A Hardy Space for Singular Integrals on Non-compact Manifolds

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1. Introduction


Motivation

- **Heisenberg groups** are typical examples.
- The boundary of unbounded **model polynomial domains** in $\mathbb{C}^N$: For example, when $N = 2$, let $P(z)$ be a real, subharmonic, non-harmonic polynomial on $\mathbb{C}$ of degree $m$ and define

\[
\Omega = \{ (z, w) \in \mathbb{C}^2 : \Im m[w] > P(z) \}.
\]

Then $M = \partial \Omega \approx \mathbb{C} \times \mathbb{R}$ and the basic $(0, 1)$ Levi vector field is

\[
\overline{Z} = \frac{\partial}{\partial \overline{z}} - i \frac{\partial P}{\partial \overline{z}} \frac{\partial}{\partial t} \equiv X_1 + iX_2.
\]
Motivation (2)

- Heisenberg group $\mathbb{H}^2 \equiv M$: $P(z) = |z|^2$; $\Omega$ is called to be the **Siegel upper half-space**; moreover, $X_1 = \frac{1}{2} \frac{\partial}{\partial x} + y \frac{\partial}{\partial t}$ and $X_2 = \frac{1}{2} \frac{\partial}{\partial y} - x \frac{\partial}{\partial t}$.

- $\mathbb{R}^n$: $X_j = \frac{\partial}{\partial x_j}$, $j = 1, \ldots, n$.

- A new theory of **singular integrals** $\implies$ **optimal estimates** for solutions of the **Kohn-Laplacian** for certain classes of **model domains** in several complex variables.
Manifolds

- Let $M$ be a non-compact manifold, and let $\mathcal{X} = \{X_1, \ldots, X_k\}$ be real smooth vector fields on $M$, which together with their commutators of order $\leq m$, span the tangent space to $M$; ($\mathcal{X}$ is called to be of finite type $m$ and a control system.)

- Control distance (or Carnot-Caratheodory distance): For $x, y \in M$ and $\delta > 0$, let $AC(x, y, \delta)$ be the set of $\varphi : [0, 1] \to M$ absolutely continuous, $\varphi(0) = x, \varphi(1) = y$, and for $a.e. \, t \in [0, 1], \varphi'(t) = \sum_{j=1}^{k} a_j X_j(\varphi(t))$ with $|a_j| \leq \delta$. Define

$$d(x, y) = \inf \{\delta > 0 : AC(x, y, \delta) \neq \emptyset\}.$$  

Then $d$ is a metric.
Manifolds (2)

- When $M = \mathbb{C} \times \mathbb{R}$, we take Lebesgue measure on $M$. Then if $B(x, \delta) = \{ y \in M : d(x, y) < \delta \}$ for $\delta \in (0, \infty)$,

$$|B(x, \delta)| \sim \left( \sum_{k=2}^{m} \Lambda_k(x) \delta^k \right) \delta^2,$$

where $\Lambda_k(x) \geq 0$ are continuous functions on $M$.

- **Doubling property**: for all $\delta > 0$ and all $x \in M$,

$$|B(x, 2\delta)| \leq C_0 |B(x, \delta)|,$$

where $C_0 \geq 1$ is independent of $\delta$ and $x$. (Thus, $(M, d)$ together with Lebesgue measure is a space of homogeneous type.)
**Balls**

- Let $V_\delta(x) = |B(x, \delta)|$ and $V(x, y) = |B(x, d(x, y))|$ for all $x, y \in M$ and $\delta > 0$ (*volume functions*). Then

$$V(x, y) \sim V(y, x)$$

via (1.2).

