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TRAJECTORY EQUIVALENCE AND CORRESPONDING INTEGRALS

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We suggest a simple approach for obtaining integrals of Hamiltonian systems if there is known a trajectorian map of two Hamiltonian systems. An explicite formila is given. As an example, it is proved that if on a manifold are given two Riemannian metrics which are geodesically equivalent then there is a big family of integrals. Our theorem is a generalization of the well-known Painlevé — Liouville theorems.

1. Introduction

Let v and $\bar v$ be Hamiltonian systems on symplectic manifolds (M^{2n},ω) and $(\bar M^{2n},\bar\omega)$ with Hamiltonians H and $\bar H$ respectively. Consider the isoenergy surfaces

$$Q \stackrel{\text{def}}{=} \left\{x \in M^{2n} : H(x) = h\right\}, \quad \bar Q \stackrel{\text{def}}{=} \left\{x \in \bar M^{2n} : \bar H(x) = \bar h\right\},$$

where h and $\bar h$ are regular values of the functions H, $\bar H$ respectively.

Definition 1. A diffeomorphism $\phi : Q \longrightarrow \bar Q$ is said to be *trajectorial*, if it takes the trajectories of the system v to the trajectories of the system $\bar v$. The systems v and $\bar v$ are called trajectory equivalent (on Q and $\bar Q$, respectively), if there exists a trajectorial diffeomorphism $\phi : Q \longrightarrow \bar Q$.

In the paper [22] it was shown, that a trajectorial diffeomorphism allows one to construct n integrals of the geodesic flow of the system v. This result could be considered as a particular result of a theory, developing in [4], [22]. More precisely, in [4] it was shown, that the existence of a vector field on Q which commutes with the Hamiltonian vector field v allows one to construct a (multi-valued in general situation) integrals of the Hamiltonian system. In the paper [22] the result of [4] was generalized to tensor fields. It was shown, that if a Hamiltonian flow preserves a tensor field, then there exist an (also multi-valued) integrals of the Hamiltonian system.

Now, the trajectorial diffeomorphism allows one to construct an invariant tensor field. Take the restriction $\bar\omega_{|\bar Q}$ of the symplectic form ω to the isoenergy surface Q, and consider the form $\phi^*\bar\omega_{|\bar Q}$ on Q. This form is preserved by the Hamiltonian flow v, see Lemma 1 in Section 2.

For the invariant form $\phi^*\bar{\omega}_{|\bar{Q}}$ the integrals are not multi-values. The explicit formulae for them are given in Theorem 4.

There are not very many examples of trajectorial diffeomorphisms of mechanical systems, see [11], and all of them are a-priory integrable.

Now let the number of the degrees of freedom of the system be two. Since trajectorial diffeomorphism allows to construct integrals, there is almost no sense (at least from symplectic point of view) to consider non-integrable trajectory equivalent systems. In the series of papers [5], [6], [7], [8], [9] it was constructed a trajectory invariant of integrable Hamiltonian systems on isoenergy surfaces. This invariant is called trajectory molecule. Two Hamiltonian systems are trajectory equivalent (on isoenergy surfaces), if and only if they have the same trajectory molecule.

A classical example of trajectory equivalence of Hamiltonian systems is geodesic equivalence of metrics. Let $g = (g_{ij})$ and $\bar{g} = (\bar{g}_{ij})$ be smooth metrics on the same manifold M^n.

Definition 2. The metrics g and $\bar{g}$ are *geodesically equivalent*, if they have the same geodesics (considered as unparameterized curves).

For geodesically equivalent metrics, a trajectorial diffeomorphism Φ is given by $\Phi(x,\xi) = (x, \frac{\|\xi\|_g}{\|\xi\|_{\bar{g}}}\xi)$. Here $(x,\xi) \in \mathcal{T}M^n$, x is a point of M^n and $\xi \in \mathcal{T}_xM^n$.

The theory of geodesically equivalent metrics is rather classical material. The setting of it is due Beltrami [1], [2]. He has proved, that the metric on a surface, geodesically equivalent to the metric of a constant curvature, is also a metric of a constant curvature. In 1869 Dini [10] formulated the problem of local description of geodesically equivalent metrics, and solved it for dimension two. In 1896 Levi-Civita [12] got a local description of geodesically equivalent metrics on manifolds of arbitrary dimension. A significant contribution to the theory of geodesically equivalent metrics was made by Aminova, Sinukov, Venzi, Mikesh, Pogorelov, see [16] for references.

Let us recall a few examples of geodesically equivalent metrics. There always exists a trivial example, which is not interesting for our consideration: an arbitrary metric g is geodesically equivalent to the metric Cg, where C is a constant.

The first non-trivial in this sense example is the following. Take the torus $T^n = S^1 \times S^1 \times \ldots \times S^1$. Consider the coordinate system $x^1, \ldots, x^n$ on T^n, assuming x^k is the cyclic coordinate on the circle number k. Now consider the metrics $ds_1^2 = \sum_{i=1}^n (dx^i)^2$ and $ds_2^2 = \sum_{i,j=1}^n a_{ij}(dx^i)(dx^j)$, where (a_{ij}) is a positive definite symmetric matrix. The metrics ds_1^2 and ds_2^2 are evidently geodesically equivalent, and they are, generally speaking, not proportional.

The second non-trivial example is due to Beltrami. Consider the standard sphere $\sum_{i=1}^{n+1}(x^i)^2 = 1$, where $x^1, \ldots, x^{n+1}$ are standard coordinates in the Euclidean space R^{n+1}. Take a non-degenerate linear transformation $L: R^{n+1} \to R^{n+1}$ of the space R^{n+1}, and consider the corresponding projective transformation l of the sphere. Let x be a point of the sphere. Consider the ray $[0,x)$, where 0 is the zero point of R^{n+1}. Evidently the image $L([0,x))$ is also a ray with origin in zero. Let the intersection of the ray $L([0,x))$ and the sphere be y. Then, by definition, put $l(x) = y$.

It is easy to see, that the mapping l preserves the geodesics of the sphere. Actually, the geodesics of the sphere are intersections of the planes, which go through the zero point, with the sphere. The linear mapping L takes the planes to the planes. Therefore the projective mapping l takes geodesics to geodesics. Then the standard metric $g_{standard}$ of the sphere and the metric $l^*g_{standard}$ are geodesically equivalent.

We would like to point out the following common property of these two examples: the geodesic flows of both metrics are completely integrable, in sense that there exist n integrals in involution. We claim that this property is common for all geodesically equivalent metrics in general position.

Denote by G the linear operator $g^{-1}\bar{g} = (g^{i\alpha}\bar{g}_{\alpha j})$. Consider the characteristic polynomial

$$det(G - \mu E) = c_0\mu^n + c_1\mu^{n-1} + \ldots + c^n.$$

The coefficients $c_1, \dots, c_n$ are smooth functions on the manifold M^n, and $c_0 \equiv (-1)^n$. Consider functions $I_k : TM^n \to R$, $k = 0, \dots, n-1$, given by formulae $I_k(x, \xi) = \left(\frac{det(g)}{det(\bar g)}\right)^{\frac{k+2}{n+1}} \bar g(S_k \xi, \xi)$, where S_k is the linear operator given by $S_k \stackrel{\text{def}}{=} \sum_{i=0}^{k} c_i G^{k-i}$, and $\bar g(\nu, \xi)$ denotes the dot product of the tangent vectors (ν, ξ) in the metrics $\bar g$.

Theorem 1. *If the metrics g and $\bar g$ on M^n are geodesically equivalent, then the functions I_k are integrals of the geodesic flow of the metric g and pairwise commute.*

The metrics g, $\bar g$ are strictly non-proportionally, if the characteristic polynomial $det(G - \mu E)$ has no multiply roots. It can be shown, that if the metrics g, $\bar g$ are geodesically equivalent and strictly non-proportional at the point x, then in a neighborhood of the point the integrals I_k are functionally independent almost everywhere.

