Nijenhuis geometry and applications: finite-dimensional reductions and integration in quadratures of certain nondiagonalisable systems of hydrodynamic type

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(work in progress, paper in preparation)

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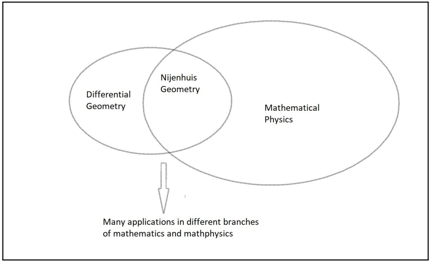
The proejct is new

- The project Nijenhuis Geometry was initiated in 2018 by ABAKVM.



- The initial plan was:
  - Develop theoretical background of Nijenhuis Geometry following the standard path in mathematics
    (A) Local description
    (B) Singular points
    (C) Global properties
  - Apply obtained results in differential geometry and mathematical physics
- The plan has to be essentially modified.

The reason for modification: First theoretic results brought too many immediate applications

Pure theoretical study of Nijenhuis Geometry

1. A. Bolsinov, A. Konyaev, V. Matveev, Nijenhuis Geometry, Adv. Math. **394**(2022)
2. A. Konyaev, Nijenhuis geometry II: left-symmetric algebras and linearization problem, Diff. Geom. App. **74**(2021)
3. A. Bolsinov, A. Konyaev, V. Matveev, Nijenehuis Geometry III: gl-regular Nijenhuis operators, Rev. Mat. Iberoam. (2023)

Applications

1. A. Bolsinov, A. Konyaev, V. Matveev, Applications of Nijenhuis geometry: Nondegenerate singular points of Poisson-Nijenhuis structures, European Journal of Mathematics (2021)
2. A. Bolsinov, A. Konyaev, V. Matveev, Applications of Nijenhuis geometry II: maximal pencils of multihamiltonian structures of hydrodynamic type, Nonlinearity **34** (2021)
3. A. Bolsinov, A. Konyaev, V. Matveev, Applications of Nijenhuis geometry III: Frobenius pencils and compatible non-homogeneous Poisson structures, arXiv:2112.09471
4. A. Konyaev, Geometry of inhomogeneous Poisson brackets, multicomponent Harry Dym hierarchies and multi-component Hunter-Saxton equations, Russ. J. Math. Physics **24**(2022)
5. A. Bolsinov, A. Konyaev, V. Matveev, Orthogonal separation of variables for spaces of constant curvature, arXiv:2212.01605
6. A. Bolsinov, A. Konyaev, V. Matveev, Applications of Nijenhuis Geometry IV: multicomponent KdV and Camassa-Holm equations, (with A. Bolsinov and A. Konyaev), Dynamics of Partial Differential Equations **20**(2023)

Plan of my talk

- I will start with a short introduction to Nijenhuis geometry emphasizing the results I will use in my talk
- I will recall the notion of (integrable) systems of hydrodynamic type
- The new results of my talk will be construction and integration in quadratures of certain (integrable) systems of hydrodynamic type

Short introduction to Nijenhuis geometry

Definition

By Nijenhuis operators we understand (1, 1)-tensors $L = \big(L^i_j(x)\big)$ with vanishing Nijenhuis torsion.

- Recall that:
  - For two vector fields ξ, η , its Nijenhuis torsion is given by:

$$\mathcal{N}_L(\xi,\eta) = L^2[\xi,\eta] + [L\xi, L\eta] - L[L\xi,\eta] - L[\xi, L\eta].$$

  - $\mathcal{N}_L$ is a (1,2)-tensor field; its local tensorial expression is

$$(\mathcal{N}_L)^i_{jk} = L^\ell_j \frac{\partial L^i_k}{\partial x^\ell} - L^\ell_k \frac{\partial L^i_j}{\partial x^\ell} - L^i_\ell \frac{\partial L^\ell_k}{\partial x^j} + L^i_\ell \frac{\partial L^\ell_j}{\partial x^k}.$$

- A manifold M endowed with such an operator is called a Nijenhuis manifold.
- Nijenhuis geometry studies Nijenhuis manifolds

Motivation of Nijenhuis Geometry:

Nijenhuis geometric structure is defined by means of a matrix $\left(L^i_j(x)\right)$ which is a tensor field of type $(1,1)$, satisfying one differential condition, Nijenhuis identity $\mathcal{N}_L = 0$:

Let me list some similarly defined geometrical structures:

- Symmetric tensors of type $(0,2)$ give rise to Riemannian geometry.
- Symmetric tensors of type $(2,0)$ give rise to sub-Riemannian geometry.
- Skew-symmetric tensors of type $(2,0)$ give rise to Poisson geometry, and of type $(0,2)$ to symplectic geometry.
- General tensors of type $(0,2)$ give rise to Kähler geometry.

Tensors of type $(1,1)$ is the only case left!

Why the Nijenhuis Identity?

