Hausdorff h-measure and box dimension of some sets

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1 Background

Definition 1. *Let* (X, d) *be a metric space.*

If $r_0 > 0$ *is a given number, then, a continuous function* $h(r)$*, defined on* $[0, r_0)$, *nondecreasing and such that* $\lim\limits_{r\to 0} h(r) = 0$ *is called* a *measure function.*

If $0 < \delta < \infty$*,* E *is a subset of* X *and* h *is a measure function, then, the Hausdorff h-measure of* E *is defined by*:

$$H_h(E) = \lim_{\delta\to 0} \inf \left\{ \sum_i h(|U_i|) : E \subseteq \bigcup_i U_i : 0 < |U_i| < \delta \right\}.$$

where $||$ *denotes the diameter of a set.*

Particularly, when $h(r) = r^s, 0 < s < \infty$*, then the s-dimensional Hausdorff measure of E, denoted by* $H^s(E)$*, is obtained.*

The Hausdorff dimension of a nonempty set $E \subset X$ *is the number defined by*

$$\dim_H E = \inf \{s : H^s(E) = 0\} = \sup \{s : H^s(E) = \infty\}.$$

Remark. There are definitions where the covering of the set E is made with balls. The relation between the new measure, denoted by H'_h and H_h is: $H_h(E) \leq H'_h(E)$. Thus,

$$H'_h(E) < \infty \Rightarrow H_h(E) < \infty,$$

$$H'_h(E) = 0 \Rightarrow H_h(E) = 0, \tag{1}$$

and

$$H_h(E) > 0 \Rightarrow H'_h(E) > 0.$$

Definition 2. *Let* β *be a positive number and* E *a nonempty and bounded subset of the metric space* (X, d)*. Let* $N_\beta(E)$ *be the smallest number of sets of* *diameters at most* β *that cover* E*. Then the upper and lower Box dimension of* E *are defined by:*

$$\overline{\dim}_B E = \overline{\lim_{\beta \to 0}} \frac{\log N_\beta(E)}{-\log \beta}; \quad \underline{\dim}_B E = \underline{\lim_{\beta \to 0}} \frac{\log N_\beta(E)}{-\log \beta}.$$

If these limits are equal, the common value is called Box dimension of E *and is denoted by* $\dim_B E$*.*

Definition 3. *Let* $\varphi_1, \varphi_2 > 0$ *be functions defined in a neighborhood of* $0 \in \mathbf{R}^n$*. We say that* φ_1 *and* φ_2 *are equivalent and we denote by:* $\varphi_1 \sim \varphi_2$*, for* $x \to 0$*, if there exist* $r > 0$*,* $Q > 0$*, satisfying:*

$$\frac{1}{Q}\varphi_1(x) \le \varphi_2(x) \le Q\varphi_1(x), (\forall) x \in \mathbf{R}^n, |x| < r,$$

where for $x \in \mathbf{R}^n$*,* $x = (x_1, ..., x_n)$*,* $|x| = \sum_{i=1}^n x_i^2$*.*

An analogous definition can be given for $x \to \infty$*. In this case,* $\varphi_1 \sim \varphi_2$ *means that the previous inequalities have place in all the space.*

Remark. In what follows, if U is a set in a metric space, particularly in $\mathbf{R}^n$, $|U|$ means the diameter of U and if $x \in \mathbf{R}^n$, $|x|$ has the significance given in the definition 3.