- From (1.1), it is easy to deduce that for all $x \in M$, $\delta > 0$ and all $s \geq 1$,

$$s^4 |B(x, \delta)| \leq |B(x, s\delta)| \leq s^{m+2} |B(x, \delta)|.$$

(But, we have not $|B(x, \delta)| \sim \delta^d$ for certain $d > 0$.)
Approximation to the identity

- The **sub-Laplacian** $\mathcal{L}$ in self-adjoint form is defined by

$$\mathcal{L} = \sum_{j=1}^{k} X^*_j X_j.$$

Consider

$$\begin{cases}
\frac{\partial u}{\partial s}(x, s) + \mathcal{L}_x u(x, s) = 0; \\
u(x, 0) = f(x).
\end{cases}$$

**Nagel & Stein** proved that for $f \in L^2(M)$ and $x \in M$,

$$u(x, s) = \left( e^{-s\mathcal{L}} \right)[f](x) \equiv \int_{M} H(s, x, y) f(y) \, dy.$$
Approximation to the identity (2)

- For $t > 0$ and $x, y \in M$, set

\begin{equation}
S_t(x, y) = H(t^2, x, y) \quad \text{and} \quad D_t(x, y) = t \frac{\partial}{\partial t} S_t(x, y).
\end{equation}

(Denote $D_t \circ D_t$ simply by $D^2_t$.)

- **Nagel & Stein** proved that for any $\epsilon \in (0, 1]$, there exists a constant $C_1 = C_1(\epsilon) > 0$ such that for all $t > 0$ and all $x, y, y' \in M$,

\begin{align*}
\text{(C1)} & \quad D_t(x, y) = D_t(y, x); \\
\text{(C2)} & \quad |D_t(x, y)| \leq C_1 \frac{1}{V_t(x) + V_t(y) + V(x, y)} \left( t + \frac{t^\epsilon}{(t + d(x, y))^{\epsilon}} \right);
\end{align*}
Approximation to the identity (2)

(C3) \[ |D_t(x, y) - D_t(x, y')| \leq C_1 \left( \frac{d(y, y')}{{t + d(x, y)}} \right)^\varepsilon \times \frac{1}{{V_t(x) + V_t(y) + V(x, y)}} t^\varepsilon (t + d(x, y))^\varepsilon \]
for \( d(y, y') \leq \frac{1}{2}(t + d(x, y)) \);

(C4) \[ \int_M D_t(x, y) \, dx = 0. \]

• \( \{D_t\}_{t>0} \leftarrow \) Approximation to the identity (for short: ATTI).

Recall

(1.4) \[ s^4 |B(x, \delta)| \leq |B(x, s\delta)| \leq s^{m+2} |B(x, \delta)|. \]
2. Hardy space $H^1(M)$

- Space of **test functions** on $M$, $G(x_1, r, \beta, \gamma)$:

**Defn 1** Let $x_1 \in M$, $r > 0$, $0 < \beta \leq 1$ and $\gamma > 0$. A function $f$ on $M$ is said to be a **test function of type** $(x_1, r, \beta, \gamma)$ if

(i) $|f(x)| \leq C' \frac{1}{V_r(x_1) + V(x_1, x)} \left(\frac{r}{r + d(x_1, x)}\right)^\gamma$;

(ii) $|f(x) - f(y)| \leq C \left(\frac{d(x, y)}{r + d(x_1, x)}\right)^\beta \frac{1}{V_r(x_1) + V(x_1, x)} \left(\frac{r}{r + d(x_1, x)}\right)^\gamma$

when $d(x, y) \leq (r + d(x_1, x))/2$.

Moreover, we denote by $G(x_1, r, \beta, \gamma)$ the set of all test functions of type $(x_1, r, \beta, \gamma)$, and if $f \in G(x_1, r, \beta, \gamma)$, we define its norm by

$$\|f\|_{G(x_1, r, \beta, \gamma)} = \inf \{C : (i) \text{ and } (ii) \text{ hold}\}.$$
Distributional spaces

- \( G_0(x_1, r, \beta, \gamma) = \{ f \in G(x_1, r, \beta, \gamma) : \int_M f(x) \, dx = 0 \} \); Fix \( x_1 \in M \) and let \( G(\beta, \gamma) = G(x_1, 1, \beta, \gamma) \).