- Nijenhuis identity is the *simplest* differential geometric condition on (1,1)-tensors (Puninskii 2014):
- (1,1)-tensors satisfying Nijenhuis identity pop up in many unrelated problems of differential geometry and mathematical physics

Local results in Nijenhuis geometry which I will need in the selected application

Definition

A (1,1) tensor L is gl-regular, if each of its eigenvalues has geometric multiplicity 1

(that is, all Jordan blocks have mutually different eigenvalues).

For the “applications” part, I will need a local description of gl-regular Nijenhuis operators near almost every point.

Splitting Lemma for Nijenhuis operators (simplified version).

Splitting Lemma. Suppose eigenvalues of a Nijenhuis operator L at $p \in M^n$ are all real and are decomposed into k disjunct subsets:

$$\mathrm{spectrum}(L)(p) = \mathrm{spectrum}_1 \dot\cup \cdots \dot\cup \, \mathrm{spectrum}_k.$$

Then, locally there exists a local coordinate system such that L is blockdiagonal, $L = \mathrm{diag}(L_1, \ldots, L_k)$, each L_i is a Nijenhuis operator depending on the own coordinates only, and at the point p we have

$$\mathrm{spectrum}(L_1)(p) = \mathrm{spectrum}_1, \ldots, \mathrm{spectrum}(L_k)(p) = \mathrm{spectrum}_k.$$

For simplicity, we will further always assume that the eigenvalues of all our Nijenhuis operators are real. In our papers we treat also complex-valued eigenvalues and this case does not pose essential difficulties.

Success report. In order to describe locally gl-regular Nijenhuis operators (with real eigenvalues), one can assume that the operator has only one Jordan block.

How the Splitting Lemma works if all Jordan blocks have dimension 1 (Haantjes Theorem)

Theorem (Haantjes 1955/Nijenhuis 1951) Suppose L is a Nijenhuis operator on M^n such that at $p \in M^n$ it has n different real eigenvalues. Then, there exists a local coordinate system $(x^1, ..., x^n)$ near p such that

$$L = \mathrm{diag}(f_1(x^1), ..., f_n(x^n)) = \begin{pmatrix} f_1(x^1) & & \\ & \ddots & \\ & & f_n(x^n) \end{pmatrix}$$

where $f_i = f_i(x^i)$ are functions of one (indicated) variable.

If all $f_i(x^i)$ are not constant, one can take them as coordinates, so

$$L = \begin{pmatrix} x^1 & & \\ & \ddots & \\ & & x^n \end{pmatrix}$$

Local description for (one) Jordan block with a constant eigenvalue (Thompson Theorem)

Theorem (Thompson 2002). Let L be a Nijenhuis operator conjugated to a single Jordan block with a constant eigenvalue λ. Then, there exists a coordinate system such that all components of L are constants, so in a certain coordinate system

$$L - \lambda \mathrm{Id} = L_c = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Local description for (one) Jordan block with a nonconstant eigenvalue

Theorem (BKM 2022). Let L be a Nijenhuis operator conjugated to a single Jordan block with eigenvalue λ such that $d\lambda \neq 0$. Then, there exists a coordinate system such that

$$L = L_{nc} = \begin{pmatrix} x^n & x^{n-1} & x^{n-2} & \dots & x^1 \\ 0 & x^n & x^{n-1} & \dots & x^2 \\ & & \ddots & \ddots & \\ 0 & \dots & 0 & x^n & x^{n-1} \\ 0 & 0 & 0 & \dots & x^n \end{pmatrix}.$$

Success report: Splitting Lemma and Theorem above give a local description of gl-regular Nijenhuis operators near almost every point such that every eigenvalue is not constant: in a coordinate system

$$L = \mathrm{diag}(L_1, ..., L_k) \text{ such that } L_i \text{ is as above.}$$

(Every L_i depends on its own coordinates)

The application selected for this talk: (integrable) systems of hydrodynamic type

- As mathematical objects, they are PDE systems of the form

$$\frac{\partial}{\partial t} u^i(t,x) = A(u)^i_s \frac{\partial}{\partial x} u^s(t,x) \quad \text{(shortly: } u_t = Au_x).$$

  Here $u(t,x) = (u^1(t,x), ..., u^n(t,x))$ is unknown vector-function of two variables (t,x) and A is a matrix depending on $(u^1, ..., u^n)$ with no explicit dependence on t and x.

- $(u^1, ..., u^n)$ should be viewed as local coordinate system on a manifold. The matrix A is then an operator (=$(1,1)$-tensor); after a transformation $(u^1, ..., u^n) \mapsto (\tilde{u}^1, ..., \tilde{u}^n)$ the equation above has the same form with $\tilde{A}(\tilde{u}) = JA(u)J^{-1}$.

Let us compare the ODE $\frac{d}{dt}u(t) = V(u)$ describing dynamics of particles with the PDE $u_t = Au_x$:

- Infinite-dimensional "particles" are curves $u(x)$; the "configuration space" is the set of all (real-analytic curves)
- In $u_t = Au_x$: the t-derivatives for fixed $t = t_0$ are expressed in terms of $u(t_0, x)$.
- For initial real-analytic data $\hat{u}(x) = u(t_0, x)$ there exists a unique local solution $u(t, x)$ by the Kovalevskaya Theorem.