Definition 4. *Let* $\delta > 0$ *and* $f : D \subset \mathbf{R}^n \to \overline{\mathbf{R}}$*.* f *is said to be a* δ *- class Lipschitz function if:*

$$|f(x+\alpha) - f(x)| < M|\alpha|^\delta, x \in D, \alpha \in \mathbf{R}^n, x + \alpha \in D, M > 0. \qquad (2)$$

f *is said to be a Lipschitz function if* $\delta = 1$*.*

Definition 5. *A set* $E \subset \mathbf{R}^n$ *is called a* k *- rectifiable set if there is a Lipschitz function* $f : \mathbf{R}^k \to \mathbf{R}^n$ *which applies a bounded subset of* $\mathbf{R}^k$ *on* E*.*

Definition 6. *The oscillation of the function* $f : [0,1] \to \mathbf{R}$ *on the interval* $[t_1, t_2] \subset [0,1]$*, denoted by* $R_f[t_1, t_2]$*, is the number*

$$R_f[t_1, t_2] = \sup_{t_1 \le t, u \le t_2} |f(t) - f(u)|.$$

We denote by $\Gamma(f)$ the graph of the function f.

In the second part of the paper we shall use the following results:

Lemma 1. ([6]) *If* E *is a set in* $\mathbf{R}^2$*, then*

$$\dim_H E \le \underline{\dim}_B E \le \overline{\dim}_B E. \qquad (3)$$

Remark. The previous lemma remains true in a nonempty compact metric space.

Lemma 2. ([6]) *Let* $f \in C[0,1]$*,* $0 < \beta < 1$ *and* m *be the least integer greater than or equal to* $1/\beta$*. If* N_β *is the number of the squares of the* $\beta-$ *mesh that intersects* $\Gamma(f)$*, then*

$$\beta^{-1} \sum_{j=0}^{m-1} R_f[j\beta, (j+1)\beta] \le N_\beta \le 2m + \beta^{-1} \sum_{j=0}^{m-1} R_f[j\beta, (j+1)\beta].$$

Lemma 3. ([3]) *If $E \subset \mathbf{R}^m$, $F \subset \mathbf{R}^n$, $f : E \to F$ is a surjective Lipschitz function, with the Lipschitz constant M and h is a measure function, then: $H_h(F) \leq H_h(M \cdot E)$.*

2 Results

2.1 Results in compact metric spaces

Theorem 1. *Let (X, d) be a nonempty compact metric space, with $\dim_H X = s$. Let h be a measure function such that there is $m > 0$, with $\frac{h(t)}{t^s} > m$. Suppose that there exist λ_0, $\alpha > 0$ such that for any set $E \subset X$, with $|E| < \lambda_0$, there is a mapping $\varphi : E \to X$ such that*:

$$\alpha d(x, y) \leq |E| d(\varphi(x), \varphi(y)), (\forall) x, y \in E.$$

Then $H_h(X) > 0$.

Proposition 1. *Let (X, d) be a nonempty compact metric space, with $\dim_H X = s$. Let h be a measure function such that there is $M > 0$, $\frac{h(t)}{t^s} < M$. Then $H_h(X) \leq M \cdot H^s(X)$.*

Remark. The theorem 1 and the proposition 1 give boundedness conditions for the Hausdorff h-measure of a compact metric space X, if $h(t) \sim t^s$.

Indeed, if $h(t) \sim t^s$, there is $Q > 0$, satisfying:

$$\frac{1}{Q} \cdot t^s \leq h(t) \leq Q \cdot t^s, (\forall) t > 0.$$

In the hypotheses of the mentioned theorems, for $m = \frac{1}{Q}$ and $M = Q$,

$$0 < \frac{1}{Q} \cdot \alpha^s \leq \frac{1}{Q} \cdot H^s(X) \leq H_h(X) \leq Q \cdot\ H^s(X).$$

Theorem 2. *Let (X, d) be a nonempty compact metric space, with $\dim_H X = s < \infty$. Suppose that there exists $a, r_0 > 0$ such that for any ball B in X of radius $r < r_0$ there is a mapping $\psi : E \to B$ such that:*

$$ard(x, y) \leq d(\psi(x), \psi(y)), (\forall) x, y \in X.$$

Let h be a measure function such that there is $M > 0$, with $\frac{h(t)}{t^s} < M$. Then $H_h(X) < Ms$.

