On interpolation of Asplund operators

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Abstract. We study the interpolation properties of Asplund operators by the complex method, as well as by general J- and K-methods.

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1. Introduction

Let A, B be Banach spaces. A bounded linear operator \( T \in \mathcal{L}(A, B) \) is said to be a Radon-Nikodým operator if for any probability measure \( \mu \), \( T \) maps each \( \mu \)-continuous \( A \)-valued measure of finite variation into a \( \mu \)-differentiable \( B \)-valued measure (see [22] and [11]). An operator \( T \in \mathcal{L}(A, B) \) is called an Asplund operator if \( T^* \) is a Radon-Nikodým operator. Asplund operators have been studied widely. See, for example, the papers by Edgar [12], Stegall [24] and the book by Diestel, Jarchow and Tonge [10]; see also the book by Pietsch [22] and the paper by Heinrich [15] where they are referred to as decomposing operators. Asplund operators \( A \) form a closed injective and surjective operator ideal in the sense of Pietsch [22].

A Banach space \( A \) is said to be an Asplund space if its identity operator \( I_A \) belongs to \( A \). This class of Banach spaces has attracted considerable attention in recent years. It originated in the work of Asplund [1] where these spaces are called strong differentiability space. In fact, a Banach space \( A \) is Asplund if every continuous convex function on \( A \) is Fréchet differentiable at each point of a dense \( G_δ \) subset of \( A \). Any reflexive space is Asplund. Further, \( c_0 \) and some other non-reflexive spaces are Asplund, but \( \ell_1 \) and \( \ell_\infty \) are not Asplund. We refer to the books by Giles
We end the paper by returning to the complex method to show that if we put \( \| \cdot \| \) is Asplund and \( T \) is an Asplund operator, then the real interpolation space \( (A_0, A_1)_{0,q} \) is Asplund for \( 0 < \theta < 1, 1 < q < \infty \), and from this result he derived easily the factorization property of Asplund operators.

More precisely, focus on Heinrich’s paper, first he showed that if the embedding \( i : A_0 \cap A_1 \rightarrow A_0 + A_1 \) is an Asplund operator, then the real interpolation space \( (A_0, A_1)_{0,q} \) is Asplund. Afterwards, in Section 3, we investigate the behaviour of Asplund operators under general \( J \) - and \( K \) -methods. Many authors have worked on these interpolation methods, which are extensions of the real method (see, for example, Peetre [20], Brudnyˇı and Krugljak [4], Cwikel and Peetre [8] and Nilsson [19]). These methods arise when one wants to describe all interpolation spaces with respect to the couple \((L_1, L_\infty)\). The usual weighted \( L_q \) norm of the real method is not enough and one should work with general lattice norms. Here we prove that if the lattice \( \Gamma \) that defines the methods is an Asplund space and \( T : A_0 \cap A_1 \rightarrow B_0 + B_1 \) is an Asplund operator, then the interpolated operators by the \( K \) - and the \( J \) -method are Asplund operators as well.

We end the paper by returning to the complex method to show that if \( T : A_0 \rightarrow B_0 \) is Asplund and \( T : A_1 \rightarrow B_1 \) is bounded, then \( T : (A_0, A_1)_{01} \rightarrow (B_0, B_1)_{01} \) is Asplund.

### 2. Complex interpolation

Let \( \tilde{A} = (A_0, A_1) \) be a Banach couple and let \( \mathcal{F}(\tilde{A}) \) be the space of all functions \( f \) from the closed strip \( S = \{ z \in \mathbb{C} : 0 \leq \text{Re} z \leq 1 \} \) into \( A_0 + A_1 \) such that

- \( f \) is bounded and continuous on \( S \) and analytic on the interior of \( S \), and
- the functions \( t \mapsto f(j + it) \) \((j = 0, 1)\) are continuous from \( \mathbb{R} \) into \( A_j \) and tend to zero as \( |t| \rightarrow \infty \).

We put \( \| f \|_\mathcal{F} = \max_{j=0,1} \sup_{t \in \mathbb{R}} \| f(j + it) \|_{A_j} \). The complex interpolation space \( (A_0, A_1)_{0\theta} \) (where \( 0 < \theta < 1 \)) consists of all \( a \in A_0 + A_1 \) such that \( a = f(\theta) \) for some \( f \in \mathcal{F}(\tilde{A}) \). The norm of \( (A_0, A_1)_{0\theta} \) is \( \| a \|_{0\theta} = \inf \{ \| f \|_\mathcal{F} : f(\theta) = a, f \in \mathcal{F}(A_0, A_1) \} \).