What is the dimension of the space of the metrics, geodesically equivalent to a given one? This question was actively discussed, see [16] for references. Even locally, there exist metrics that have no non-trivially geodesically equivalent. Even locally, the dimension of space of metrics, geodesically equivalent to a given one, does not exceed $\frac{(n+1)(n+2)}{2}$ and is equal to $\frac{(n+1)(n+2)}{2}$ only for the metrics of constant curvature.

If the metrics g, $\bar g$ are geodesically equivalent, then there exists an one-parameter family of metrics, geodesically equivalent to g. Note, that the integrals I_k from Theorem 1 are quadratic in velocities. Then there exists symmetric bilinear forms $\tilde I_k$, such that for any $k \in \{0, .., n-1\}$ $\tilde I_k(\xi, \xi) = I_k(\xi)$.

Take a real number α and consider the form

$$f_\alpha \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} (-\alpha)^i \tilde I_i.$$

The form is positive definite for positive α, and therefore can be positive definite for small negative α. Consider the metric

$$g_\alpha \stackrel{\text{def}}{=} \left(\frac{det(g)}{det(f_\alpha)}\right)^{\frac{2}{n-1}} f_\alpha.$$

Theorem 2. *If the metrics g, $\bar g$ are geodesically equivalent, then for any α, such that the metric g_α is positive definite, the metric g_α is geodesically equivalent to the metric g.*

Let the manifold M^n be closed, and let the metrics g, $\bar g$ be strictly non-proportional at almost everywhere dense set of points. Then the geodesic flow of the metric g is completely integrable, and almost all trajectories lie at the corresponding Liouville tori. Suppose, that the geodesic flow is non-resonant. Then the Liouville foliation is unique definite, and any integral of the geodesic flow commutes with the integrals $I_0, \dots, I_{n-1}$. Assume in additional, that there are sufficiently many caustics of the Liouville tori of the geodesic flow: almost each point of the surface is an intersection of n caustics. Then the Levi-Civita coordinates are unique definite, and the dimension of the space of the metric, geodesically equivalent to the metric g, is equal one.

The geodesic flow of the metric of an ellipsoid satisfies all these conditions. First of all, it admits non-trivially geodesically equivalent metric

Consider the ellipsoid

$$\sum_{i=1}^{n} \frac{(x^i)^2}{a_i} = 1, \text{ where } a_i > 0, i = 1, \dots, n.$$

Theorem 3. *The restriction of the metric $\sum_{i=1}^n (dx^i)^2$ to the ellipsoid $\sum_{i=1}^n \frac{(x^i)^2}{a_i} = 1$ is geodesically equivalent to the restriction of the metric*

$$\frac{1}{\sum_{i=1}^n \left(\frac{x^i}{a_i}\right)^2} \left(\sum_{i=1}^n \frac{(dx^i)^2}{a_i}\right)$$

to the ellipsoid.

A very beautiful construction that allows one to find the metric, that is geodesically equivalent to the metric of ellipsoid, is due to Tabachnikov [20].

If we apply Theorem 1 to the metrics from Theorem 3, then the integrals $I_0, \dots, I_{n-1}$ are linear combinations of the integrals from [17]. In [17] it was shown, that the geodesic flow of the metric on the ellipsoid is non-resonance, and almost each point is the point of intersection of n caustics.

The paper is organized as follows. In Section 2 we prove Theorem 4 that gives an explicit formula for an one-parameter family of first integrals, if there exist a trajectorial diffeomorphism between two Hamiltonian systems. In Section 3, for readers convenience, we formulate Levi-Civita's results about the local form of geodesically equivalent metrics. In Section 4 we apply Theorem MainTh to geodesically equivalent metrics. As the result we get the formulae for the integrals I_k. In Section 5 we prove that the integrals I_k are in involution. In Sections 6, 7 we prove Theorems 2, 3.

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2. Trajectorial diffeomorphism and integrals

Let v and $\bar v$ be Hamiltonian systems on symplectic manifolds (M, ω) and $(\bar M, \bar\omega)$ with Hamiltonians H and $\bar H$ respectively. Consider the isoenergy surfaces

$$Q \stackrel{\text{def}}{=} \{x \in M : H(x) = h\}, \quad \bar Q \stackrel{\text{def}}{=} \left\{x \in \bar M : \bar H(x) = \bar h\right\},$$

where h and $\bar h$ are regular values of the functions H, $\bar H$ respectively. Let $U(Q) \subset M$ and $U(\bar Q) \subset \bar M$ be neighborhoods of the isoenergy surfaces Q and $\bar Q$.

Definition 3. A diffeomorphism $\Phi : U(Q) \longrightarrow U(\bar Q)$, $\Phi(Q) = \bar Q$, is said to be *trajectorial on* Q, if the restriction $\Phi|_Q$ takes the trajectories of the system v to the trajectories of the system $\bar v$.

Denote the restriction $\Phi|_Q$ by ϕ. Since ϕ takes the trajectories of v to the trajectories of $\bar v$, it takes the vector field v to the vector field that is proportional to $\bar v$. Denote by $a_1 : Q \to R$ the coefficient of proportionality, i.e. $\phi_*(v) = a_1 \bar v$. Since Φ takes Q to $\bar Q$, it takes the differential dH to a form that is proportional to $d\bar H$. Denote by $a_2 : Q \to R$ the coefficient of proportionality, i.e. $\phi_* \mathrm{dH} = \mathrm{a_2 d\bar H}$. By a we denote the product $a_1 a_2$. We denote the Pfaffian of a skew-symmetric matrix X by $\mathrm{Pf}(X)$.

Theorem 4. *Let a diffeomorphism $\Phi : U(Q) \to U(\bar Q)$, $\Phi(Q) = \bar Q$, be trajectorial on Q. Then for each value of the parameter t the polynomial*

$$\mathcal{P}^{n-1}(t) \stackrel{\text{def}}{=} \frac{\mathrm{Pf}\,(\Phi^* \bar\omega - t\omega)}{\mathrm{Pf}\,(\omega)\,(t - a)}$$

is an integral of the system v on Q. In particular, all the coefficients of the polynomial $\mathcal{P}^{n-1}(t)$ are integrals.

Proof.

Denote by σ, $\bar{\sigma}$ the restrictions of the forms $\omega, \bar{\omega}$ to Q, $\bar{Q}$ respectively. Consider the form $\phi^*\bar{\sigma}$ on Q.

Lemma 1. *[22] The flow v preserves the form $\phi^*\bar{\sigma}$.*

Proof of Lemma 1.

The Lie derivative L_v of the form $\phi^*\bar{\sigma}$ along the vector field v satisfies

$$L_v\phi^*\bar{\sigma} = \mathrm{d}\left[\imath_{\mathrm{v}}\phi^*\bar{\sigma}\right] + \imath_{\mathrm{v}}\mathrm{d}\left[\phi^*\bar{\sigma}\right].$$

On the right side both terms vanish. More precisely, for an arbitrary vector $u \in \mathcal{T}_xQ$ at an arbitrary point $x \in Q$ we have

$$\begin{array}{rcccl} \imath_v\phi^*\bar{\sigma}(u) & = & \bar{\sigma}(\phi_*(v), \phi_*(u)) & = & \\ & = & \bar{\sigma}(a_1\bar{v}, \phi_*(u)) & = & \\ & = & -a_1\mathrm{d}\bar{H}(\phi_*(\mathrm{u})) & = & 0. \end{array}$$

Since the form $\bar{\omega}$ is closed, the form $\bar{\sigma}$ is also closed and $\mathrm{d}\left[\phi^*\bar{\sigma}\right] = \phi^*(\mathrm{d}\bar{\sigma}) = 0$. ■

It is obvious that the kernels of the forms σ and $\phi^*\bar{\sigma}$ coincide (in the space $\mathcal{T}_xQ$ at each point $x \in Q$) with the linear span of the vector v. Therefore these forms induce two non-degenerate tensor fields on the quotient bundle $\mathcal{T}Q/\langle v\rangle$. We shall denote the corresponding forms on $\mathcal{T}Q/\langle v\rangle$ also by the letters $\sigma, \bar{\sigma}$.