Examples of systems of hydrodynamic type

- Systems of hydrodynamic type naturally arise in continuum mechanics, in the theory of nonlinear dispersive waves and in the theory of shock waves. In mathematics it was studied before, e.g. by Riemann and by Hopf.
- I will give two mathematical examples.

Example: Cauchy-Riemann conditions $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ can be written in the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_y = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x .$$

This is a system of hydrodynamic type with $n = 2$, $t = y$ and $A = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$.

Of course in this example the matrix A does not depend on u, and is relatively simple. In many mathematical problems A is complicated.

A more complicated example: polynomial integrals for geodesic flows on surfaces

Example: Consider a 2-dimensional Riemannian metric in the conformal form $g = \lambda(x_1, x_2)(dx_1^2 + dx_2^2)$ and let us look for an integral F of the geodesic flow which is homogeneous in momenta of degree 3. Then, by a known (e.g. Kolokoltsov 1982) coordinate transformation we may think that

$$F = {p_1}^3 a(x_1, x_2) + {p_1}^2 p_2 b(x_1, x_2) + p_1 {p_2}^2 (a(x_1, x_2) - 1) + {p_2}^3 b(x_1, x_2)$$

The condition that $\{F, H\} = 0$ is then the system $u_t = Au_x$ of hydrodynamic type with $n = 3$, $t = x_1$, $x_2 = x$,

$$u = \begin{pmatrix} \lambda(x_1, x_2) \\ a(x_1, x_2) \\ b(x_1, x_2) \end{pmatrix} \text{ and } A = \begin{bmatrix} -3\,\frac{b}{a-1} & 0 & 2\,\frac{\lambda}{a-1} \\ -\frac{1}{2}\,\frac{b(8a+1)}{\lambda(a-1)} & 0 & 3\,\frac{a}{a-1} \\ -\frac{3b^2 - a^2 + 2a - 1}{\lambda(a-1)} & -1 & 2\,\frac{b}{a-1} \end{bmatrix}.$$

What does integrability mean in this context?

- Let me first recall the definition in the finite-dimensional case.
  - The established definition is as follows: an ODE-system $\dot{u} = V(u)$ is integrable, if it has sufficiently many
  - functions $f_1, f_2, \ldots, f_{k-1}$ that are constant on the solutions,
  - vectors fields $W_1, \ldots, W_{n-k}$ which commute with V.
- In the finite-dimensional case, integrability implies that one can reduce solving of the system to integration in quadratures, i.e., to integration of closed 1-forms and solving the systems of functional equations (S. Lie 1884; sometimes called Arnold-Kozlov Theorem).
- The infinite-dimensional case:
  - There exists an established definition, which is visually similar (= uses similar words) to the finite-dimensional case.
  - There is also a collection of methods to deal with integrable infinite-dimensional integrable systems
  - Unfortunately in most cases this collection of methods does not help: it merely reduces one problem to an equally complicated another one
- **In my talk I will call a system of hydrodynamic type integrable, if one can integrate it in quadratures (for most initial data, or at least for sufficiently many initial data).**
- This definition of integrable systems was used by classical mathematicians (e.g., Jacobi, Lie, Pailnleve, Cartan, Eisenhard) and physicists

The list of known integrable systems of hydrodynamic type, in the sense of my definition, is very short; moreover, in all known cases the operator A is diagonal in a coordinate system

- Simplest nontrivial example: assume L is Nijenhuis operator with n different eigenvalues. Then, the equation $u_t = L u_x$ can be integrated in quadratures.
  - **Proof (Folklore).** By Haantjes Theorem, we may assume $L = \mathrm{diag}(u^1, \dots, u^n)$, since one can do it by a diffeomorphic coordinate change. The equation $u_t = L u_x$ decouples then in n independent Hopf equations $\begin{cases} u^1_t = u^1 u^1_x \\ \vdots \\ u^n_t = u^n u^n_x \end{cases}$
- Bi-Hamiltonian systems are integrable (Dubrovin-Novikov, Tsarev). Many examples of Bi-Hamiltonian systems were constructed by Ferapontov et al and Magri et al.
- Weakly-nonlinear integrable systems (of hydrodynamic type) are integrable in my sense (Ferapontov 1991–1992). Our main result will be generalisation of these results to the nondiagonal case.

Nijenhuis geometry allowed us to generalise all results from the list above to nondiagonal (gl-regular) operators

- The case $u_t = L u_x$, where L is a gl-regular Nijenhuis operator, possibly nondiagonalisable, was done in BKM 2023.
- Nondiagonal bi-Hamiltonian examples were constructed in BKM 2021
- **The remaining part of this talk will be about the nondiagonal analogy of the weakly-nonlinear case.**

What problem I will discuss (solve and give proofs) in the remaining part of the talk

- I start with gl-regular Nijenhuis operator L with nonconstant eigenvalues (we locally described them in the first part).
- I consider the PDE-system $u_t = Au_x$ with

$$A = \det(L)L^{-1}.$$

- I explain how to solve this PDE in quadratures, that is, using integration of closed 1-forms and solving of systems of functional equations
- If time allows, I will speak about finite-dimensional reductions of such systems.