Full details on complex interpolation can be found in [2] and [25]. We only recall that \( A_0 \cap A_1 \) is dense in \( (A_0, A_1)_{0\theta} \), and that, if \( A_0 \cap A_1 \) is dense in \( A_0 \) and in \( A_1 \), then the dual space \( (A_0, A_1)_{0\theta}^* \) of \( (A_0, A_1)_{0\theta} \) coincides with the upper complex space \( (A_0^*, A_1^*)_{0\theta}^* \).
In order to determine whether \((A_0, A_1)_{\theta[0]}\) is Asplund, we shall need the following characterization of Asplund operators. Given any Banach space \(A\), we denote by \(U_A\) the closed unit ball of \(A\). If \(D \subseteq A\) is a bounded set, let \(\mu_D\) denote the seminorm on \(A^*\) given by
\[
\mu_D(f) = \sup \{|f(x)| : x \in D\}, \quad f \in A^*.
\]

**Theorem 2.1** (see [3], Thm. 5.2.11). Let \(T \in \mathcal{L}(A, B)\) be a bounded linear operator from the Banach space \(A\) into the Banach space \(B\). Then \(T\) is Asplund if and only if the seminormed space \((B^*, \mu_{T,\theta[0]}\mu_D)\) is separable whenever \(D \subseteq U_A\) is countable.

Now we are ready to describe the behaviour of Asplund spaces under complex interpolation.

**Theorem 2.2.** Let \(\mathcal{A} = (A_0, A_1)\) be a Banach couple and let \(0 < \theta < 1\). Then the complex interpolation space \((A_0, A_1)_{\theta[0]}\) is Asplund whenever \(A_0\) or \(A_1\) is Asplund.

**Proof.** Let \(A_0^*\) be the closure of \(A_0 \cap A_1\) in \(A_j\). Since \((A_0, A_1)_{[\theta]} = (A_0^*, A_1^*)_{[\theta]}\) and any closed subspace of an Asplund space is Asplund, without loss of generality we may assume that \(A_0 \cap A_1\) is dense in \(A_0\) and in \(A_1\). Hence
\[
(A_0, A_1)_{[\theta]} = (A_0^*, A_1^*)_{[\theta]} = (A_0^*, A_1^*)_{[\theta]}.
\]
The last equality holds because \(A_1^*\) has the Radon-Nikodým property for \(j = 0\) or \(j = 1\) (see [21]). In particular, we get that
\[
(A_0 + A_1)^* = A_0^* \cap A_1^* \text{ is dense in } (A_0, A_1)_{[\theta]}.
\]
On the other hand, in the following diagram
\[
\begin{array}{ccc}
A_0 & \xrightarrow{A_0} & A_0 + A_1 \\
\downarrow i & & \downarrow i \\
A_1 & \xrightarrow{A_1} & A_0 + A_1
\end{array}
\]
any of these two embeddings is Asplund. Whence, by [15], Prop. 1.7, we obtain that the embedding
\[
i : (A_0, A_1)_{[\theta]} \longrightarrow A_0 + A_1 \quad \text{is Asplund.} \tag{2.2}
\]
Consequently, given any \(D \subseteq U_{(A_0, A_1)_{[\theta]}}\) with \(D\) countable, it follows from (2.2) that \(((A_0 + A_1)^*, \mu_D^\theta)\) is separable. Combining this fact with (2.1), we conclude that \(((A_0, A_1)^*_{[\theta]}, \mu_D^\theta)\) is separable. Using Theorem 2.1, this implies that \((A_0, A_1)_{[\theta]}\) is Asplund. \(\Box\)

**Remark 2.1.** If we only assume that the embedding \(i : A_0 \cap A_1 \longrightarrow A_0 + A_1\) is Asplund, then it is not true in general that \((A_0, A_1)_{[\theta]}\) is Asplund. Indeed, Mastylo has constructed in [18], page 161, a couple of Lorentz spaces \((\Lambda_\psi, \Lambda_\psi)\) such that the embedding \(i : \Lambda_\psi \cap \Lambda_\psi \longrightarrow \Lambda_\psi + \Lambda_\psi\) is weakly compact and so it is Asplund, but \((\Lambda_\psi, \Lambda_\psi)_{[\theta]}\) contains a subspace isomorphic to \(\ell_1\) and therefore \((\Lambda_\psi, \Lambda_\psi)_{[\theta]}\) is not an Asplund space.

