Spaces on sets

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Abstract
The paper deals with spaces $\mathcal{B}_{pq}^s$ and $\mathcal{F}_{pq}^s$ of positive smoothness $s > 0$, based on $L_p$-spaces with $0 < p \leq \infty$ and reproducing formulas for smooth functions. We compare these spaces with other $B$-spaces and $F$-spaces obtained by different means.

2000 Math. Subject Classification: primary 46E35, secondary 42B35, 28A80, 43A85

keywords: function spaces, reproducing formula

Contents

1 Introduction and reproducing formula 2

2 Fourier-analytical spaces 3
   2.1 Basic notation ................. 3
   2.2 Quarkonial decompositions and wavelet frames .......... 5

3 Proof of Theorem 1, further comments 8
   3.1 Proof of Theorem 1 ............. 8
   3.2 Multiresolution property ........ 10

4 Spaces on $\mathbb{R}^n$ 11
   4.1 Definition and basic assertions ........ 11
   4.2 Properties ................... 15

5 Spaces on sets 17
   5.1 Preliminaries, sequence spaces ............ 17
   5.2 Function spaces .................. 21
   5.3 Properties, comments, snowflaked transforms .......... 23
1 Introduction and reproducing formula

The theory of the classical Nikol’skij-Besov spaces $B^{s}_{pq}$, Sobolev spaces $W^s_p$ and Bessel-potential spaces $H^s_p$ of positive smoothness $s > 0$ as it has been developed in [24, 23, 3], is based on $L^p$-spaces with $1 \leq p \leq \infty$, (fractional) derivatives and differences of functions. In this paper we describe an approach to spaces of type $B^{s}_{pq}$ and $F^{s}_{pq}$ again of positive smoothness $s > 0$, now based on $L^p$-spaces with $0 < p \leq \infty$ and reproducing formulas for smooth functions. This method applies to $\mathbb{R}^n$, domains in $\mathbb{R}^n$, closed sets in $\mathbb{R}^n$ and abstract quasi-metric spaces. We compare the outcome with other approaches.

Let $k$ be a non-negative $C^\infty$ function in $\mathbb{R}^n$ with

$$\text{supp } k \subset \{ y \in \mathbb{R}^n : |y| < 2^{J-\varepsilon}, y > 0 \}$$

for some $\varepsilon > 0$ and $J \in \mathbb{N}$ (one may fix $J = n$), and

$$\sum_{m \in \mathbb{Z}^n} k(x - m) = 1, \quad x \in \mathbb{R}^n.\quad (2)$$

Then

$$k^\beta(x) = (2^{-J}x)^\beta k(x) \geq 0 \quad \text{if} \quad x \in \mathbb{R}^n \quad \text{and} \quad \beta \in \mathbb{N}_0^n.\quad (3)$$

Let

$$k^\beta_{jm}(x) = k^\beta(2^j x - m) \quad \text{where} \quad j \in \mathbb{N}_0 \quad \text{and} \quad m \in \mathbb{Z}^n.\quad (4)$$

Let $K$ be a second function of this type with (1), (2), and let $K^\beta$ and $K^\beta_{jm}$ as in (3), (4). Let $(f, g) = f(g)$ with $f \in S'(\mathbb{R}^n)$ and $g \in S(\mathbb{R}^n)$ be the usual dual pairing.

**Theorem 1.** There are universal functions $\Psi_{\beta, \gamma}^{\beta, \gamma}_{j,r;m,l} \in S(\mathbb{R}^n)$ with $\beta \in \mathbb{N}_0^n$, $\gamma \in \mathbb{N}_0^n$; $j \in \mathbb{N}_0$, $r \in \mathbb{N}_0$; $m \in \mathbb{Z}^n$, $l \in \mathbb{Z}^n$, such that for any couple of functions $k, K$ as above,

$$K^\gamma_{rl}(x) = \sum_{\beta,j \geq r,m} \left( K^\gamma_{rl}, \Psi_{j,r;m,l}^{\beta, \gamma} \right) k^\beta_{jm}(x), \quad x \in \mathbb{R}^n,\quad (5)$$

unconditional convergence being in $S(\mathbb{R}^n)$. For any $\varkappa > 0$, $M > 0$, $b > 0$, there is a positive constant $c = c_{\varkappa,M,b}$ (depending on $k$, $K$) such that

$$\left| \left( K^\gamma_{rl}, \Psi_{j,r;m,l}^{\beta, \gamma} \right) \right| \leq c 2^{-|\gamma|} 2^{-|\beta|} 2^{-(j-r)M} (1 + |m - 2^j r l|)^{-b}.\quad (6)$$


