one.

Actually, this fact has been essentially proved in [15]. But since in the paper [15] the proof is hidden in the third part of the proof of the main theorem, we will repeat it here.

Let us denote the eigenvalues of L at $x \in T^3$ by $\lambda_1(x), \lambda_2(x), \lambda_3(x)$. By Theorem 5, without loss of generality we can assume that

$$\lambda_1(x) < \lambda_2(x) = \lambda_3(x) = const \stackrel{\text{def}}{=} \lambda$$

for almost each point $x \in T^3$. Suppose the eigenvalues bifurcate so that there exists $y \in T^3$ such that

$$\lambda_1(y) = \lambda_2(y) = \lambda_3(y).$$

Let us show that it is possible only on the sphere S^3 or on the Projective Space $\mathbb{RP}^3$. As in Section 2, at each point $x \in T^3$, we can find a basis of the space T_xT^3 such that in this basis the metric g is given by the diagonal matrix $\mathrm{diag}(1,1,1)$ and the matrix L is given by the diagonal matrix $\mathrm{diag}(\lambda_1(x), \lambda_2(x), \lambda_3(x))$. In this basis, the polynomial $I_t(x,\xi)$ given by (3) reads

$$I_t = (\lambda - t)^2\xi_1^2 + (\lambda_1(x) - t)(\lambda - t)(\xi_2^2 + \xi_3^2).$$

Therefore, for every t, the functions

$$\tilde{I}_t \stackrel{\text{def}}{=} \frac{I_t}{(\lambda - t)} = (\lambda - t)\xi_1^2 + (\lambda_1(x) - t)(\xi_2^2 + \xi_3^2)$$

is an integral for the geodesic flow of g. Substituting $t = \lambda$ we get that the function

$$\tilde{I}_\lambda = (\lambda_1(x) - \lambda)(\xi_2^2 + \xi_3^2)$$

is also an integral

By Lemma 1, for every geodesic γ passing through the point y we have $\tilde{I}_\lambda(\gamma, \dot\gamma) \equiv 0$. Then, for each $x \in \gamma$ such that $\lambda_1(x) \neq \lambda$ the sum $(\xi_2^2 + \xi_3^2)$ is zero. Therefore, the velocity vector $\dot\gamma(x)$ is an eigenvector of L with the eigenvalue $\lambda_1(x)$. Then, two geodesics passing through y can transversally intersect only in the points z where $\lambda_1(z) = \lambda$. Then, there can be maximum two of such points and T^3 is homeomorphic either to S^3 or to $\mathbb{RP}^3$. The contradiction shows that if number of eigenvalues of L is equal to two at least at one point of T^3, then it is so at each point. **Theorem 10 is proved.**

References

[1] E. Beltrami, *Resoluzione del problema: riportari i punti di una superficie sopra un piano in modo che le linee geodetische vengano rappresentante da linee rette*, Ann. Mat., **1**(1865), no. 7. 185–204.

[2] A. V. Bolsinov, V. S. Matveev, A. T. Fomenko, *Two-dimensional Riemannian metrics with an integrable geodesic flow. Local and global geometries,* Sb. Math. **189**(1998), no. 9-10, 1441–1466.

[3] U. Dini, *Sopra un problema che si presenta nella theoria generale delle rappresetazioni geografice di una superficie su un'altra*, Ann. Mat., ser.2, **3**(1869), 269–293.

[4] I. Hasegawa, S. Fujimura, *On holomorphically projective transformations of Kaehlerian manifolds,* Math. Japon. **42**(1995), no. 1, 99–104.

[5] K. Kiyohara, *Compact Liouville surfaces*, J. Math. Soc. Japan **43**(1991), 555-591.

[6] T. Levi-Civita, *Sulle trasformazioni delle equazioni dinamiche*, Ann. di Mat., serie 2^a, **24**(1896), 255–300.

[7] A. Lichnerowicz, *Sur les applications harmoniques,* C. R. Acad. Sci. Paris Sér. A–B **267**(1968), A548–A553.

[8] V. S. Matveev, P. J. Topalov, *Trajectory equivalence and corresponding integrals*, Regular and Chaotic Dynamics, **3**(1998), no. 2, 30–45.

[9] V. S. Matveev, *Quantum integrability of the Beltrami-Laplace operator for geodesically equivalent metrics*, Russian Math. Doklady **61**(2000), no 2, 216–219.

[10] V. S. Matveev, P. J. Topalov, *Geodesic equivalence of metrics on surfaces, and their integrability,* Dokl. Akad. Nauk **367**(1999), no. 6, 736–738.

[11] V. S. Matveev, P. J. Topalov, *Quantum integrability for the Beltrami-Laplace operator as geodesic equivalence,* Math. Z. **238**(2001), 833–866.

[12] V. S. Matveev and P. J. Topalov, *Quantum integrability for the Beltrami-Laplace operator for geodesically equivalent metrics. Integrability criterium for geodesic equivalence. Separation of variables*, Preprint IHES M/**00**/17.

[13] Vladimir S. Matveev, *Quantum integrability and complete separation of variables for projectively equivalent metrics on the torus,* In the book: Geometry, integrability and quantization (Varna 2000), 228–244.

[14] Vladimir S. Matveev, *Commuting operators and separation of variables for Laplacians of projectively equivalent metrics,* Let. Math. Phys., **119**(2001), 193–201.

[15] Vladimir S. Matveev, *Geschlossene hyperbolische 3-Mannigflatigkeiten sind geodätisch starr,* Manuscripta Math. 105 (2001), no. 3, 343–352.

[16] J. Mikes, *Geodesic mappings of affine-connected and Riemannian spaces. Geometry, 2.,* J. Math. Sci. **78**(1996), no. 3, 311–333.

[17] I. A. Taimanov, *Topological obstructions to the integrability of geodesic flow on nonsimply connected manifold*, Math.USSR-Izv., **30**(1988), no. 2, 403–409.

[18] P. J. Topalov and V. S. Matveev, *Geodesic equivalence and integrability,* Preprint of Max-Planck-Institut f. Math. no. 74(**1998**).

[19] P. Venzi, *Geodätische Abbildungen in Riemannschen Mannigfaltigkeiten,* Tensor (N.S.) **33**(1979), no. 3, 313–321.