We will come back to Theorem 2.2 at the end of the next section and we will show that it can be extended to general couples \(\mathcal{A}, \mathcal{B}\) and any operator \(T\) with \(T : A_0 \longrightarrow B_0\) Asplund and \(T : A_1 \longrightarrow B_1\) bounded.
3. Real interpolation

In this section we will show that the behaviour under interpolation of Asplund operators improves when one works with the extensions of the real method.

By a \( \mathbb{Z} \)-lattice \( \Gamma \) we mean a Banach space of real valued sequences with \( \mathbb{Z} \) as index set, such that it contains all sequences with only finitely many non-zero coordinates, and moreover \( \Gamma \) satisfies that whenever \( |\xi_m| \leq |\mu_m| \) for each \( m \in \mathbb{Z} \) and \( \{\mu_m\} \in \Gamma \), then \( \{\xi_m\} \in \Gamma \) and \( \|\{\xi_m\}\| \Gamma \leq \|\{\mu_m\}\| \Gamma \).

A \( \mathbb{Z} \)-lattice \( \Gamma \) is said to be regular if for any \( \{\tau_n\}_{n \in \mathbb{N}} \subseteq \Gamma \) with \( \tau_n \downarrow 0 \) it follows that \( \|\tau_n\| \Gamma \rightarrow 0 \). The associated space \( \Gamma' \) of \( \Gamma \) is formed by all sequences \( \{\eta_m\} \) for which

\[
\|\{\eta_m\}\| \Gamma' = \sup \left\{ \sum_{m=-\infty}^{\infty} |\eta_m\xi_m| : \|\{\xi_m\}\| \Gamma \leq 1 \right\} < \infty.
\]

The space \( \Gamma' \) is also a \( \mathbb{Z} \)-lattice.

The \( \mathbb{Z} \)-lattice \( \Gamma \) is said to be \( K \)-non-trivial if \( \{\min(1,2^m)\} \in \Gamma \). It is called \( J \)-non-trivial if

\[
\sup \left\{ \sum_{m=-\infty}^{\infty} \min(1,2^{-m})|\xi_m| : \|\xi\| \Gamma \leq 1 \right\} < \infty.
\]

Let \( \tilde{A} = (A_0, A_1) \) be a Banach couple and let \( \Gamma \) be a \( K \)-non-trivial \( \mathbb{Z} \)-lattice. The \( K \)-space \( \tilde{A}_{\Gamma;K} = (A_0, A_1)_{\Gamma;K} \) consists of all \( a \in A_0 + A_1 \) such that \( (K(2^m,a)) \in \Gamma \). Here

\[
K(2^m,a) = K(2^m,a;A_0, A_1) = \inf \{\|a_0\|_A + 2^m\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j \}.
\]

The norm in \( (A_0, A_1)_{\Gamma;K} \) is given by \( \|a\|_{\tilde{A}_{\Gamma;K}} = \|K(2^m,a)\| \Gamma \).

If \( \Gamma \) is \( J \)-non-trivial, the \( J \)-space \( \tilde{A}_{\Gamma;J} = (A_0, A_1)_{\Gamma;J} \) consists of all sums \( a = \sum_{m=-\infty}^{\infty} u_m \) (convergence in \( A_0 + A_1 \)) where \( \{u_m\} \subseteq A_0 \cap A_1 \) and \( (J(2^m,u_m)) \in \Gamma \).

Here

\[
J(2^m,u_m) = J(2^m,u_m;A_0, A_1) = \max\{\|u_m\|_{A_0}, 2^m\|u_m\|_{A_1}\}.
\]

We put

\[
\|a\|_{\tilde{A}_{\Gamma;J}} = \inf \left\{ \|J(2^m,u_m)\|_{\Gamma} : a = \sum_{m=-\infty}^{\infty} u_m \right\}.
\]

Spaces \( \tilde{A}_{\Gamma;K} \) and \( \tilde{A}_{\Gamma;J} \) are Banach spaces.

Let \( \bar{B} = (B_0, B_1) \) be another Banach couple. We write \( T \in \mathcal{L}(\tilde{A}, \bar{B}) \) to mean that \( T \) is a linear operator from \( A_0 + A_1 \) into \( B_0 + B_1 \) whose restriction to each \( A_j \) defines a bounded operator from \( A_j \) into \( B_j \) \((j = 0, 1)\). We put \( \|T\|_{\tilde{A}, \bar{B}} = \max \{\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\} \).
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If \( T \in \mathcal{L}(\tilde{A}, \tilde{B}) \), then it is clear that the restriction of \( T \) to \( \tilde{A}_{\Gamma,K} \) defines a bounded operator \( T : \tilde{A}_{\Gamma,K} \to \tilde{B}_{\Gamma,K} \) with \( \| T \|_{\tilde{A}_{\Gamma,K},\tilde{B}_{\Gamma,K}} \leq \| T \|_{\tilde{A},\tilde{B}}. \) A similar estimate holds for the \( J \)-method. In other words, \( \tilde{K} \) and \( J \)-method are exact interpolation methods.