Lemma 2. *The characteristic polynomial of the operator $(\sigma)^{-1}(\phi^*\bar{\sigma})$ on $\mathcal{T}Q/\langle v\rangle$ is preserved by the flow v.*

Proof of Lemma 2.

Since the flow v preserves the Hamiltonian H and the form ω, the flow v preserves the form σ. Since the flow v preserves both forms, it preserves the characteristic polynomial of the operator $(\sigma)^{-1}(\phi^*\bar{\sigma})$. ■

Since both forms are skew-symmetric, each root of the characteristic polynomial of the operator $(\sigma)^{-1}(\Phi^*\bar{\sigma})$ has an even multiplicity. Then the characteristic polynomial is the square of a polynomial $\delta^{n-1}(t)$ of degree $n-1$. Hence the polynomial $\delta^{n-1}(t)$ is also preserved by the flow v. It is obvious that

$$\delta^{n-1}(t) = (-1)^{n-1}\frac{\mathrm{Pf}\,(\phi^*\bar{\sigma} - t\sigma)}{\mathrm{Pf}\,(\sigma)}.$$

The last step of the proof is to verify that

$$(t-a)\delta^{n-1} = \frac{\mathrm{Pf}\,(\Phi^*\bar{\omega} - t\omega)}{\mathrm{Pf}\,(\omega)} \stackrel{\mathrm{def}}{=} \Delta^n.$$

Take an arbitrary point $x \in Q$. Consider the form $\Phi^*\bar{\omega} - a\omega$ on $\mathcal{T}_xM$. The form $\imath_v(\Phi^*\bar{\omega} - a\omega)$ equals zero. More precisely, for any vector $u \in \mathcal{T}_xM$ we have

$$\begin{array}{rcl} \imath_v(\Phi^*\bar{\omega} - a\omega) & = & \bar{\omega}(\Phi_*(v), \Phi_*(u)) - a\omega(v,u) = \\ & = & \bar{\omega}(a_1v, \Phi_*(u)) - a\omega(v,u) = \\ & = & -a_1\mathrm{d}\bar{\mathrm{H}}(\Phi_*(\mathrm{u})) + \mathrm{adH} = \\ & = & -a\mathrm{dH} + \mathrm{adH} = 0. \end{array}$$

There exists a vector $A \in \mathcal{T}_xM$ such that $\omega(A, v) \neq 0$ and the restriction of the form $\imath_A(\Phi^*\bar{\omega} - a\omega)$ to the space $\mathcal{T}_xM$ equals zero. More precisely, since the forms $\Phi^*\bar{\omega}$, ω are skew-symmetric, then the kernel $K_{\Phi^*\bar\omega-a\omega}$ of the form $\Phi^*\bar\omega-a\omega$ has an even dimension, and the kernel of the restriction of the form $\Phi^*\bar\omega-a\omega$ to $\mathcal{T}_xQ$ has an odd dimension. Thus the intersection $K_{\Phi^*\bar\omega-a\omega}\cap(\mathcal{T}_xM\setminus\mathcal{T}_xQ)$ is not empty. For each vector A from the intersection we obviously have $\omega(A,v)\neq 0$ and $\imath_A(\Phi^*\bar\omega-a\omega)=0$. Without loss of generality we can assume $\omega(A,v)=1$.

Consider a basis $(v,e_1,\dots,e_{2n-2})$ for the space $\mathcal{T}_xQ$. The set $(A,v,e_1,\dots,e_{2n-2})$ is a basis for the space $\mathcal{T}_xM$. In this basis we have

$$\begin{aligned}\det(\Phi^*\bar\omega-t\omega) &= \det\left|\begin{array}{cc|c} 0 & a-t & (*)\\ -(a-t) & 0 & 0\cdots 0\\ \hline -(*) & 0 & (\Phi^*\bar\omega-t\omega)_{\langle e_1,\dots,e_{2n-2}\rangle}\end{array}\right| =\\ &= (a-t)^2\det((\Phi^*\bar\omega-t\omega)_{\langle e_1,\dots,e_{2n-2}\rangle})\\ &= (a-t)^2\det(\phi^*\bar\sigma-t\sigma),\end{aligned}$$

where $(\Phi^*\bar\omega-t\omega)_{\langle e_1,\dots,e_{2n-2}\rangle}$ is the matrix of the form $\Phi^*\bar\omega-t\omega$ in the basis $(e_1,\dots,e_{2n-2})$. Finally, $\delta^{n-1}=\mathcal{P}^{n-1}$. ■

3. Levi-Civita theorem

Let g and $\bar g$ be smooth metrics on a manifold M^n. Denote by $\rho^1,\dots,\rho^m$ $(1\le m\le n)$ the common eigenvalues of the metrics g and $\bar g$. Suppose the functions $\rho^1,\dots,\rho^m$ are differ at every point of an open domain $\mathcal{D}\subset M^n$. In the paper [12], Levi-Civita proved that for every point $P\in\mathcal{D}$ there is an open neighborhood $\mathcal{U}(P)\subset\mathcal{D}$ and a coordinate system $\bar x=(\bar x_1,\dots,\bar x_m)$ (in $\mathcal{U}(P)$), where $\bar x_i=(x_i^1,\dots,x_i^{k_i})$, $(1\le i\le m)$, such that the quadratic forms of the metrics g and $\bar g$ have the following form:

$$\begin{aligned} g(\dot{\bar x},\dot{\bar x}) &= \Pi_1(\bar x)A_1(\bar x_1,\dot{\bar x}_1)+\Pi_2(\bar x)A_2(\bar x_2,\dot{\bar x}_2)+\dots+\\ &+ \Pi_m(\bar x)A_m(\bar x_m,\dot{\bar x}_m), \qquad (1)\\ \bar g(\dot{\bar x},\dot{\bar x}) &= \rho^1\Pi_1(\bar x)A_1(\bar x_1,\dot{\bar x}_1)+\rho^2\Pi_2(\bar x)A_2(\bar x_2,\dot{\bar x}_2)+\dots+\\ &+ \rho^m\Pi_m(\bar x)A_m(\bar x_m,\dot{\bar x}_m), \qquad (2)\end{aligned}$$

where $A_i(\bar x_i,\dot{\bar x}_i)$ are positive-definite quadratic forms in the velocities $\dot{\bar x}_i$ with coefficients depending on $\bar x_i$,

$$\Pi_i \stackrel{\text{def}}{=} (\phi_i-\phi_1)\cdots(\phi_i-\phi_{i-1})(\phi_{i+1}-\phi_i)\cdots(\phi_m-\phi_i) \qquad (3)$$

and $\phi_1,\phi_2,\dots,\phi_m$, $0<\phi_1<\phi_2<\ldots<\phi_m$, are smooth functions such that

$$\phi_i=\begin{cases}\phi_i(\bar x_i), & \text{if}\quad k_i=1\\ \text{constant}, & \text{else.}\end{cases}$$

It is easy to see that the functions ρ^i as functions of ϕ_i and the function ϕ_i as functions of ρ^i are given by

$$\begin{aligned}\rho^i &= \frac{1}{\phi_1\ldots\phi_m}\frac{1}{\phi_i}\\ \phi_i &= \frac{1}{\rho_i}(\rho_1\rho_2\ldots\rho_m)^{\frac{1}{m+1}}\end{aligned}$$

Definition 4. Let metrics g and $\bar g$ be given by formulae (1) and (2) in a coordinate chart $\mathcal{U}$. Then we say that the metrics g and $\bar g$ have *Levi-Civita local form (of type m)*, and the coordinate chart $\mathcal{U}$ is *Levi-Civita* coordinate chart (with respect to the metrics).