Preliminary work: conservation laws of the Nijenhuis operators

Def. Let L be a Nijenhuis operator. By its conservation law we understand a function $f : M \to \mathbb{R}$ such that $d(L^* df) = 0$.

The name "conservation law" will not be explained; they are infinite-dimensional analogues of integrals (=functions constant on trajectories) for finite-dimensional integrable systems and their first examples are related to the term "conservation laws" used in physics, e.g. in thermodynamics.

Theorem (Folklore). If f is conservation law for a Nijenhuis operator L, then it is conservation law for L^k with any k.

Starting with one conservation law , we then construct infinitely many of them (from which we will need the first n):
$f_1 := f$, f_2 satisfying $df_2 = L^* df_1$, f_3 satisfying $df_3 = L^* df_2 = (L^2)^* df_1, \ldots$

We will call the sequence of functions satisfying $df_i = L^* df_{i-1}$ hierarchy of conservation laws.

Our first new result is a local description of conservation laws for gl-regular operators. I will first give a series of examples and then explain that there are no other examples.

“Splitting Lemma” for conservation laws

Splitting Lemma for Nijenhuis operators. Suppose eigenvalues of a Nijenhuis operator L at $p \in M^n$ are decomposed into k disjunct subsets:

$$\mathrm{spectrum}(L)(p) = \mathrm{spectrum}_1 \dot\cup \cdots \dot\cup \, \mathrm{spectrum}_k.$$

Then, locally there exists a local coordinate system such that L is block-diagonal, $L = \mathrm{diag}(L_1, \ldots, L_k)$, each L_i is a Nijenhuis operator depending on the own coordinates only, and at the point p we have

$$\mathrm{spectrum}(L_1)(p) = \mathrm{spectrum}_1, \ldots, \mathrm{spectrum}(L_k)(p) = \mathrm{spectrum}_k.$$

Theorem. Consider the “blockdiagonal” gl-regular Nijenhuis operator $L = \mathrm{diag}(L_1, \ldots, L_k)$. Then, every conservation law for L is the sum $h_1 + ... + h_k$, where for every $i = 1, ..., k$ the function h_i depends on the coordinates of the ith block and is a conservation law for L_i.

Success report: By Theorem above, in order to describe all conservation laws for gl-regular Nijenhuis operators in a neighbourhood of almost every point, it is sufficient to describe conservation laws for L which is conjugate to a Jordan block. I will explain how it works first in the diagonal case and then generally.

“Diagonal” example (well known)

Conservation law is a function $f : M \to \mathbb{R}$ such that $d(L^* df) = 0$. For gl-regular $L = diag(L_1, \dots, L_k)$ every conservation law is the sum of conservation laws for L_i.

In dimension 1, a Nijenhuis operator with nonconstant eigenvalue is up to a coordinate change $udu \otimes \frac{\partial}{\partial u} = (u)$.
Any function $h(u)$ is a conservation law since any 1-form is closed. The hierarchy of the conservation laws is then $h(u)$, $\int_{\hat{u}}^{u} \xi h'(\xi) d\xi$, $\int_{\hat{u}}^{u} \xi^2 h'(\xi) d\xi$,..., $\int_{\hat{u}}^{u} \xi^k h'(\xi) d\xi$,... .

Corollary. Suppose $L = \mathrm{diag}(u^1, ..., u^n)$. Then, near a point where $u^1, ..., u^n$ are mutually different, every conservation law has the following form

$$f = h_1(u^1) + ... + h_n(u^n)$$

Then, corresponding hierarchy is

$$f_2 = \int_{\hat{u}^1}^{u^1} \xi h'_1(\xi) d\xi + ... + \int_{\hat{u}^n}^{u^n} \xi h'_n(\xi) d\xi, \ \dots$$

$$f_k = \int_{\hat{u}^1}^{u^1} \xi^{k-1} h'_1(\xi) d\xi + \dots + \int_{\hat{u}^n}^{u^n} \xi^{k-1} h'_n(\xi) d\xi, \ \dots$$

Conservation laws for the Jordan-block case

Now consider the Nijenhuis operators in the form

$$L_c = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad L_{nc} = \begin{pmatrix} u^n & u^{n-1} & u^{n-2} & \dots & u^1 \\ 0 & u^n & u^{n-1} & \dots & u^2 \\ & & \ddots & \ddots & \\ 0 & \dots & 0 & u^n & u^{n-1} \\ 0 & 0 & 0 & \dots & u^n \end{pmatrix}.$$

Next, for any n functions $h_1, \dots, h_n$ of one variables we consider the functions $f_1, \dots, f_n$ given by

$$h_1(L_{nc})L_c^{n-1} + \dots + h_{n-1}(L_{nc})L_c + h_n(L_{nc}) = f_1 L_c^{n-1} + f_2 L_c^{n-2} + \dots + f_n \mathrm{Id}$$

Theorem. The functions $f_1, \dots, f_n$ are conservation laws both for L_c and for L_{nc}. Moreover, every conservation law of L_c or of L_{nc} can be constructed in this way.