If \( \Gamma \) is \( K \)- and \( J \)-non-trivial \( \mathbb{Z} \)-lattice, then \( \tilde{A}_{\Gamma,K} \hookrightarrow \tilde{A}_{\Gamma,J} \). But it is not true in general that \( \tilde{A}_{\Gamma,K} \) coincides with \( \tilde{A}_{\Gamma,J} \). We refer to \([19]\) and \([4]\) for full details on these interpolation methods. For \( \Gamma = \ell_q(2^{-m}) \), the space \( \ell_q \) with the weight \( (2^{-m})_+ \), \( K \)- and \( J \)-methods coincide with the real method

\[
(A_0, A_1)_{0,q} = (A_0, A_1)_{\ell_q(2^{-m})}, \quad K = (A_0, A_1)_{\ell_q(2^{-m})}; \quad \text{see } [2] \text{ and } [25].
\]

Here \( 0 < \theta < 1, 1 \leq q \leq \infty \). In a more general way, if \( f \) is a function parameter and \( \Gamma = \ell_q(1/f(2^m)) \) then we recover the real method with a function parameter

\[
(A_0, A_1)_{f,q} = (A_0, A_1)_{\ell_q(1/f(2^m))}; \quad K = (A_0, A_1)_{\ell_q(1/f(2^m))}; \quad \text{see } [20] \text{ and } [16].
\]

For \( t > 0 \), let \( t \mathbb{R} \) be \( \mathbb{R} \) with the norm \( \| t \|_{t \mathbb{R}} = t |\lambda| \). The characteristic function \( \varphi_K \) of the \( K \)-method is defined by

\[
(\mathbb{R}, (1/t)\mathbb{R})_\Gamma, K = (1/\varphi_K(t))\mathbb{R} \quad \text{see } [16].
\]

The characteristic function \( \varphi_J \) of the \( J \)-method is defined analogously. It turns out that \( \varphi_K(t) = \| \{ t/2^m \} \|^2_1 \) and

\[
\varphi_J(t) = \sup \left\{ \sum_{m=-\infty}^{\infty} \min(1, t/2^m) |\xi_m| : \| \{ \xi_m \} \|_\Gamma \leq 1 \right\}, \quad t > 0,
\]

(see [6], Lemma 2.4). Characteristic functions are quasiconcave. Later on we will need to know the behaviour of these functions at 0 and \( \infty \). We write \( \varphi \in \mathcal{P}_0 \) if

\[
\min(1, 1/t)\varphi(t) \to 0 \quad \text{as } t \to 0 \quad \text{or as } t \to \infty.
\]

We put \( \varphi^*(t) = 1/\varphi(1/t) \). We also recall that the functions (see [5])

\[
\psi_{\tilde{A}_{\Gamma,K}}(t) = \sup \{ \psi_{\tilde{A}_{\Gamma,K}}(t, a) : \| a \|_{\tilde{A}_{\Gamma,K}} = 1 \},
\]

\[
\rho_{\tilde{A}_{\Gamma,J}}(t) = \inf \{ \psi_{\tilde{A}_{\Gamma,J}}(t, a) : a \in A_0 \cap A_1, \| a \|_{\tilde{A}_{\Gamma,J}} = 1 \},
\]

are related with the characteristic functions by the inequalities

\[
\psi_{\tilde{A}_{\Gamma,K}}(t) \leq \varphi_K(t) \quad \text{for all } t > 0, \tag{3.1}
\]

\[
\rho_{\tilde{A}_{\Gamma,J}}^*(t) \leq \varphi_J^*(t) \quad \text{for all } t > 0, \tag{3.2}
\]

(see [6], Lemma 2.1).

Given any sequence of Banach spaces \( (E_m) \), we put

\[
\Gamma(E_m) = \{ \{ x_m \} : x_m \in E_m \ \text{and} \ \| \{ x_m \} \|_\Gamma(E_m) = \| \{ x_m \} \|_E \ < \infty \}\}.
\]

We denote by \( Q_k : \Gamma(E_m) \to E_k \) the projection \( Q_k \{ x_m \} = x_k \), and by \( P_r : E_r \to \Gamma(E_m) \) the embedding \( P_r x = [\delta'(r)] \) where \( \delta'(r) \) is the Kronecker delta.