This reproduction of one system \( \{ K_{\gamma} \} \) by a second system \( \{ k_{\beta}^{jm} \} \) at the expense of complex numbers with (6) paves the way to introduce spaces

\[
\mathcal{B}_{pq}^{s}(\mathbb{R}^{n}) \quad \text{and} \quad \mathcal{F}_{pq}^{s}(\mathbb{R}^{n}) \quad \text{for} \quad s > 0, \ 0 < p \leq \infty, \ 0 < q \leq \infty, \quad (7)
\]

(with \( p < \infty \) for the \( \mathcal{F} \)-spaces) as subspaces of \( L_{p}(\mathbb{R}^{n}) \) via representations

\[
f = \sum_{\beta,j,m} \lambda_{jm}^{\beta} k_{jm}^{\beta}, \quad (8)
\]

where \( \{ \lambda_{jm}^{\beta} \} \subset \mathbb{C} \) belongs to some sequence spaces. This will be done in Section 4. But first we collect in Section 2 what is known in this respect for the (nowadays almost classical) Fourier-analytically defined spaces \( B_{pq}^{s}(\mathbb{R}^{n}) \) and \( F_{pq}^{s}(\mathbb{R}^{n}) \). Based on these results we prove Theorem 1 in Section 3. Then it comes out that for \( 0 < p \leq \infty \) (\( p < \infty \) for \( F \)-spaces) and \( 0 < q \leq \infty \),

\[
\mathcal{B}_{pq}^{s}(\mathbb{R}^{n}) = B_{pq}^{s}(\mathbb{R}^{n}) \quad \text{if} \quad s > \sigma_{p} = n \left( \frac{1}{p} - 1 \right)_{+}, \quad (9)
\]

and

\[
\mathcal{F}_{pq}^{s}(\mathbb{R}^{n}) = F_{pq}^{s}(\mathbb{R}^{n}) \quad \text{if} \quad s > \sigma_{pq} \left( \frac{1}{\min(p,q)} - 1 \right)_{+}, \quad (10)
\]

whereas \( \mathcal{B}_{pq}^{s}(\mathbb{R}^{n}), \mathcal{F}_{pq}^{s}(\mathbb{R}^{n}) \) with \( p < 1 \) and \( 0 < s < \sigma_{p} \) as subspaces of \( L_{p}(\mathbb{R}^{n}) \) on the one hand, and \( B_{pq}^{s}(\mathbb{R}^{n}), F_{pq}^{s}(\mathbb{R}^{n}) \) as subspaces of \( S'(\mathbb{R}^{n}) \) (containing singular distributions) on the other hand, cannot coincide. Finally we deal in Section 5 with spaces of type (7) on sets in \( \mathbb{R}^{n} \), preferably on domains (= open sets) \( \Omega \) and on compact sets \( \Gamma \) which are supports of finite Radon measures in \( \mathbb{R}^{n} \). Again we compare the outcome with spaces introduced by other means and we formulate some problems worth to be considered in the sequel.

## 2 Fourier-analytical spaces

### 2.1 Basic notation

We use standard notation: \( \mathbb{R}^{n} \) (euclidean \( n \)-space), \( \mathbb{R} = \mathbb{R}^{1} \), \( \mathbb{C} \) (complex plane), \( \mathbb{Z}^{n} \) (lattice of all points in \( \mathbb{R}^{n} \) with integer-valued components), \( \mathbb{N} \) (natural numbers), \( \mathbb{N}_{0} \) (non-negative integers), \( \mathbb{N}_{n} \) (multi-indices), \( x^{\beta} = \)
\[ x_1^{\beta_1} \cdots x_n^{\beta_n} \text{ (monomials with } x \in \mathbb{R}^n \text{ and } \beta \in \mathbb{N}_0^n), \quad S(\mathbb{R}^n) \text{ (Schwartz space),} \]
\[ S'(\mathbb{R}^n) \text{ (tempered distributions),} \quad \hat{f} = Ff \text{ (Fourier transform of } f \in S'(\mathbb{R}^n)), \]
\[ f^\vee = F^{-1}f \text{ (related inverse Fourier transform).} \]

Let
\[
\| f \|_{L_p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p}, \quad 0 < p \leq \infty,
\]  
Obviously modified if \( p = \infty \). We assume that the reader is familiar with basic assertions for the spaces \( B_{pq}^s(\mathbb{R}^n) \) and \( F_{pq}^s(\mathbb{R}^n) \). All what one needs (and more) can be found in \([27, 28, 35]\).

Let \( \varphi_0 \in S(\mathbb{R}^n) \) with \( \varphi_0(x) = 1 \) if \( |x| \leq 1 \) and \( \varphi_0(y) = 0 \) if \( |y| \geq 3/2 \). Let
\[
\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \quad x \in \mathbb{R}^n, \quad k \in \mathbb{N}.
\]  
Then \( \sum_{j=0}^{\infty} \varphi_j(x) = 1 \) is a dyadic resolution of unity and \( (\varphi_j \hat{f})^\vee(x) \) is an entire analytic function for any \( f \in S'(\mathbb{R}^n) \).