We see that the “freedom” is the same as in the “diagonal case”: a conservation law is determined by n functions $h_1, \dots, h_n$ of 1 variable each.

Success report. We described all conservation laws of gl-regular Nijenhuis operators (at almost every point). Starting from one of them (we call it h), one constructs the whole hierarchy by integrating of closed forms $L^* dh$, $(L^2)^* dh$, $(L^3)^* dh$.

"Splitting lemma" for conservation laws and preliminary success report:

Splitting Lemma for Nijenhuis operators. Suppose eigenvalues of a Nijenhuis operator L at $p \in M^n$ are decomposed into k disjunkt subsets:

$$\mathrm{spectrum}(L)(p) = \mathrm{spectrum}_1 \dot\cup \cdots \dot\cup \mathrm{spectrum}_k.$$

Then, locally there exists a local coordinate system such that L is block-diagonal, $L = \mathrm{diag}(L_1, \dots, L_k)$, each L_i is a Nijenhuis operator depending on the own coordinates only, and at the point p we have

$$\mathrm{spectrum}(L_1)(p) = \mathrm{spectrum}_1, \dots, \mathrm{spectrum}(L_k)(p) = \mathrm{spectrum}_k.$$

Theorem. Let $L = \mathrm{diag}(L_1, \dots, L_k)$ where L_i are gl-regular Nijenhuis operators with pairwise disjunct spectra (= the operator L is gl-regular).

Then, every conservation law of L is the sum of conservation laws of L_i.

Success report. Theorem and examples above describe all possible pairs (gl-regular Nijenhuis operator, its conservation laws) near almost every point: Splitting Lemma describes all gl-regular Nijenhuis operators, and Theorem above all its conservation laws.

An integrable system of hydrodynamic type, its symmetries and integration in quadratures

Let L be a gl-regular Nijenhuis operator. Define $(1,1)$ tensors A_i via

$$\det(\lambda \mathrm{Id} - L)(\lambda \mathrm{Id} - L)^{-1} = \lambda^{n-1} \underbrace{\mathrm{Id}}_{A_n} + \lambda^{n-2} A_1 + \cdots + (-1)^{n-1} \underbrace{\det(L) L^{-1}}_{A_1}.$$

Theorem (BKM 2022). The systems of hydrodynamic type corresponding to the operators $A_1, \ldots, A_n$ commute, that is, the system of n^2 PDEs on n functions $u^1(t_1, \ldots, t_n)$ given by

$$\frac{\partial u}{\partial t_i} = A_i \frac{\partial u}{\partial x} \qquad (*)$$

is compatible, that is, for any initial real analytic curve $\hat{u}(x)$ there exists a real-analytic solution $u(t_1, \ldots t_{n-1}, t_n)$ such that $u(0, \ldots, 0, x) = \hat{u}(x)$.

Remark. The variable t_n above should be viewed as the variable x. That is, the ith equation is actually $\frac{\partial u}{\partial t_i} = A_i \frac{\partial u}{\partial t_n}$. Note that since $A_n = \mathrm{Id}$, the nth equation from Theorem above reads $\frac{\partial u}{\partial t_n} = \frac{\partial u}{\partial x}$.

Historical remark and credits. For diagonal L, Theorem was known in 1990th due to a series of works of Ferapontov and Pavlov. For arbitrary operator L, Theorem can be obtained from the results of Lorenzoni-Magri 2005.

Integration in quadratures of the system $(*)$

$$\frac{\partial u}{\partial t_i} = A_i \frac{\partial u}{\partial x} \qquad (*)$$

Main Theorem. Let L be a gl-regular Nijenhuis operator and $f_1, \dots, f_n$ the first n elements of a hierarchy of conservation laws (i.e., $df_i = L^* df_{i-1}$). Then, near the points where functions $f_1, \dots, f_n$ are functionally independent, any vector-valued function $u(t_1, \dots, t_n)$ satisfying $(**)$ is a solution of $(*)$.

$$f_i(u(t_1, \dots, t_n)) = t_i + \mathrm{const}_i \qquad (**)$$

Moreover, for almost every solution $u(t_1, \dots, t_n)$ of $(*)$ there exists a hijerarchy of the conservation laws $f_1, \dots, f_n$ such that they are functionally independent and such that $u(t_1, \dots, t_n)$ satisfies $(**)$.

Success report: We reduced solving of the PDE-system $(*)$ to solving the system $(**)$, that is, to algebraic operations and integrations. In the classical literature, it refers to as "solving, or integrating, in quadratures". Almost all solutions $(u(t_1, \dots, t_n))$ of $(*)$ can be obtained by this procedure.

Proof of Main Theorem

Main Theorem. Let L be a gl-regular Nijenhuis operator and $f_1, \dots, f_n$ the first n elements of a hierarchy of conservation laws such that the functions $f_1, \dots, f_n$ are functionally independent. Then, any vector-valued function $u(t_1, \dots, t_n)$ satisfying $f_i(u(t, \dots, t_n)) = t_i + \mathrm{const}_i$ is a solution of $\frac{\partial u}{\partial t_i} = A_i \frac{\partial u}{\partial x}$ with $x = t_n$.