In the following results we use again the characterization of Asplund operators given in Theorem 2.1.
Theorem 3.1. Let $\Gamma$ be a $K$-non-trivial Asplund $\mathbb{Z}$-lattice with $\varphi_K \in \mathcal{P}_0$, let $\tilde{A} = (A_0, A_1)$, $\tilde{B} = (B_0, B_1)$ be Banach couples and let $T \in \mathcal{L}(A, \tilde{B})$. Then $T : \tilde{A}_{\Gamma,K} \rightarrow \tilde{B}_{\Gamma,K}$ is Asplund if and only if $T : A_0 \cap A_1 \rightarrow B_0 + B_1$ is Asplund.

Proof. Factorization

$$A_0 \cap A_1 \xrightarrow{\tilde{A}_{\Gamma,K}} \tilde{A}_{\Gamma,K} \xrightarrow{T} \tilde{B}_{\Gamma,K} \xrightarrow{} B_0 + B_1$$

yields that $T : A_0 \cap A_1 \rightarrow B_0 + B_1$ is Asplund if $T : \tilde{A}_{\Gamma,K} \rightarrow \tilde{B}_{\Gamma,K}$ is Asplund.

Conversely, assume that $T : A_0 \cap A_1 \rightarrow B_0 + B_1$ is Asplund. Since $\varphi_K \in \mathcal{P}_0$ it follows from (3.1) that

$$\lim_{t \rightarrow 0} \psi_{\tilde{A}_{\Gamma,K}}(t) = 0 = \lim_{t \rightarrow \infty} (1/t)\psi_{\tilde{A}_{\Gamma,K}}(t).$$

Then, by [7], Thm. 3.3, we get that $T : \tilde{A}_{\Gamma,K} \rightarrow B_0 + B_1$ is Asplund. Let $F_m$ be the space $B_0 + B_1$ normed by $K(2^m, \cdot)$, $m \in \mathbb{Z}$, and let $\tilde{T} : \tilde{A}_{\Gamma,K} \rightarrow \Gamma(F_m)$ be the operator defined by $T(a) = \{ \ldots, Ta, Ta, Ta, \ldots \}$. Since any two norms of the family $K(2^m, \cdot)$ are equivalent on $B_0 + B_1$, and since $Q_m \tilde{T} = T$, we have that $Q_m \tilde{T} : \tilde{A}_{\Gamma,K} \rightarrow F_m$ is Asplund for each $m \in \mathbb{Z}$. We claim that

$$\tilde{T} : \tilde{A}_{\Gamma,K} \rightarrow \Gamma(F_m) \quad \text{is Asplund.} \quad (3.3)$$

Indeed, let $D \subseteq \bigcup \tilde{A}_{\Gamma,K}$ be a countable set. For each $m \in \mathbb{Z}$, write $W_m = Q_m \tilde{T}(D)$. We know that $(F_m^*, \mu_{\eta_m})$ is separable. Let $\Lambda_m$ be a countable set that is dense in $(F_m^*, \mu_{\eta_m})$. To establish (3.3) it suffices to show that the countable set

$$\Lambda = \left\{ \sum_{k=-N}^{N} P_k g_k : g_k \in \Lambda_k, \ N \in \mathbb{N} \right\}$$

is dense in $(\Gamma(F_m)^*, \mu_{\tilde{T}(D)}).$ Since $\Gamma$ is Asplund, $\Gamma$ does not contain a copy either of $\ell_\infty$ or $\ell_1$. Hence, by [26], Thms. 117.3 and 117.2, the spaces $\Gamma$ and $\Gamma'$ are regular. By regularity of $\Gamma$ and [18], Prop. 3.1, we get that the dual space $\Gamma(F_m)^*$ of $\Gamma(F_m)$ is isometrically isomorphic to $\Gamma'(F_m^*)$. Take any $h \in \Gamma(F_m)^*$ $\Gamma'(F_m^*)$ and put $h_m = Q_m h \in F_m^*$. Using that $\Gamma'$ is regular, given any $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that $\|h - \sum_{|k| \leq N} P_k g_k\|_{\Gamma'(F_m^*)} \leq \varepsilon/2 \|T\|_{\tilde{A}, \tilde{B}}$. Choose $g_k \in \Lambda_k$ such that