Levi-Civita proved that the metrics g and $\bar{g}$ given by formulae (1) and (2) are geodesically equivalent. If we replace ϕ_i by $\phi_i + c,\ i = 1, \dots, m$, where c is a (positive for simplicity) constant, in (1) and (2), we obtain the following one-parameter family of metrics, geodesically equivalent to g:

$$g_c(\dot{\bar{x}}, \dot{\bar{x}}) = \frac{1}{(\phi_1 + c) \cdots (\phi_m + c)} \left\{ \frac{1}{\phi_1 + c} \Pi_1 A_1 + \dots + \frac{1}{\phi_m + c} \Pi_m A_m \right\}.$$

The next theorem is essentially due to Painlevé, see [12].

Theorem 5. *If the metrics g and $\bar{g}$ are geodesically equivalent, then the function*

$$I_0 \stackrel{\text{def}}{=} \left(\frac{\det(g)}{\det(\bar{g})} \right)^{\frac{2}{n+1}} \bar{g}(\dot{\bar{x}}, \dot{\bar{x}}), \qquad (4)$$

is an integral of the geodesic flow of the metric g.

Substituting g_c instead of $\bar{g}$ in (4), we obtain the following one-parameter family of integrals

$$\begin{aligned} I_c &\stackrel{\text{def}}{=} \left(\frac{\det(g)}{\det(g_c)} \right)^{\frac{2}{n+1}} g_c(\dot{\bar{x}}, \dot{\bar{x}}) = \\ &= C[(\phi_1 + c) \cdots (\phi_m + c)] \left\{ \frac{1}{\phi_1 + c} \Pi_1 A_1 + \dots + \frac{1}{\phi_m + c} \Pi_m A_m \right\} \\ &= C\{L_1 c^{m-1} + L_2 c^{m-2} + \dots + L_m\}, \end{aligned}$$

where

$$\begin{aligned} L_1 &= \Pi_1 A_1 + \dots + \Pi_m A_m \quad \text{— the twice energy integral,} \\ L_2 &= \sigma_1(\phi_2, \dots, \phi_m) \Pi_1 A_1 + \dots + \sigma_1(\phi_1, \dots, \phi_{m-1}) \Pi_m A_m, \\ L_3 &= \sigma_2(\phi_2, \dots, \phi_m) \Pi_1 A_1 + \dots + \sigma_2(\phi_1, \dots, \phi_{m-1}) \Pi_m A_m, \\ &\ \vdots \\ L_m &= (\phi_2 \dots \phi_m) \Pi_1 A_1 + \dots + (\phi_1 \dots \phi_{m-1}) \Pi_m A_m, \end{aligned}$$

σ_k denotes the elementary symmetric polynomial of degree k, and
$C \stackrel{\text{def}}{=} \left[(\phi_1 + c)^{k_1 - 1} \cdots (\phi_m + c)^{k_m - 1} \right]^{\frac{2}{n+1}}$ is a constant. Therefore the functions $L_k,\ k = 1, \dots, m$, are integrals of the geodesic flows of the metric g. We call these integrals *Levi-Civita integrals.*

From the results of [18] it follows that Levi-Civita integrals are in involution. More precisely, let $D = (d_j^i)$ be an $m \times m$ matrix. Suppose that for any i, j the element d_j^i depends only on the variables $\bar{x}_j$. Denote by Δ the determinant of the matrix D and by Δ_j^i the minor of the element d_j^i. In the paper [18] it was shown that, for arbitrary functions $A_i(\bar{x}_i, \dot{\bar{x}}_i)$, quadratic in velocities $\dot{\bar{x}}_i$, the Lagrangian system with Lagrangian

$$T_1 = \Delta \left(\frac{A_1(\bar{x}_1, \dot{\bar{x}}_1)}{\Delta_1^1} + \frac{A_2(\bar{x}_2, \dot{\bar{x}}_2)}{\Delta_2^1} + \dots + \frac{A_m(\bar{x}_m, \dot{\bar{x}}_m)}{\Delta_m^1} \right)$$

admits $(m-1)$ integrals

$$T_i = \Delta \left(A_1(\bar{x}_1, \dot{\bar{x}}_1) \frac{\Delta_1^i}{(\Delta_1^1)^2} + A_2(\bar{x}_2, \dot{\bar{x}}_2) \frac{\Delta_2^i}{(\Delta_2^1)^2} + \dots + A_m(\bar{x}_m, \dot{\bar{x}}_m) \frac{\Delta_m^i}{(\Delta_m^1)^2} \right),$$

where $i = 2, \dots, m$, and if we identify the tangent and cotangent bundles the Lagrangian T_1 and consider the standard symplectic form on the cotangent bundle, then the integrals are in involution.

If we take $d_j^i = (\phi_j)^{m-i}$, then Δ and Δ_j^i are given by

$$\Delta_j^i = (-1)^{m-1}\sigma^{i-1}(\phi_1, \phi_2, \dots, \phi_{j-1}, \phi_{j+1}, \dots, \phi_m) \prod_{\alpha>\beta\geq 1, \alpha\neq j, \beta\neq j} (\phi_\alpha - \phi_\beta),$$

$$\Delta = (-1)^m \prod_{\alpha>\beta\geq 1} (\phi_\alpha - \phi_\beta).$$

Therefore,

$$\frac{\Delta\Delta_j^i}{(\Delta_j^1)^2} = \sigma^{i-1}(\phi_1, \phi_2, \dots, \phi_{j-1}, \phi_{j+1}, \dots, \phi_m)\Pi_j,$$

so $T_i = -L_i$ and thus the integrals L_i are in involution. ■

4. Geodesic equivalence and corresponding integrals

Let metrics g and $\bar{g}$ on a manifold M (of dimension n) be geodesically equivalent. By definition, put

$$U_g^r M \stackrel{\text{def}}{=} \{(x, \xi) \in \mathcal{T}M : \quad ||\xi||_g = r\},$$

where $x \in M$, $\xi \in \mathcal{T}_x M$ and $||\xi||_g \stackrel{\text{def}}{=} \sqrt{g(\xi,\xi)} = \sqrt{g_{ij}\xi^i\xi^j}$ is the norm of the vector ξ in the metric g.

By the geodesic flow of the metric g we mean the Lagrangian system of differential equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$ on $\mathcal{T}M$ with Lagrangian $L \stackrel{\text{def}}{=} \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j$. Because of the Legendre transformation, the geodesic flow could be considered as a Hamiltonian system on $\mathcal{T}M$ (as a symplectic form we take $\omega_g \stackrel{\text{def}}{=} \mathrm{d}[\mathrm{g_{ij}}\xi^{\mathrm{j}}\mathrm{dx^i}]$) with the Hamiltonian $H_g \stackrel{\text{def}}{=} \frac{1}{2}g_{ij}\xi^i\xi^j$.

Since the metrics g, $\bar{g}$ are geodesically equivalent, the mapping $\Phi : \mathcal{T}M \to \mathcal{T}M$, $\Phi(x,\xi) = \left(x, \xi\frac{||\xi||_g}{||\xi||_{\bar{g}}}\right)$, takes the trajectories of the geodesic flow of the metric g to the trajectories of the geodesic flow of the metric $\bar{g}$. This mapping is a diffeomorphism (for $r \neq 0$), takes $U_g^r M$ to $U_{\bar{g}}^r M$ and is trajectorial on $U_g^r M$. Obviously the surfaces U_g^r, $U_{\bar{g}}^r$ are regular isoenergy surfaces $\{H_g = \frac{r}{2}\}$, $\{H_{\bar{g}} = \frac{r}{2}\}$.

By Theorem 4, in order to obtain a family of first integrals we have to find the polynomial $\Delta^n(t)$ and divide it by $(t-a)$. In our case $H_g = H_{\bar{g}} \circ \Phi$. Therefore the function a from Theorem 4 equals to $\frac{||\xi||_{\bar{g}}}{||\xi||_g}$.