- For given \( \epsilon \in (0, 1] \), let \( \hat{G}(\beta, \gamma) \) be the completion of the space \( \hat{G}(\epsilon, \epsilon) \) in \( \hat{G}(\beta, \gamma) \) when \( 0 < \beta, \gamma < \epsilon \). (Hans Triebel)

- Let \( (\hat{G}(\beta, \gamma))' \) be the set of all bounded linear functionals from \( \hat{G}(\beta, \gamma) \) to \( \mathbb{C} \).
Calderón reproducing formula

**Thm 1**  Let $\epsilon \in (0, 1]$ and $0 < \beta, \gamma < \epsilon$. Then

(2.1) \[ f = \int_0^\infty D_t^2(f) \frac{dt}{t} \]

holds, respectively, in $L^p(M)$ with $p \in (1, \infty)$, $\mathcal{G}_0(\beta, \gamma)$, and $\left(\mathcal{G}_0(\beta, \gamma)\right)'$.

- Nagel & Stein, $L^2(M)$
For $\alpha > 0$ and $x \in M$, define **Lusin-area function** (or called the **Littlewood-Paley $S$-function**) $S_\alpha(f)(x)$ by

$$S_\alpha(f)(x) = \left\{ \int_{\Gamma_\alpha(x)} |D_t(f)(y)|^2 \frac{dydt}{V_t(y)t} \right\}^{1/2},$$

where $\Gamma_\alpha(x) = \{ y \in M : d(y, x) \leq \alpha t \}$.

**Littlewood-Paley theorem**

**Prop 1** Let $\alpha > 0$ and $p \in (1, \infty)$. Then there exists a constant $C_p > 0$ such that for all $f \in L^p(M)$,

$$C_p^{-1} \| f \|_{L^p(M)} \leq \| S_\alpha(f) \|_{L^p(M)} \leq C_p \| f \|_{L^p(M)}.$$
**Hardy space** $H^1(M)$

**Defn 2** Let $\alpha > 0$, $\epsilon \in (0, 1]$ and $\beta, \gamma \in (0, \epsilon)$. The Hardy space $H^1(M)$ is defined by

$$H^1(M) = \left\{ f \in \mathcal{G}^1(\beta, \gamma) : S_\alpha(f) \in L^1(M) \right\},$$

and, moreover, define $\|f\|_{H^1(M)} = \|S_\alpha(f)\|_{L^1(M)}$.

**Defn 3** (Coifman & Weiss, Bull. AMS, 1977) A function $a$ on $M$ is called to be an $H^1(M)$-atom if

1. $\text{supp } a \subset B(x_0, r)$ for some $x_0 \in M$ and some $r > 0$;
2. $\|a\|_{L^2(M)} \leq |B(x_0, r)|^{-1/2}$;
3. $\int_M a(x) \, dx = 0$. 

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Atomic decomposition of $H^1(M)$

**Thm 2** Let $\epsilon \in (0, 1]$ and $\beta, \gamma \in (0, \epsilon)$. Then $f \in H^1(M)$ if and only if there exist $\{\lambda_k\}_{k=-\infty}^{\infty} \subset \mathbb{C}$ with $\sum_{k=-\infty}^{\infty} |\lambda_k| < \infty$ and a sequence $\{a_k\}_{k=-\infty}^{\infty}$ of $H^1(M)$-atoms such that

$$f = \sum_{k=-\infty}^{\infty} \lambda_k a_k$$

in $\left( \mathcal{G}(\beta, \gamma) \right)'$. Moreover, in this case,

$$\|f\|_{H^1(M)} \sim \inf \left\{ \sum_{k=-\infty}^{\infty} |\lambda_k| \right\}.$$
Two remarks

**Cor 1** \( H^1(M) \subsetneq L^1(M) \). Moreover, there exists a constant \( C > 0 \) such that for all \( f \in H^1(M) \),

\[
\|f\|_{L^1(M)} \leq C\|f\|_{H^1(M)}.
\]

**Rem 1** Cor 1 & Thm 2 tell us that \( H^1(M) \) is independent of \( \alpha > 0 \) and \( (\tilde{G}_0(\beta, \gamma))' \) with \( \beta, \gamma \in (0, \varepsilon) \). Based on these facts, letting \( \alpha > 0 \), we can re-define the Hardy space \( H^1(M) \) by

\[
H^1(M) = \{ f \in L^1(M) : S_\alpha(f) \in L^1(M) \}
\]

and

\[
\|f\|_{H^1(M)} = \|S_\alpha(f)\|_{L^1(M)}.
\]
Two remarks

Rem 2 By Thm 2, we know that $H^1(M)$ coincides with the atomic Hardy space of Coifman and Weiss (1977), if we regard $M$ as a space of homogeneous type by (1.2).