$$\mu_{\tilde{T}(D)}(h - \sum_{|k| \leq N} P_k g_k) \leq \varepsilon/(4N + 2).$$

Then

$$\mu_{\tilde{T}(D)}(h - \sum_{|k| \leq N} P_k h_k) \leq \mu_{\tilde{T}(D)}(h - \sum_{|k| \leq N} P_k g_k) + \sum_{|k| \leq N} \mu_{\tilde{T}(D)}(P_k(h_k - g_k)) \leq \|T\|_{\tilde{A}, \tilde{B}} \|h - \sum_{|k| \leq N} P_k h_k\|_{\Gamma'(F_m^*)} + \sum_{|k| \leq N} \mu_{\tilde{T}(D)}(h_k - g_k) \leq \varepsilon.$$

This establishes (3.3).

The operator $\tilde{T}$ can be factorized as $\tilde{T} = jT$, where $j : \tilde{B}_{\Gamma,K} \rightarrow \Gamma(F_m)$ is the isometric embedding defined by $j b = \{ \ldots, b, b, b, \ldots \}$. Using that Asplund operators form an injective operator ideal, it follows from (3.3) that $T : \tilde{A}_{\Gamma,K} \rightarrow \tilde{B}_{\Gamma,K}$ is Asplund and completes the proof. \qed
Theorem 3.2. Let $\Gamma$ be a $J$-non-trivial Asplund $\mathcal{Z}$-lattice with $\varphi^*_J \in \mathcal{P}_0$, let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be Banach couples and let $T \in \mathcal{L}(\bar{A}, \bar{B})$. Then $T : \bar{A}_{\Gamma, J} \rightarrow \bar{B}_{\Gamma, J}$ is Asplund if and only if $T : A_0 \cap A_1 \rightarrow B_0 + B_1$ is Asplund.

Proof. Clearly, if $T : \bar{A}_{\Gamma, J} \rightarrow \bar{B}_{\Gamma, J}$ is Asplund then $T : A_0 \cap A_1 \rightarrow B_0 + B_1$ is Asplund because $A_0 \cap A_1 \hookrightarrow \bar{A}_{\Gamma, J}$ and $\bar{B}_{\Gamma, J} \hookrightarrow B_0 + B_1$.

Conversely, assume that $T : A_0 \cap A_1 \rightarrow B_0 + B_1$ is Asplund. Since $\varphi^*_J \in \mathcal{P}_0$, (3.2) implies that $\rho^*_{\mathcal{P}_{\Gamma, J}} \in \mathcal{P}_0$. Using [7], Thm. 3.1, we obtain that $T : A_0 \cap A_1 \rightarrow \bar{B}_{\Gamma, J}$ is Asplund. Moreover, since $\Gamma$ is an Asplund space, we know by [26], Thms. 117.3 and 117.2, that $\Gamma$ and $\Gamma$ are regular. Let $G_m$ be the space $A_0 \cap A_1$ endowed with the norm $J(2^m, \cdot)$, $m \in \mathbb{Z}$, and let $\bar{T} : \Gamma(G_m) \rightarrow \bar{B}_{\Gamma, J}$ be the operator given by

$$\bar{T}(u_m) = T(\sum_{m=-\infty}^{\infty} u_m).$$

For each $m \in \mathbb{Z}$, the operator $\bar{T} P_m = T : G_m \rightarrow \bar{B}_{\Gamma, J}$ is Asplund because any two norms of the family $\{J(2^m, \cdot)\}$ are equivalent on $A_0 \cap A_1$. We claim that

$$\bar{T} : \Gamma(G_m) \rightarrow \bar{B}_{\Gamma, J} \text{ is Asplund.} \quad (3.4)$$

Indeed, take any $D \subseteq U_{\Gamma(G_m)}$ countable. For each $m \in \mathbb{Z}$, put $D_m = Q_m(D) \subseteq U_{G_m}$. We have that $(\bar{T}^* | \Gamma_{\mathcal{P}_{\Gamma, J}, J}(G_m^*), \mu_{\mathcal{P}_{\Gamma, J}, J}(G_m^*))$ is separable, so there is a countable set $\Omega_m$ that is dense. Put

$$\Omega = \left\{ \sum_{|k| \leq N} P_k Q_k g_k : g_k \in \Omega_k, N \in \mathbb{N} \right\}.$$