Example. *Self-similar sets.* For $i = 1, ..., k$, let $\psi_i : \mathbf{R}^n \to \mathbf{R}^n$ be contracting similarity transformations, i.e.

$$d(\psi_i(x), \psi_i(y)) = c_i d(x, y),$$

where $0 < c_i < 1$ and d is the Euclidean metric. Then, there is an unique nonempty compact set $F \subset \mathbf{R}^n$ that is self-similar ([8]), i.e.

$$F = \bigcup_{i=1}^{k} \psi_i(F).$$

If $s = \dim_H(F)$ and h is a measure function as in the theorem 2, then $H_h(F) < \infty$.

Theorem 3. *Let (X, d) be a nonempty metric space, $E \subset X$, $E \neq \emptyset$, compact and h be a measure function such that $H'_h(E) < \infty$. Let $\boldsymbol{F}$ be the family of the closed sets in the topology induced by the metric. Suppose that there is $\varphi : \mathbf{F} \to \mathbf{R}_+$ such that φ is subadditive and:*

a. $\varphi(F) \geq 0, (\forall) F \subset \mathbf{F}$,

b. If $F \supset E$, then $\varphi(G) \geq b > 0$, where b is a constant,

c. There is a constant, $k \neq 0$, such that $\varphi(F) \leq kh(|F|)$.

Then, $H'_h(E) \geq b/k$.

Remark. The previous theorem remains true if $\mathbf{F}$ is replaced with the set $\mathbf{G}$ of the open sets.

The theorem 3 is a generalization of the sufficiency of the theorem 1 [9].

Theorem 4. *Let (X, d) be a nonempty metric space, $E \subset X$, $E \neq \emptyset$, compact and h be a measure function such that $H'_h(E) < \infty$* and $h(t) \sim P(t)e^{T(t)}, t \geq 0$, *where P and T are the polynomials:*

$$P(t) = \sum_{j=1}^{p} a_j t^j, p \geq 1, a_1 \neq 0, T(t) = \sum_{j=0}^{m} b_j t^j,$$

with positive coefficients. Then $H'_h(E) > 0$.

The result remains true if $p \geq 2, a_1 = 0$ and $\delta > 0$.

2.2 Results in $\mathbf{R}^n$, $n \in \mathbf{N}^*$

Theorem 5. *Let $\delta \geq 1$ and $f : [0, 1] \to \overline{\mathbf{R}}$ be a $\delta-$ class Lipschitz function. Suppose that h is a measure function such that: $h(t) \sim P(t)e^{T(t)}, t \geq 0$, where P and T are the polynomials:*

$$P(t) = \sum_{j=1}^{p} a_j t^j, p \geq 1, a_1 \neq 0, T(t) = \sum_{j=0}^{m} b_j t^j,$$

with positive coefficients. Then $0 < H'_h(\Gamma(f)) < \infty$.

Proof. In [1] we proved that in the hypotheses of the theorem, $H'_h(\Gamma(f)) < \infty$. Using the theorem 1 [9], we obtain that $H'_h(\Gamma(f)) > 0$.

Theorem 6. *Let $\delta > 0$ and $f : [0, 1] \to \overline{\mathbf{R}}$ be a δ- class Lipschitz function. If h is a measure function such that $h(t) \sim e^{-t} t^p, p \geq 2$, then $H_h(\Gamma(f)) < \infty$. The assertion remains true if $p \geq 1$ and $\delta \geq 1$.*

Remark. In the hypotheses of the previous theorem, if $P(t) = t^p, T(t) = -t$, then $P'(t) + P(t)T'(t) = t^{p-1}(p - t)$, which is not always positive, so the case treated differs from that studied in [3].