1) Thus, the dual space of $H^1(M)$ is the classical space $BMO(M)$ by a result of Coifman & Weiss.

2) Moreover, a molecular characterization and some other atomic characterizations of $H^1(M)$, and some interpolation theorems for operators acting on $H^1(M)$ and $L^p(M)$ with $p \in [1, \infty]$ can also be deduced from results of Coifman & Weiss, respectively.
3. Singular integrals

- Let $C_0(M)$ be the space of continuous functions on $M$ of compact support. For $0 < \eta \leq 1$, define

$$C_0^\eta(M) = \left\{ f \in C_0(M) : \|f\|_{C_0^\eta(M)} < \infty \right\},$$

where

$$\|f\|_{C_0^\eta(M)} = \sup_{x, y \in M, x \neq y} \frac{|f(x) - f(y)|}{d(x, y)^\eta}.$$

We also denote its dual space by $(C_0^\eta(M))'$. 
**Thm 3** Let $\epsilon \in (0, 1]$, and $T : C^0_0(M) \to (C^0_0(M))'$ for all $\eta \in (0, \epsilon]$ with *distribution kernel* $K(x, y)$ which is continuous on $M \times M \setminus \{(x, x) : x \in M\}$, moreover, there exists a constant $C > 0$ such that when $d(y, y') \leq d(x, y)/2$,

\[
|K(x, y) - K(x, y')| + |K(y, x) - K(y', x)| \\
\leq C \frac{d(y, y')^\epsilon}{V(x, y)d(x, y)^\epsilon}.
\]

If $T$ is bounded on $L^2(M)$, then $T$ is also bounded from $H^1(M)$ to $L^1(M)$, namely, there exists a constant $C > 0$ such that for all $f \in H^1(M)$,

\[
\|Tf\|_{L^1(M)} \leq C\|f\|_{H^1(M)}.
\]
Cor 2  Let $T$ be a singular integral of Nagel & Stein in [Rev. Mat. Ibero. 20(2004), 531-561]. Then $T$ is bounded from $H^1(M)$ to $L^1(M)$.

- Boundedness in Hardy space: $T^*1 = 0$ means that for any $H^1(M)$-atom $a$,

$$\int_M Ta(x) \, dx = 0.$$  

(By Thm 3, $Ta \in L^1(M)$. Moreover, if $Ta \in H^1(M)$, then it is also necessary to have $T^*1 = 0$.)
Singular integrals (4)

**Thm 4** Let $\epsilon \in (0, 1]$, $T$ be the same as in Thm 3 with kernel $K$ satisfying (3.1) and that for all $x, y \in M$,

$$|K(x, y)| \leq C \frac{1}{V(x, y)}.$$

If $T$ is bounded on $L^2(M)$ and $T^*1 = 0$, then $T$ is also bounded on $H^1(M)$. Moreover, there exists a constant $C > 0$ such that for all $f \in H^1(M)$,

$$\|Tf\|_{H^1(M)} \leq C\|f\|_{H^1(M)}.$$
Some remarks

Cor 3  Let $T$ be a singular integral of Nagel & Stein in [Rev. Mat. Ibero. 20(2004), 531-561]. If $T^* 1 = 0$, then $T$ is bounded on $H^1(M)$.

• Some open problems:

1) Littlewood-Paley $g$-function characterizations and maximal function characterizations for $H^1(M)$?

2) A theory for Hardy spaces $H^p(M)$ when $p < 1$?

3) A theory of Besov and Triebel-Lizorkin spaces when $p \leq 1$ in this setting?

(A theory of product Hardy space $H^1(M \times M)$ and a theory for Besov and Triebel-Lizorkin spaces when $p > 1$ are in progress.)