We are going to show that the countable set $\Omega \subseteq \Gamma(G_m^*)$ is dense in $(\bar{T}^* | \Gamma_{\mathcal{P}_{\Gamma, J}, J}(G_m^*), \mu_{\mathcal{P}_{\Gamma, J}, J})$. Take any $f \in \Gamma_{\mathcal{P}_{\Gamma, J}, J}$ and any $\varepsilon > 0$. Using the regularity of $\Gamma'$ we can find $N \in \mathbb{N}$ such that

$$\left\| f \bar{T} - \sum_{|k| \leq N} P_k Q_k f \bar{T} \right\|_{\Gamma_{\mathcal{P}_{\Gamma, J}, J}(G_m^*)} \leq \varepsilon/2.$$

Now, for each $|k| \leq N$, choose $g_k \in \Omega_k$ such that $\mu_{\mathcal{P}_{\Gamma, J}, J}(f \bar{T} - g_k) \leq \varepsilon/(4N + 2)$. Then we obtain

$$\mu_{\mathcal{P}_{\Gamma, J}, J}(f \bar{T} - \sum_{|k| \leq N} P_k Q_k g_k) \leq \sum_{|k| \leq N} \mu_{\mathcal{P}_{\Gamma, J}, J}(f \bar{T} - g_k) \leq \varepsilon/2 + \sum_{|k| \leq N} \mu_{\mathcal{P}_{\Gamma, J}, J}(f \bar{T} - g_k) \leq \varepsilon.$$

This proves (3.4).
The operator \( \tilde{T} \) is the composition of \( T \) with the metric surjection \( \pi : \Gamma(G_m) \rightarrow \tilde{A}_{\Gamma;J} \) defined by \( \pi \{ u_m \} = \sum_{m=-\infty}^{\infty} u_m \). Consequently, using surjectivity of Asplund operators, we derive that \( T : \tilde{A}_{\Gamma;J} \rightarrow \tilde{B}_{\Gamma;J} \) is Asplund.

Corollary 3.1. Let \( \Gamma \) be an Asplund \( \mathbb{Z} \)-lattice and let \( \tilde{A} = (A_0, A_1) \) be a Banach couple with the embedding \( i : A_0 \cap A_1 \rightarrow A_0 + A_1 \) being an Asplund operator.

(i) If \( \Gamma \) is \( K \)-non-trivial with \( \phi_K \in P_0 \), then the space \( \tilde{A}_{\Gamma;K} \) is Asplund.

(ii) If \( \Gamma \) is \( J \)-non-trivial with \( \phi_J^* \in P_0 \), then the space \( \tilde{A}_{\Gamma;J} \) is Asplund.

If one of the spaces in the couple is Asplund, then we can weaken the assumptions on \( \phi_K \) and \( \phi_J \). Indeed, repeating the proofs of Theorems 3.1 and 3.2 but using now [5], Thms. 3.1 and 3.2 instead of [7], Thms. 3.3 and 3.1, we obtain:

Corollary 3.2. Let \( \Gamma \) be an Asplund \( \mathbb{Z} \)-lattice and let \( \tilde{A} = (A_0, A_1) \) be a Banach couple. Assume that \( A_0 \) is Asplund.

(i) If \( \Gamma \) is \( K \)-non-trivial and \( \lim_{t \to \infty} \phi_K(t)/t = 0 \), then the space \( \tilde{A}_{\Gamma;K} \) is Asplund.

(ii) If \( \Gamma \) is \( J \)-non-trivial and \( \lim_{t \to 0} t/\phi_J(t) = 0 \), then the space \( \tilde{A}_{\Gamma;J} \) is Asplund.

The behaviour at 0 and \( \infty \) of the functions \( \phi_K \), \( \phi_J \) can be controlled by the norms of shift operators on \( \Gamma \). For \( k \in \mathbb{Z} \), the shift operator \( \tau_k \) is defined by \( \tau_k \{ \xi_m \}_{m \in \mathbb{Z}} = \{ \xi_{m+k} \}_{m \in \mathbb{Z}} \). It turns out (see [6], Lemma 2.5) that if

\[
2^{-\theta} \| \tau_n \|_{\Gamma, J} \rightarrow 0 \quad \text{and} \quad \| \tau_{-n} \|_{\Gamma, J} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]

then \( \phi_K \in P_0 \) and \( \phi_J^* \in P_0 \). In particular, these conditions are satisfied for \( \Gamma = \ell_q(2^{-\theta m}) \) or \( \Gamma = \ell_q(1/f(2^m)) \), \( f \) being a function parameter.