Proof. Since $f_1, ..., f_n$ are functionally independent, they form a coordinate system. In this coordinate system, the covectors $df_1, ..., df_n$ are given by

$$\begin{aligned} df_1 &= (1 \quad 0 \quad \dots \quad 0) \\ df_2 &= (0 \quad 1 \quad \dots \quad 0) \\ &\vdots \\ df_2 &= (0 \quad 0 \quad \dots \quad 1)\ . \end{aligned}$$

The condition $L^* df_i = df_{i+1}$ implies that in this coordinate system

$$L = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & 1 \\ \sigma_n & \sigma_{n-1} & \dots & \sigma_2 & \sigma_1 \end{pmatrix}$$

The matrices of these form (called companion matrices by Frobenius (Germ: Begleitmatrix)) are frequently used e.g. in the standard University course of basic linear algebra, in particular because the characteristic polynomial of such a matrix is given by

$$\chi_L = \det(\lambda \mathrm{Id} - L) = \lambda^n - \sigma_1 \lambda^{n-1} - \dots - \sigma_n.$$

Preliminary algebraic Lemma

Lemma. For L given by

$$L = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ \sigma_n & \sigma_{n-1} & \dots & \sigma_2 & \sigma_1 \end{pmatrix}$$

define $A_1, \dots, A_n$ by

$$\det(\lambda \mathrm{Id} - L)(\lambda \mathrm{Id} - L)^{-1} = \lambda^{n-1} \underbrace{\mathrm{Id}}_{A_n} + \lambda^{n-2} A_{n-1} + \dots + (-1)^{n-1} \underbrace{\det(L) L^{-1}}_{A_1}.$$

Then, the last column of A_i is $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1_i \\ \vdots \\ 0 \end{pmatrix}$.

Proof is an exercise for the 1st year students in linear algebra which you give to them when you read the linear algebra course next time.

Next, consider our system $(*)$:

$$\frac{\partial u}{\partial t_i} = A_i \frac{\partial u}{\partial x} \qquad (*)$$

In our coordinate system, since the last column of A_i is e_i, we have that $A_i e_n = e_i$. This implies that the vector valued functions

$$\begin{pmatrix} f_1(t_1, ..., t_n) \\ \vdots \\ f_n(t_1, ..., t_n) \end{pmatrix} \quad \text{given by} \quad \begin{pmatrix} f_1(t_1, ..., t_n) \\ \vdots \\ f_n(t_1, ..., t_n) \end{pmatrix} = \begin{pmatrix} t_1 + \text{const}_1 \\ \vdots \\ t_n + \text{const}_n \end{pmatrix}$$

solves the system $(*)$. Indeed, $\frac{\partial}{\partial t_i} \begin{pmatrix} t_1 + \text{const}_1 \\ \vdots \\ t_n + \text{const}_n \end{pmatrix} = e_i.$

Next, note that the condition $(**)$ reads, in coordinates $f_1, ..., f_n$,

$$f_i(u(t_1, \dots, t_n)) = t_i + \text{const}_i \qquad (**)$$

and solving these condition with respect to $f_1, ..., f_n$ we obtain

$$\begin{pmatrix} f_1(t_1, ..., t_n) \\ \vdots \\ f_n(t_1, ..., t_n) \end{pmatrix} = \begin{pmatrix} t_1 + \text{const}_1 \\ \vdots \\ t_n + \text{const}_n \end{pmatrix}.$$

We see that in the coordinate system $f_1, ..., f_n$ every solution of $(**)$ is a solution of $(*)$. Since the property to be a solutions of $(**)$ or a solution of $(*)$ does not depend on the coordinate system, in any coordinate system every solution of $(**)$ is a solution of $(*)$. **Main Theorem is proved.**

Initial value problem

Natural question. Initially we wanted to solve the system $u_t = A_1 u_x$ for the unknown function $u(t,x) = (u^1(t,x), \dots, u^{n-1}(t,x))$ with the initial conditions $u(0,x) = \hat{u}(x)$. We did the following:

- We enlarged the system up to $(*)$: we considered the system $u_{t_i} = A_i u_x$. The solutions of the initial system and of $(*)$ are in 1:1 correspondence. In particular, for any fixed $t_1, \dots, t_{n-1}$, the solution $u(t_1, \dots, t_n)$ of $(*)$ gives us a solution of $u_t = A_{n-1} u_x$ (we think $x = t_n$, $t = t_1$ and set all other $t_i = 0$) and any solution of $u_t = A_{n-1} u_x$ can be obtained by this procedure.
- Any solution of the algebraic system $(**)$ is a solution of $(*)$.
- But what about the initial value problem? Can we obtain, by this method, any initial curve $\hat{u}(x)$ as $u(0, \dots, 0, x)$? In other words, given an initial curve $\hat{u}(x)$, can one find a hierarchy of conservation laws $f_0, \dots, f_{n-1}$ such that

$$f_1(\hat{u}(x)) = \dots = f_{n-1}(\hat{u}(x)) = 0 \text{ and } f_n(\hat{u}(x)) = x \text{ ?} \qquad (***)$$

Theorem. For a generic (we understand the precise meaning) initial curve $\hat{u}(x)$ there exists a hierarchy of conservation laws such that $(***)$ is fulfilled. In order to find the hierarchy, one needs to solve an explicit system of n algebraic equations on n functions of one variable x. The coefficients of this system come from the first at most n $\frac{\partial}{\partial x}$-derivatives of $\hat{u}(x)$.