In coordinates we have

$$\omega_g = \mathrm{d}[\mathrm{g_{ij}}\xi^{\mathrm{j}}\mathrm{dx^i}]$$

and

$$\omega_{\bar{g}} = \mathrm{d}[\bar{\mathrm{g}}_{\mathrm{ij}}\xi^{\mathrm{j}}\mathrm{dx^i}].$$

Therefore,

$$\begin{aligned} \Phi^*\omega_{\bar{g}} &= \mathrm{d}\left[\frac{||\xi||_{\mathrm{g}}}{||\xi||_{\bar{\mathrm{g}}}}\bar{\mathrm{g}}_{\mathrm{ij}}\xi^{\mathrm{j}}\mathrm{dx^i}\right] = \\ &= \frac{\partial}{\partial x^k}\left[\frac{||\xi||_g}{||\xi||_{\bar{g}}}\bar{g}_{ij}\xi^j\right] dx^k \wedge dx^i - \frac{\partial}{\partial \xi^k}\left[\frac{||\xi||_g}{||\xi||_{\bar{g}}}\bar{g}_{ij}\xi^j\right] dx^i \wedge d\xi^k. \end{aligned}$$

It is easy to see that at a point $\xi \in \mathcal{T}_x M$ the quantities

$$A_{ik} \stackrel{\text{def}}{=} -\frac{\partial}{\partial \xi^k}\left[\frac{||\xi||_g}{||\xi||_{\bar{g}}}\bar{g}_{ij}\xi^j\right]$$

form an element of $\mathcal{T}_x M \otimes \mathcal{T}_x M$. Without loss of generality we can assume that in the space $\mathcal{T}_x M$ the metrics g and $\bar{g}$ are given in principal axes. Then

$$\begin{aligned} A_{ij} &\stackrel{\text{def}}{=} -\rho^i(x)\frac{\partial}{\partial \xi^j}\left(\xi^i \frac{\sqrt{{\xi^1}^2+\ldots+{\xi^n}^2}}{\sqrt{\rho^1{\xi^1}^2+\ldots+\rho^n{\xi^n}^2}}\right) = \\ &= \rho^i\delta^i_j\frac{||\xi||_g}{||\xi||_{\bar{g}}} - \rho^i\xi^i\left(\frac{\frac{||\xi||_{\bar{g}}}{||\xi||_g} - \rho^j\frac{||\xi||_g}{||\xi||_{\bar{g}}}}{||\xi||^2_{\bar{g}}}\xi^j\right) = \\ &= \operatorname{diag}(\mu_1,\ldots,\mu_n) - A\otimes B. \end{aligned}$$

Here $\rho^i, i=1,\ldots,n$ are common eigenvalues (here we allow ρ^i to be equal to ρ^j for some i,j) of the metrics g and $\bar{g}$, $\mu_i \stackrel{\text{def}}{=} -\rho^i\frac{||\xi||_g}{||\xi||_{\bar{g}}}$, $A_i \stackrel{\text{def}}{=} \rho^i\xi^i$ and

$$B_i \stackrel{\text{def}}{=} \frac{\frac{||\xi||_{\bar{g}}}{||\xi||_g} - \rho^i\frac{||\xi||_g}{||\xi||_{\bar{g}}}}{||\xi||^2_{\bar{g}}}\xi^i.$$

We have

$$\begin{aligned} \det(\Phi^*\omega_{\bar{g}} - t\omega_g) &= \det\left|\begin{array}{c|c} (*) & (A_{ij}+t\delta_{ij}) \\ \hline -(A_{ij}+t\delta_{ij}) & 0 \end{array}\right| \\ &= \det(A_{ij}+t\delta_{ij})^2. \end{aligned}$$

Therefore,

$$\Delta^n(t) = \det\left(\operatorname{diag}(t+\mu_1,\ldots,t+\mu_n) - a\otimes b\right).$$

Lemma 3. *The following relation holds:*

$$\begin{aligned} \Delta^n(t) &= (t+\mu_1)\cdots(t+\mu_n) - (a_1b_1)(t+\mu_2)\cdots(t+\mu_n) - \ldots \\ &- (t+\mu_1)\cdots(t+\mu_{n-1})(a_nb_n). \end{aligned} \tag{5}$$

The lemma follows from induction considerations.

To divide the polynomial by $(t-a)$ we shall use the Horner scheme. Suppose that $\Delta^n(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$ and $\delta^{n-1}(t) = t^{n-1} + b_{n-2}t^{n-2} + \cdots + b_0$. Then we have

$$b_{n-1} = a_n = 1, \tag{6}$$

$$b_{n-2} = a_{n-1} + a, \tag{7}$$

$$\ldots$$

$$b_k = a_{k+1} + ab_{k+1}, \tag{8}$$

$$\ldots$$

$$0 = a_0 + ab_0. \tag{9}$$

It follows from lemma 3 that

$$\begin{aligned} a_0 &= (\mu_1\ldots\mu_n) - (A_1B_1)(\mu_2\ldots\mu_n) - \cdots - (\mu_1\ldots\mu_{n-1})A_nB_n = \\ &= (-1)^n\left(\frac{||\xi||_g}{||\xi||_{\bar{g}}}\right)^n(\rho^1\cdots\rho^n). \end{aligned}$$

Combining with (9) we get

$$b_0 = -\frac{a_0}{a} = (-1)^{n+1} \left(\frac{||\xi||_g}{||\xi||_{\bar g}} \right)^{n+1} (\rho^1 \cdots \rho^n).$$

Since $\frac{1}{2} g_{ij} \xi^i \xi^j$ is an integral of the geodesic flow of the metric g, the function

$$I_0 \stackrel{\rm def}{=} (\rho^1 \cdots \rho^n)^{-\frac{2}{n+1}} \bar g(\xi, \xi) \tag{10}$$

is also an integral of the geodesic flow of the metric g. Using Lemma 3 we have

$$\begin{aligned} a_{n-1} &= (\mu_1 + \ldots + \mu_n) - (A_1 B_1 + \ldots + A_n B_n) = \\ &= \frac{||\xi||_g}{||\xi||^3_{\bar g}} \Big\{ ({\rho^1}^2 {\xi^1}^2 + \ldots + {\rho^n}^2 {\xi^n}^2) - \\ &- (\rho^1 + \ldots + \rho^n)(\rho^1 {\xi^1}^2 + \ldots + \rho^n {\xi^n}^2) \Big\} - \frac{||\xi||_{\bar g}}{||\xi||_g}. \end{aligned}$$

Using (7) we get

$$\begin{aligned} b_{n-2} &= a_{n-2} + a = \\ &= \frac{||\xi||_g}{||\xi||^3_{\bar g}} \Big\{ ({\rho^1}^2 {\xi^1}^2 + \ldots + {\rho^n}^2 {\xi^n}^2) - (\rho^1 + \ldots + \rho^n)(\rho^1 {\xi^1}^2 + \ldots + \rho^n {\xi^n}^2) \Big\}. \end{aligned}$$

Therefore, the function

$$\begin{aligned} I_1 &\stackrel{\rm def}{=} (\rho^1 \cdots \rho^n)^{-\frac{3}{n+1}} \Big\{ ({\rho^1}^2 {\xi^1}^2 + \ldots + {\rho^n}^2 {\xi^n}^2) - \\ &- (\rho^1 + \ldots + \rho^n)(\rho^1 {\xi^1}^2 + \ldots + \rho^n {\xi^n}^2) \Big\} \end{aligned}$$

is an integral. (It is easy to see that $\frac{||\xi||^2_g}{||\xi||^2_{\bar g}} = (\rho^1 \cdots \rho^n)^{-\frac{2}{n+1}} \frac{||\xi||^2_g}{I_0}$.)