Writing down Corollary 3.1 for \( \Gamma = \ell_q(2^{-\theta m}) \) with \( 0 < \theta < 1 \) and \( 1 < q < \infty \), we recover a result of Heinrich [15], Cor. 2.5(ii):

Corollary 3.3. Let \( 0 < \theta < 1 \), \( 1 < q < \infty \) and let \( \tilde{A} = (A_0, A_1) \) be a Banach couple. Then the following are equivalent:

(a) \( (A_0, A_1)_{\theta, q} \) is Asplund.

(b) The embedding \( i : A_0 \cap A_1 \rightarrow A_0 + A_1 \) is an Asplund operator.

As we have seen in Remark 2.1, a similar result to Corollary 3.3 does not hold for the complex method.

The next result refers to the limit cases \( q = 1 \) and \( q = \infty \).

Proposition 3.1. Let \( 0 < \theta < 1 \) and \( q = 1 \) or \( q = \infty \). Let \( \tilde{A} = (A_0, A_1) \) be a Banach couple. Then the following are equivalent:

(a) \( (A_0, A_1)_{\theta, q} \) is Asplund.

(b) The embedding \( i : A_0 \cap A_1 \rightarrow A_0 + A_1 \) is an Asplund operator and its range is closed.
Proof. If \((A_0, A_1)_\theta, q\) is Asplund then it does not contain a subspace isomorphic to \(\ell_q\) (because \(q = 1\) or \(q = \infty\)) and so, using [17], Thm. 1, we have that \(A_0 \cap A_1\) is closed in \(A_0 + A_1\). In other words, \(i : A_0 \cap A_1 \to A_0 + A_1\) has closed range. Moreover, \(i\) factorizes through the identity of \((A_0, A_1)_\theta, q\), therefore \(i : A_0 \cap A_1 \to A_0 + A_1\) is Asplund. This shows that (a) implies (b). The converse implication follows from [6], Prop. 4.7.

We end the paper by returning to the complex method to establish Theorem 2.2 in its general form.

**Corollary 3.4.** Let \(0 < \theta < 1\), let \(\tilde{A} = (A_0, A_1), \tilde{B} = (B_0, B_1)\) be Banach couples and let \(T \in \mathcal{L}(\tilde{A}, \tilde{B})\). If \(T : A_0 \to B_0\) is Asplund, then \(T : (A_0, A_1)_{\theta} \to (B_0, B_1)_{\theta}\) is also Asplund.

**Proof.** We can factorize \(T : A_0 \to B_0\) as

\[
\begin{array}{ccc}
A_0 & \xrightarrow{T} & B_0 \\
\Phi & & \downarrow I_{B_0} \\
A_0/Ker(T) & \xrightarrow{j_0} & B_0
\end{array}
\]

where \(\Phi(x) = [x]\) is the quotient mapping and \(j_0[x] = Tx\). Put \(E = A_0/Ker(T)\). The map \(j_0 : E \to B_0\) is a continuous embedding, and so \((E, B_0)\) is a Banach couple. Since the ideal of Asplund operators is surjective, it follows from the fact that \(T : A_0 \to B_0\) is Asplund that \(j_0 : E \to B_0\) is Asplund. Hence, by Corollary 3.3, we get that \(W = (E, B_0)_{1/2, 2}\) is an Asplund space. The operator \(T : A_0 \to B_0\) admits the factorization

\[
\begin{array}{ccc}
A_0 & \xrightarrow{T} & B_0 \\
\downarrow T & & \downarrow I \\
W & \xrightarrow{I} & B_0
\end{array}
\]

Consequently, the interpolated operator by the complex method can be factorized as
and Theorem 2.2 implies that $T : (A_0, A_1)_{\theta} \rightarrow (B_0, B_1)_{\theta}$ is Asplund. □

Combining Corollary 3.4 with the reiteration theorem for the complex method, we obtain:

**Corollary 3.5.** Let $\tilde{A} = (A_0, A_1)$, $\tilde{B} = (B_0, B_1)$ be Banach couples and let $T \in \mathcal{L}(\tilde{A}, \tilde{B})$. If there exists $0 < \theta_0 < 1$ such that $T : (A_0, A_1)_{\theta_0} \rightarrow (B_0, B_1)_{\theta_0}$ is Asplund, then $T : (A_0, A_1)_{\theta} \rightarrow (B_0, B_1)_{\theta}$ is Asplund for all $0 < \theta < 1$.

Interpolation properties of other operator ideals have been investigated by the authors in [6]. There, we have studied the case of weakly compact operators, Rosenthal operators and Banach-Saks operators.

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**References**