Known example (Ferapontov 1991, 1992 and partially Ferapontov-Fordy 1997)

Assume $L = \mathrm{diag}(u^1, \ldots, u^n)$. The conservation laws are:

$$\begin{array}{rcl} f_0 & = & h_1(u^1) + \cdots + h_n(u^n) \\ f_1 & = & \int_0^{u^1} \xi h_1'(\xi) d\xi + \cdots + \int_0^{u^n} \xi h_n'(\xi) d\xi \\ & \vdots & \\ f_{n-1} & = & \int_0^{u^1} \xi^{n-1} h_1'(\xi) d\xi + \cdots + \int_0^{u^n} \xi^{n-1} h_n'(\xi) d\xi. \end{array}$$

The Jacobi matrix $\left(\frac{\partial f}{\partial u}\right)$ is the product of the diagonal matrix and of the Wandermonde matrix:

$$\left(\frac{\partial f}{\partial u}\right) = \begin{pmatrix} h_1'(u^1) & & & \\ & h_2'(u^2) & & \\ & & \ddots & \\ & & & h_n'(u^n) \end{pmatrix} \begin{pmatrix} 1 & u^1 & \cdots & (u^1)^{n-1} \\ 1 & u^2 & \cdots & (u^2)^{n-1} \\ \vdots & & & \vdots \\ 1 & u^n & \cdots & (u^n)^{n-1} \end{pmatrix}.$$

We see that for different $u^1, \ldots, u^n$ and all $h_i' \neq 0$ the system

$$f_0(\hat{u}(x)) = \mathrm{const}_1, \cdots, f_{n-1}(\hat{u}(x)) = \mathrm{const}_{n-1} \qquad (***)$$

can be solved with respect to the functions h_i'. For these functions h_i', the vector-function $u(t_1, \ldots, t_n)$ implicitly given by

$$f_i(u(t_1, \ldots, t_n)) = t_i + \mathrm{const}_i \qquad (**)$$

solves the system $u_{t_i} = A_i u_{t_n}$, and in particular $u_t = \det(L) L^{-1} u_x$.

Finite-dimensional reduction of infinite-dimensional integrable systems

- There is no formal definition, different authors see it differently and unfortunately most of them do not give any definition
- Informally, one would like to embed a finite-dimensional integrable system, together with its symmetries, in the initial system. One then hope to use the initial finite-dimensional integrable system to understand better the behaviour of the infinite-dimensional one
- The famous example is the finite-gap solutions of KdV. As observed by Moser (1978) and Trubowitz et al (1976), and discussed by many authors including Al'ber (1979, 1982), Mamford (1979), Veselov (1980) and more recently (2022) by Blaszak and Szablikowski. These clever guys observed that finite-gap solutions are in 1:1 correspondence to a certain finite dimensional Hamiltonian system of the form $H = K_g + U$ admitting n commuting integrals which are quadratic in momenta (so the metric admits n commuting Killing tensors) .
- We will do a similar thing in our set-up: we show that our system $(*)$ restricted to a 2n-dimensional class of curves is in 1:1 correspondence to a finite-dimensional system of the form $H = K_g$. The metric admits n Killing tensors.

What do we need to do:

- We start with a gl-regular Nijenhuis operator L and consider the system $(*)$:

$$\frac{\partial u}{\partial t_i} = A_i \frac{\partial u}{\partial x} \qquad (*)$$

- We need to specify a class of curves such that it is invariant with respect to the system $(*)$.
- And relate the dynamics restricted to this class of curve to a finite-dimensional system, namely to a geodesic flows having n commuting Killing tensors.

I will do it on the next slides

Geodesically equivalent metrics compatible with a gl-nondegenerate Nijenhuis operator

Def. Let g be a metric (of any signature) and L be a g-selfadjoint $(1,1)$-tensor. We say that the pair (g, L) is geodesically compatible, if the following equation is fulfilled:

$$\nabla^g_k L_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik}$$

with $\lambda_i = \frac{1}{2} d \mathrm{trace}_g\, L$

Remark. The name "geodesically compatible" is due to the following fact: assume $\det(L) \neq 0$ and consider the metric $\bar{g}(\ ,\ ) := \frac{1}{\det(L)} g(L^{-1}\ ,\ )$. Then, the metric $\bar{g}$ has the same (unparameterized) geodesics with g if and only if (g, L) are geodesically compatible.

Fact (BM 2002). If L is geodesically compatible to g, then L is a Nijenhuis operator.

Given a gl-regular Nijenhuis operator L, how many metrics are geodesically compatible to L?