Arguing as above, we see that the functions

$$\begin{aligned} I_k &\stackrel{\rm def}{=} (\rho^1 \cdots \rho^n)^{-\frac{k+2}{n+1}} \Big\{ ({\rho^1}^{k+1} {\xi^1}^2 + \ldots + {\rho^n}^{k+1} {\xi^n}^2) - \\ &- (\rho^1 + \ldots + \rho^n)({\rho^1}^k {\xi^1}^2 + \ldots + {\rho^n}^k {\xi^n}^2) + \cdots \\ &+ (-1)^k \sigma_k(\rho^1, \ldots, \rho^n)(\rho^1 {\xi^1}^2 + \ldots + \rho^n {\xi^n}^2) \Big\}, \end{aligned}$$

are integrals of the geodesic flow of the metric g, where by σ_k we denote the elementary symmetric polynomial of degree k. It is obvious that $(-1)^k \sigma_k = c_k$ from Theorem 1, and therefore $I_k = \left(\frac{\det(g)}{\det(\bar g)} \right)^{\frac{k+2}{n+1}} \bar g(S_k \xi, \xi)$. Thus I_k, $k = 0, \ldots, n-1$, are integrals of the geodesic flow of the metric g. ■

5. Liouville integrability

The last step of the proof of Theorem 1 is to verify that the integrals $I_0, \ldots, I_{n-1}$ are in involution. We proceed along the following plan. First we show that it is sufficient to prove the involutivity in each Levi-Civita chart. Then we prove that in each Levi-Civita chart the integrals $I_0, \ldots, I_{n-1}$ are linear combinations of Levi-Civita integrals, and therefore commute.

Let $g, \bar{g}$ be metrics on M. A point $x \in M$ is called *stable*, if in a neighborhood of x the number of different eigenvalues of the metrics g, $\bar{g}$ does not depend of a point.

Denote by $\mathcal{M}$ the set of stable points of M. The set $\mathcal{M}$ is an open subset of M. Obviously

$$\mathcal{M} = \bigsqcup_{1 \le q \le n} \mathcal{M}^q, \tag{11}$$

where $\mathcal{M}^q$ denotes the set of stable points whose number of distinct common eigenvalues equals q. Points $x \in M \setminus \mathcal{M}$ are called *points of bifurcation.*

Lemma 4. *The set $\mathcal{M}$ is everywhere dense in M.*

Proof of Lemma 4.

Denote by $N(x)$ the number of distinct common eigenvalues of the metrics $g, \hat{g}$ at a point x. Recall that the common eigenvalues of the metrics g, $\hat{g}$ at a point $x \in M$ are roots of the characteristic polynomial $P_x(t) = \det\left(G - tE\right)_{|x}$, where $G = \left(g^{i\alpha}\bar{g}_{\alpha j}\right)$. In particular, all roots of $P_x(t)$ are real.

Let us prove that, for a sufficiently small neighborhood of an arbitrary point $x \in M$, for any y from the neighborhood the number $N(x)$ is no less than $N(y)$. Take a small $\epsilon > 0$ and an arbitrary root ρ of $P_x(t)$. Let us prove that for a sufficiently small neighborhood $U(x) \subset M$, for any $y \in U(x)$ there is a root ρ_y, $\rho - \epsilon < \rho_y < \rho + \epsilon$, of the polynomial $P_y(t)$. If ϵ is small, then for a sufficiently small neighborhood $U(x)$ of the point x, for any $y \in U(x)$ the numbers $\rho + \epsilon$ and $\rho - \epsilon$ are not roots of $P_y(t)$. Consider the circle $S_\epsilon \stackrel{\text{def}}{=} \{z \in C : \quad |z - \rho| = \epsilon\}$ on the complex plane C. Clearly the number of roots (with multiplicities) of the polynomial P_y inside the circle is equal to

$$\frac{1}{2\pi i}\int_{S_\epsilon} \frac{P_y'(z)}{P_y(z)} dz.$$

Since for any $y \in U(x)$ there are no roots of P_y on the circle S_ϵ, then the function

$$\frac{1}{2\pi i}\int_{S_\epsilon} \frac{P_y'(z)}{P_y(z)} dz$$

continuously depends on $y \in U(x)$, and therefore is a constant. Clearly it is positive. Thus for any $y \in U(x)$ there is at least one root of P_y that lies between $\rho + \epsilon$ and $\rho - \epsilon$. Then for any y from a sufficiently small neighborhood of x we have $N(y) \ge N(x)$.

Now let us prove the lemma. Evidently the set $\mathcal{M}$ is an open subset of M. Then it is sufficient to prove that for any open subset $U \subset M$ there is a stable point $x \in U$. Suppose otherwise, i.e. let all the points of U be points of bifurcation. Take a point $y \in M$ with maximal value of the function N on it. We have that in a neighborhood $U(y)$ of the point y the function N is constant and equals $N(y)$. Then the point y is a stable point, and we get a contradiction. ∎

Now let the metrics $g, \bar{g}$ be geodesically equivalent. Since the set of points of bifurcation is nowhere dense, it is sufficient to prove the involutivity in each Levi-Civita chart. Let the metrics g and $\bar{g}$ be given by

$$\begin{aligned} g(\dot{\bar{x}}, \dot{\bar{x}}) &= \Pi_1(\bar{x})A_1(\bar{x}_1, \dot{\bar{x}}_1) + \Pi_2(\bar{x})A_2(\bar{x}_2, \dot{\bar{x}}_2) + \cdots + \\ &+ \Pi_m(\bar{x})A_m(\bar{x}_m, \dot{\bar{x}}_m), \end{aligned} \tag{12}$$

$$\begin{aligned} \bar{g}(\dot{\bar{x}}, \dot{\bar{x}}) &= \rho^1\Pi_1(\bar{x})A_1(\bar{x}_1, \dot{\bar{x}}_1) + \rho^2\Pi_2(\bar{x})A_2(\bar{x}_2, \dot{\bar{x}}_2) + \cdots \\ &+ \rho^m\Pi_m(\bar{x})A_m(\bar{x}_m, \dot{\bar{x}}_m). \end{aligned} \tag{13}$$

We show that the integrals I_k are linear combinations of the Levi-Civita integrals. We have

$$\bar{G} = \operatorname{diag}(\underbrace{\rho_1, \ldots, \rho_1}_{k_1}, \ldots, \underbrace{\rho_m, \ldots, \rho_m}_{k_m}), \tag{14}$$

where $\rho_k = \frac{1}{(\phi_1 \ldots \phi_m)} \frac{1}{\phi_k}$. It is easy to check that

$$S_k = (-1)^k \operatorname{diag}(\underbrace{\sigma_k^1, \ldots, \sigma_k^1}_{k_1}, \ldots, \underbrace{\sigma_k^m, \ldots, \sigma_k^m}_{k_m}), \tag{15}$$

where

$$\sigma_k^l \stackrel{\text{def}}{=} \sigma_k(\underbrace{\rho_1, \ldots, \rho_1}_{k_1}, \ldots, \underbrace{\rho_l, \ldots, \rho_l}_{k_l - 1}, \ldots, \underbrace{\rho_m, \ldots, \rho_m}_{k_m}). \tag{16}$$

We have

$$\sigma_k^1 = \frac{1}{(\phi_1 \ldots \phi_m)^k} \sigma_k \Big(\underbrace{\frac{1}{\phi_1}, \ldots, \frac{1}{\phi_1}}_{k_1 - 1}, \ldots, \underbrace{\frac{1}{\phi_m}, \ldots, \frac{1}{\phi_m}}_{k_m} \Big) = \tag{17}$$

$$= \frac{1}{(\phi_1 \ldots \phi_m)^k} \sum_{|\alpha| = k} \binom{k_1 - 1}{\alpha_1} \binom{k_2}{\alpha_2} \cdots \binom{k_m}{\alpha_m} \frac{1}{\phi_1^{\alpha_1}} \frac{1}{\phi_2^{\alpha_2}} \cdots \frac{1}{\phi_m^{\alpha_m}}, \tag{18}$$