In our papers ([1] − [3]), the following functions were introduced:

$$g(x) = \begin{cases} 2x & ,0 \le x < \frac{1}{2} \\ -2\,(x-1) & ,\frac{1}{2} \le x < \frac{3}{2} \\ 2(x-2) & ,\frac{3}{2} \le x < 2 \end{cases} \tag{4}$$

$$f(x) = \sum_{i=1}^{\infty} \lambda_i^{s-2} g(\lambda_i x), (\forall) x \in [0,1]\,, \tag{5}$$

where g is given in (11), $s > 0$ and $\{\lambda_i\}_{i\in\mathbf{N}^*}$ is a sequence such that

$$(\exists)\,\varepsilon > 1 : \lambda_{i+1} \ge \varepsilon\lambda_i > 0, (\forall)\;\; i \in \mathbf{N}^*. \tag{6}$$

Theorem 7. *Let us consider the function f, defined above,* $s \in [0,2)$, *and the measure function h, such that* $h(t) \sim e^{-t}t^p, p \ge 2, t > 0$. *Then* $H_h(\Gamma(f)) < \infty$.

Theorem 8. *If* $f : [0,1] \to \mathbf{R}$ *is a* δ*-class Lipschitz function,* $\delta > 0$ *and h is a measure function such that* $\frac{h(t)}{t^p}$ *is bounded (or* $h(t) \sim t^p$*),* $p > 2$*, then* $H_h(\Gamma(f)) = 0$.

The assertion remains true if $p \ge 1$ *and* $\delta > 1$.

Remark. The previous theorem gives a better result as the theorem 6 [1], where it was proved that in the same hypotheses, $H_h'(\Gamma(f)) < \infty$.

Theorem 9. *If* $f : [0,1] \to \mathbf{R}$ *is a* δ*-class Lipschitz function,* $\delta > 0$ *and h is a measure function such that* $\frac{h(t)}{e^t t^p}$ *is bounded (or* $h(t) \sim e^t t^p$*) ,* $p \ge 2$*, then* $H_h(\Gamma(f)) = 0$.

The assertion remains true if $p \ge 1$ *and* $\delta > 1$.

Theorem 10. *If* $f : [0,1] \to \mathbf{R}$ *is a* δ*-class Lipschitz function,* $\delta > 0$ *and h is a measure function such that* $\frac{h(t)}{e^{-t} t^p}$ *is bounded (or* $h(t) \sim e^{-t} t^p$*),* $p \ge 2$*, then* $H_h(\Gamma(f)) = 0$.

The assertion remains true if $p \ge 1$ *and* $\delta > 1$.

Theorem 11. *If* $E \subset \mathbf{R}^n$ *is a k-rectifiable set and h is a measure function such that* $h(t) \sim t^p$*,* $p > 2$*, then* $H_h(E) = 0$.

Theorem 12. *If* $E \subset \mathbf{R}^n$ *is a k-rectifiable set and h is a measure function such that* $h(t) \sim e^t t^p$*,* $p > 2$*, then* $H_h(E) = 0$.

Theorem 13. *If* $E \subset \mathbf{R}^n$ *is a k-rectifiable set and h is a measure function such that* $h(t) \sim e^{-t} t^p$*,* $p > 2$*, then* $H_h(E) = 0$.

Theorem 14. *Let A be any Hausdorff measurable set, with* $\dim_H(A) = p$*,* $0 < p \le 1$ *on Ox axis and B any Lebesgue measurable set on Oy axis, such that* $0 < m_1(B) < \infty$*. If h is a measure function such that* $h(t) \sim e^t t^{p+1}$*, then there is a constant, c, such that*

$$H_h(A \times B) < c \cdot H^p(A) \cdot m_1(B),$$

where m_1 *is the linear Lebesgue measure.*

Theorem 15. *Let A be any Hausdorff measurable set, with $\dim_H(A) = p$, $0 < p \leq 1$ on Ox axis and B any Lebesgue measurable set on Oy axis, such that $0 < m_1(B) < \infty$. If h is a measure function such that $h(t) \sim e^{-t}t^{p+1}$, then there is a constant, c, such that*

$$H_h(A \times B) < c \cdot H^p(A) \cdot m_1(B),$$

where m_1 is the linear Lebesgue measure.

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