Theorem. Near almost every point, the space of metrics geodesically compatible to a gl-regular L is (explicitly) parameterized by n functions of one variable.

Example with Jordan block (BM 2015): Suppose

$$L = \begin{pmatrix} u_3 & u_2 & u_1 \\ & u_3 & u_2 \\ & & u_3 \end{pmatrix}.$$

Then, for any functions $f_1(u_3), f_2(u_3), f_1(u_3)$ of one variable u_3 the metric

$$g = \begin{pmatrix} 0 & 0 & f_1 u_2^2 \\ 0 & f_1 u_2^2 & u_2(f_1' u_2^2 + f_2 u_2 + f_1 u_1) \\ f_1 u_2^2 & u_2(f_1' u_2^2 + f_2 u_2 + f_1 u_1) & \frac{1}{2} f_1'' u_2^4 + f_2' u_2^3 + 2 f_1' u_1 u_2^2 + u_2^2 f_3 + f_2 u_1 u_2 + f_1 u_1^2 \end{pmatrix}$$

is geodesically compatible to L. Moreover, any metric geodesically compatible to L above has this form.

Remark. Splitting Lemma for L-geodesically compatible metrics g, L (e.g., BM 2012) shows that it is sufficient to study L-geodesically compatible metrics for one block only since the answer for many blocks is "glued" from the answers for single blocks.

Diagonal example is well known

Theorem (Levi-Civita 1896). Assume $L = \mathrm{diag}(u^1, ..., u^n)$. Then, the metric

$$g = \sum_i f_i(u^i) \left(\prod_{j \neq i} (u^i - u^j) \right) (du^i)^2$$

is geodesically compatible to L. Moreover, any metric geodesically compatible to that L has this form.

The freedom is n functions $f_1, ..., f_n$ of one variable.

Integrability of the geodesic flow of metric g admitting a geodesically compatible L

Theorem (M$\sim$-Topalov 1997, BM 2002). Suppose (g, L) is geodesically compatible. Then, for all λ, the functions

$$F_\lambda : TM \to \mathbb{R} \ , \quad F_\lambda(u, \xi) = \det(\lambda \mathrm{Id} - L) g\left((\lambda \mathrm{Id} - L)^{-1}\xi, \xi\right)$$

are commuting integrals of the geodesic flow of g.

Explanation of terminology: We identify by g the tangent and cotangent bundles; this gives us a symplectic structure on TM. For every parameter λ the Hamiltonian vector fields X_{F_λ} commute, and therefore define the (local) action of the group $\mathbb{R}^n$ on TM^n.

Remark. The family of integrals F_λ is polynomial in λ and is given by

$$F_\lambda = \lambda^{n-1} \underbrace{F_n}_{H_g} + \lambda_{n-2} F_{n-1} + ... + F_1$$

for quadratic integrals $F_n, ..., F_1$ (please excuse the ambiguity with the notations; F_i means the $i+1$st coefficient of the polynomial F_λ).

The system $(*)$ as the geodesic flow of a metric g

$$\frac{\partial u}{\partial t_i} = A_i \frac{\partial u}{\partial x} \qquad (*)$$

Theorem. Consider all (parameterised) geodesics of a metric g which is geodesically compatible with a gl-regular operator L. Then, these curves are invariant with respect to the system $(*)$, that is, if we start with a geodesic $\hat{u}(t_n)$ then for any fixed $\hat{t} = (\hat{t}_1, ..., \hat{t}_{n-1})$ the solution $u(t, x)$ of $(*)$ with the initial curve $u(0, ..., 0, x) = \hat{u}(x)$ the curve $t_n \mapsto u(\hat{t}, t_n)$ is also a geodesic.
Moreover, the flows of $(*)$ on this family of curves are equivalent to the flows of the Hamiltonian vector fields X_{F_i} on TM defined on the previous slide.

Theorem. For a generic initial curve $\hat{u}(x)$ there exists a metric geodesically compatible to L such that $\hat{u}(x)$ is its geodesic.

Remark. The embedding of TM to the space of geodesics is the most natural one: point (u, ξ) is sent to the geodesic starting from u in the direction ξ.

Conclusion

- Nijenhuis geometry is
  - a natural research direction which is useful for applications
  - new so there are many results which are easy to obtain and which later will be fundamental. In my talk I concentrated on demonstration of the strength of the Splitting Lemma; in our series of "Application of Nijenhuis geometry" papers we used virtually every single result of the "Nijenhuis geometry" papers.
- it is covariant, so one can prove the results in the adapted coordinate system
  - our description of conservation laws uses the coordinates coming from the splitting lemma
  - proof of the main result uses coordinates in which L has the 2nd companion form
- It allows to work with nondiagonal operators; in particular we constructed first nondiagonal system of hydrodynamic type integrable in quadratures
  - Our results are generalisations of the results of Levi-Civita 1896 and Ferapontov (Pavlov, Fordy) 1991 to nondiagonalisable operators L.
  - One of the philosophical ideas which lead to results is as follows: we rewrote the "diagonal" result in the terms of Nijenhuis operator, and then modified all the steps such that they go through also in the nondiagonal case.