(19)

where $|\alpha| \stackrel{\text{def}}{=} \alpha_1 + \cdots + \alpha_m$ and $\alpha_i \geq 0$. Substituting $\binom{k_l - 1}{\alpha_l} + \binom{k_l - 1}{\alpha_l - 1}$ for $\binom{k_l}{\alpha_l}$ (we assume that $\binom{k}{0} = 1$, $\binom{k}{-1} = 0$, $k \geq 0$) for $2 \leq l \leq m$ we obtain

$$\begin{aligned} \sigma_k^1 &= \frac{1}{(\phi_1 \ldots \phi_m)^k} \left(B_k + B_{k-1} \sigma_1 \Big(\frac{1}{\phi_2}, \ldots, \frac{1}{\phi_m} \Big) + \cdots + \right. \\ &+ \left. B_{k-m+1} \sigma_{m-1} \Big(\frac{1}{\phi_2}, \ldots, \frac{1}{\phi_m} \Big) \right), \end{aligned}$$

where

$$B_k \stackrel{\text{def}}{=} \sum_{|\alpha| = k} \binom{k_1 - 1}{\alpha_1} \cdots \binom{k_m - 1}{\alpha_m} \frac{1}{\phi_1^{\alpha_1}} \cdots \frac{1}{\phi_m^{\alpha_m}}. \tag{20}$$

Note that

$$\left(\frac{\det(g)}{\det(\bar{g})} \right)^{\frac{k+2}{n+1}} = C_k (\phi_1 \ldots \phi_m)^{k+2}, \tag{21}$$

where $C_k = [\phi_1^{k_1 - 1} \ldots \phi_m^{k_m - 1}]^{\frac{k+2}{n+1}}$. Therefore,

$$\begin{aligned} I_k &\stackrel{\text{def}}{=} \left(\frac{\det(g)}{\det(\bar{g})} \right)^{\frac{k+2}{n+1}} \bar{g}(S_k \dot{x}, \dot{x}) = \\ &= (-1)^k C_k (\phi_1 \ldots \phi_m)^{k+2} \left\{ \rho^1 \sigma_k^1 \Pi_1 A_1 + \cdots + \rho^m \sigma_k^m \Pi_m A_m \right\} = \\ &= (-1)^k C_k (\phi_1 \ldots \phi_m)^{k+2} \left\{ \frac{1}{\phi_1 \ldots \phi_m} \frac{1}{\phi_1} \left\{ \frac{1}{(\phi_1 \ldots \phi_m)^k} \left(B_k + \right. \right. \right. \\ &+ \left. \left. \left. \cdots + B_{k-m+1} \sigma_{m-1} \Big(\frac{1}{\phi_2}, \ldots, \frac{1}{\phi_m} \Big) \right) \right\} \Pi_1 A_1 + \cdots \right\} = \\ &= (-1)^k C_k \left\{ B_k L_m + B_{k-1} L_{m-1} + \cdots + B_{k-m+1} L_1 \right\}, \end{aligned} \tag{22}$$

where L_i are Levi-Civita integrals.

Finally, since the integrals $I_0, \dots, I_{n-1}$ are linear combinations of Levi-Civita integrals with constant coefficients, and since Levi-Civita integrals commute, then the integrals $I_0, \dots, I_{n-1}$ also commute. ■

REMARK 1. *Let m be the number of distinct common eigenvalues of geodesically equivalent metrics g, $\bar{g}$ at a point x. Then in a neighborhood U of the point x the number of functionally independent almost everywhere Levi-Civita integrals is no less than m. Therefore the dimension of the space generated by the differentials $(dI_0, dI_1, \dots, dI_{n-1})$ no less than m in almost all points of $\mathcal{T}U$.*

6. A family of geodesically equivalent metrics

Lemma 5. *Let A be the diagonal $n\times n$ matrix $\mathrm{Diag}(\frac{1}{a_1 a}, \frac{1}{a_2 a}, \dots, \frac{1}{a_i a})$, where a is $\prod_{i=1}^n a_i$, and a_i are positive. Let the characteristic polynomial $det(A-\mu E)$ be $c_0\mu^n + c_1\mu^{n-1} + \ldots + c^n$. Then for any α the matrix*

$$A\sum_{k=0}^{n-1}(-\alpha)^k\left(\frac{1}{det(A)}\right)^{\frac{k+2}{n+1}}\sum_{i=0}^{k}A^{k-i}c_i$$

is equal to $\prod_{i=0} n(a_i+\alpha)Diag(\frac{1}{a_1+\alpha}, \frac{1}{a_2+\alpha}, \dots, \frac{1}{a_n+\alpha})$.

Proof.

It is clear that $\left(\frac{1}{det(A)}\right)^{\frac{k+2}{n+1}}$ equals a^{k+2}, and that $c_k = \sigma^k\left(\frac{-1}{aa_1}, \frac{-1}{aa_2}, \dots, \frac{-1}{aa_n}\right)$, where σ^k denotes the symmetric polynomial of degree k. Then,

$$A\sum_{k=0}^{n-1}(-\alpha)^k\left(\frac{1}{det(A)}\right)^{\frac{k+2}{n+1}}\sum_{i=0}^{k}A^{k-i}c_i = A\sum_{k=0}^{n-1}(-\alpha)^k a^{k+2}\sum_{i=0}^{k}A^{k-i}\sigma^i\left(\frac{-1}{aa_1}, \dots, \frac{-1}{aa_n}\right).$$

The matrix

$$A\sum_{k=0}^{n-1}(-\alpha)^k a^{k+2}\sum_{i=0}^{k}A^{k-i}\sigma^i\left(\frac{-1}{aa_1}, \dots, \frac{-1}{aa_n}\right)$$

is evidently diagonal. The element number l on the diagonal is given by

$$\begin{aligned}
\frac{1}{aa_l}\sum_{k=0}^{n-1}(-\alpha)^k a^{k+2}\sum_{i=0}^{k}\left(\frac{1}{aa_l}\right)^{k-i}\sigma^i\left(\frac{-1}{aa_1}, \dots, \frac{-1}{aa_n}\right) &= \\
\frac{a}{a_l}\sum_{k=0}^{n-1}(-\alpha)^k\sum_{i=0}^{k}\sigma^i\left(\frac{-1}{a_1}, \dots, \frac{-1}{a_n}\right)\left(\frac{1}{a_l}\right)^{k-i} &= \\
\frac{a}{a_l}\sum_{i=0}^{n-1}\sigma^i\left(\frac{-1}{a_1}, \dots, \frac{-1}{a_n}\right)\sum_{k=i} n-1\frac{(-\alpha)^k}{a_l^{k-i}} &= \\
\frac{a}{a_l}\sum_{i=0}^{n-1}a_l^i\sigma^i\left(\frac{-1}{a_1}, \dots, \frac{-1}{a_n}\right)\frac{\left(\frac{-\alpha}{a_l}\right)^n - \left(\frac{-\alpha}{a_l}\right)^i}{-\frac{\alpha}{a_l}-1} &= \\
-\frac{a}{a_l+\alpha}\sum_{i=0}^{n}a_l^i\sigma^i\left(\frac{-1}{a_1}, \dots, \frac{-1}{a_n}\right)\left(\left(\frac{-\alpha}{a_l}\right)^n - \left(\frac{-\alpha}{a_l}\right)^i\right) &= \\
-\frac{a(-\alpha)^n}{a_l+\alpha}\sum_{i=0}^{n}\frac{1}{a_l^{n-i}}\sigma^i\left(\frac{-1}{a_1}, \dots, \frac{-1}{a_n}\right) + \frac{a}{a_l+\alpha}\sum_{i=0}^{n}\sigma^i\left(\frac{-1}{a_1}, \dots, \frac{-1}{a_n}\right) &=
\end{aligned} \tag{23}$$

$$\frac{-a(-\alpha)^n}{a_l+\alpha}\prod_{i=0}^{n}\left(\frac{1}{a_l}-\frac{1}{a_i}\right)+\frac{a}{a_l+\alpha}\prod_{i=1}^{n}\left(1+\frac{\alpha}{a_i}\right) = 0+\frac{1}{a_l+\alpha}\prod_{i=1}^{n}(a_i+\alpha),$$

■

Proof of Theorem 2.

Because of Lemma 4, it is sufficient to prove the theorem only in Levi-Civita chart. Let the metrics g and $\bar{g}$ be given by

$$\begin{aligned} g(\dot{\bar{x}},\dot{\bar{x}}) &= \Pi_1(\bar{x})A_1(\bar{x}_1,\dot{\bar{x}}_1)+\Pi_2(\bar{x})A_2(\bar{x}_2,\dot{\bar{x}}_2)+\cdots+\\ &+ \Pi_m(\bar{x})A_m(\bar{x}_m,\dot{\bar{x}}_m), \end{aligned} \tag{24}$$

$$\begin{aligned} \bar{g}(\dot{\bar{x}},\dot{\bar{x}}) &= \frac{1}{\phi_1(\phi_1^{k_1}\phi_2^{k_2}\dots\phi_m^{k_m})}\Pi_1(\bar{x})A_1(\bar{x}_1,\dot{\bar{x}}_1)+\frac{1}{\phi_2(\phi_1^{k_1}\phi_2^{k_2}\dots\phi_m^{k_m})}\Pi_2(\bar{x})A_2(\bar{x}_2,\dot{\bar{x}}_2)+\cdots+\\ &+ \frac{1}{\phi_1(\phi_m^{k_1}\phi_2^{k_2}\dots\phi_m^{k_m})}\Pi_m(\bar{x})A_m(\bar{x}_m,\dot{\bar{x}}_m), \end{aligned} \tag{25}$$

For this metrics the operator $\bar{G}\stackrel{\text{def}}{=}(g\bar{g}^{-1})$ is given by the diagonal matrix $\mathrm{Diag}(\underbrace{\phi_1,\phi_1,\dots,\phi_1}_{k_1},\underbrace{\phi_2,\phi_2,\dots,\phi_2}_{k_2},\dots,\underbrace{\phi_m,\phi_m,\dots,\phi_m}_{k_m})$. It is easy to see that for any $\xi,\nu\in\mathcal{T}_xM^n$ we have $\bar{g}(\xi,\nu)=g(G\xi,\nu)$. Then the formula for $\tilde{I}_k$ is

$$\tilde{I}_k(\xi,\xi)=\left(\frac{1}{det(G)}\right)^{\frac{k+2}{n+1}}g(G\sum_{i=0}^{k}G^{k-i}c_i\xi,\xi),$$

and f_α is given by

$$\begin{aligned} f_\alpha &= \sum_{k=0}^{n-1}(-\alpha)^k\tilde{I}_k\\ &= \sum_{k=0}^{n-1}\left(\frac{1}{det(G)}\right)^{\frac{k+2}{n+1}}g(G\sum_{i=0}^{k}G^{k-i}c_i\xi,\xi)\\ &= g(G\sum_{k=0}^{n-1}\left(\frac{1}{det(G)}\right)^{\frac{k+2}{n+1}}\sum_{i=0}^{k}G^{k-i}c_i\xi,\xi). \end{aligned} \tag{26}$$

Combining (26) with Lemma 5, we have that the form f_α is given by

$$\begin{aligned} f_\alpha(\dot{\bar{x}},\dot{\bar{x}}) &= \frac{(\phi_1+\alpha)^{k_1}(\phi_2+\alpha)^{k_2}\dots(\phi_m+\alpha)^{k_m}}{\phi_1+\alpha}\Pi_1(\bar{x})A_1(\bar{x}_1,\dot{\bar{x}}_1)\\ &+ \frac{(\phi_1+\alpha)^{k_1}(\phi_2+\alpha)^{k_2}\dots(\phi_m+\alpha)^{k_m}}{\phi_2+\alpha}\Pi_2(\bar{x})A_2(\bar{x}_2,\dot{\bar{x}}_2)+\cdots+\\ &+ \frac{(\phi_1+\alpha)^{k_1}(\phi_2+\alpha)^{k_2}\dots(\phi_m+\alpha)^{k_m}}{\phi_m+\alpha}\Pi_m(\bar{x})A_m(\bar{x}_m,\dot{\bar{x}}_m). \end{aligned} \tag{27}$$

Then the metric

$$g_\alpha\stackrel{\text{def}}{=}\left(\frac{det(g)}{det(f_\alpha)}\right)^{\frac{2}{n-1}}f_\alpha$$

is given by the formula

$$g_\alpha(\dot{x},\dot{x}) = \frac{1}{(\phi_1+\alpha)^{k_1}\cdots(\phi_m+\alpha)^{k_m}}\left\{\frac{1}{\phi_1+\alpha}\Pi_1 A_1 + \cdots + \frac{1}{\phi_m+\alpha}\Pi_m A_m\right\},$$

and is evidently geodesically equivalent to g, ∎

7. Geodesically equivalent metrics on the ellipsoid.

Proof of Theorem 3.

We show that in the elliptic coordinate system the restriction of the metrics

$$ds^2 \stackrel{\text{def}}{=} \sum_{i=1}^{n}(dx^i)^2 \text{ and } dr^2 \stackrel{\text{def}}{=} \frac{1}{\sum_{i=1}^{n}\left(\frac{x^i}{a_i}\right)^2}\left(\sum_{i=1}^{n}\frac{(dx^i)^2}{a_i}\right)$$

to the ellipsoid $\sum_{i=1}^{n}\dfrac{(x^i)^2}{a_i}=1$ have Levi-Civita local form, and therefore are geodesically equivalent. More precisely, consider elliptic coordinates $\nu^1,\dots,\nu^n$. Without loss of generality we can assume that $a^1<a^2<\dots<a^n$. Then the relation between the elliptic coordinates $\bar{\nu}$ and the Cartesian coordinates $\bar{x}$ is given by

$$x^i = \sqrt{\frac{\prod_{j=1}^{n}(a^i-\nu^j)}{\prod_{j=1,j\neq i}^{n}(a^i-a^j)}}.$$

Recall that the elliptic coordinates are non-degenerate almost everywhere, and the set

$$\{\nu^1=0, a_1<\nu^2<a_2, a_2<\nu^3<a_3,\dots,a_{n-1}<\nu^n<a^n\}$$

is the part of the ellipsoid $\{x^1>0,x^2>0,\dots,x^n>0\}$. Since for any i the symmetry $x^i\to -x^i$ takes the ellipsoid to the ellipsoid and preserves the metrics ds^2 and dr^2, it is sufficient to check the statement of the theorem only in the quadrant $\{x^1>0,x^2>0,\dots,x^n>0\}$.

In the elliptic coordinates the restriction of the metric ds^2 to the ellipsoid has the following form

$$\sum_{i=1}^{n}\Pi_i A_i (d\nu^i)^2,$$

where $\Pi_i \stackrel{\text{def}}{=} \prod_{j=1,j\neq i}^{n}(\nu^i-\nu^j)$, and $A_i \stackrel{\text{def}}{=} \frac{\nu^i}{\prod_{j=1}^{n}(a^j-\nu^i)}$. The restriction of the metric dr^2 to the ellipsoid is

$$(a^1a^2\dots a^n)\sum_{i=1}^{n}\rho^i\Pi_i A_i(d\nu^i)^2,$$

where $\rho^i \stackrel{\text{def}}{=} \dfrac{1}{\nu^i(\nu^1\nu^2\dots\nu^n)}$. We see that the metrics ds^2, dr^2 have Levi-Civita local form, and therefore are geodesically equivalent. ∎

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Предложен простой подход для нахождения интегралов гамильтоновых систем, если известны траекторные отображения двух гамильтоновых систем. Приводится точная формула. В качестве примера доказано, что если на многообразии заданы две римановы метрики, являющиеся геодезически эквивалентными, то существует большой набор интегралов. Доказанная теорема является обобщением теоремы Пенлеве – Лиувилля.