**Definition 2.** Let \( \varphi = \{ \varphi_j \}_{j=0}^{\infty} \) as above. Let \( 0 < p \leq \infty \) (with \( p < \infty \) for the \( F \)-spaces), \( 0 < q \leq \infty, \quad s \in \mathbb{R} \). Then \( B_{pq}^s(\mathbb{R}^n) \) is the collection of all \( f \in S'(\mathbb{R}^n) \) such that
\[
\| f \|_{B_{pq}^s(\mathbb{R}^n)} = \left( \sum_{j=0}^{\infty} 2^{jsq} \left\| (\varphi_j \hat{f})^\vee \right\|_{L_p(\mathbb{R}^n)}^q \right)^{1/q} < \infty
\]
and \( F_{pq}^s(\mathbb{R}^n) \) is the collection of all \( f \in S'(\mathbb{R}^n) \) such that
\[
\| f \|_{F_{pq}^s(\mathbb{R}^n)} = \left\| \sum_{j=0}^{\infty} 2^{jsq} \left| (\varphi_j \hat{f})^\vee (\cdot) \right|^q \right\|_{L_p(\mathbb{R}^n)}^{1/q} < \infty.
\]

**Remark 3.** If \( q = \infty \) then one has to modify in the usual way. Furthermore, \( L_p(\mathbb{R}^n) \) with \( 0 < p \leq \infty \) are the usual spaces quasi-normed by \((11)\). The spaces \( B_{pq}^s(\mathbb{R}^n) \) and \( F_{pq}^s(\mathbb{R}^n) \) are quasi-Banach spaces, they are independent of \( \varphi \) (equivalent quasi-norms). As for special cases (classical and fractional Sobolev spaces, classical Nikolskij-Besov spaces, Hardy spaces) one may consult the above literature. We only mention that
\[
C^\sigma(\mathbb{R}^n) = B_{\infty \infty}^\sigma(\mathbb{R}^n), \quad \sigma \in \mathbb{R},
\]  
are the Hölder-Zygmund spaces (extended to \( \sigma \leq 0 \)).
2.2 Quarkonial decompositions and wavelet frames

Let \( \chi_{jm} \) with \( j \in \mathbb{N}_0 \) and \( m \in \mathbb{Z}^n \) be the characteristic function of the cube \( Q_{jm} \) in \( \mathbb{R}^n \) with sides parallel to the axes of coordinates, centred at \( 2^{-j}m \) and with side-length \( 2^{-j+1} \).

**Definition 4.** Let \( \varrho \geq 0 \), \( s \in \mathbb{R} \), \( 0 < p \leq \infty \), \( 0 < q \leq \infty \) and

\[
\lambda = \left\{ \chi_{jm}^\beta \in \mathbb{C} : \beta \in \mathbb{N}_0^n, j \in \mathbb{N}_0, m \in \mathbb{Z}^n \right\}.
\]

Then

\[
\| \lambda \|_{b_{pq}^s, \varrho} = \sup_{\beta \in \mathbb{N}_0^n} 2^{\varrho|\beta|} \left( \sum_{j=0}^{\infty} 2^{j(s-n/p)q} \left( \sum_{m \in \mathbb{Z}^n} |\chi_{jm}^\beta|^p \right)^{q/p} \right)^{1/q}
\]

and

\[
\| \lambda \|_{f_{pq}^{s, \varrho}} = \sup_{\beta \in \mathbb{N}_0^n} 2^{\varrho|\beta|} \left\| \left( \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{jsq}|\chi_{jm}^\beta|^q \chi_{jm}(\cdot) \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)}
\]

with the usual modifications if \( p = \infty \) and/or \( q = \infty \).

**Remark 5.** Corresponding sequence spaces \( b_{pq}^{s, \varrho} \) and \( f_{pq}^{s, \varrho} \) play a role in the theory of quarkonial decompositions of function spaces. We refer in particular to [30, Section 2]. They originate from atomic decompositions in function spaces. References and explanations may be found in Section 1 both in [28] and [35], but also in [29].

Next we describe the basic ingredients of quarkonial decompositions in the spaces \( B_{pq}^s(\mathbb{R}^n) \) and \( F_{pq}^s(\mathbb{R}^n) \).

**Definition 6.** Let \( \omega \) be a \( C^\infty \) function in \( \mathbb{R}^n \) with

\[
supp \omega \subset (-\pi, \pi)^n \quad \text{and} \quad \omega(x) = 1 \quad \text{if} \quad |x| \leq 2.
\]

Let for \( J \in \mathbb{N} \) as in (1)-(3),

\[
\omega^\beta(x) = \frac{j! |\beta| !}{(2\pi)^n \beta!} x^\beta \omega(x), \quad x \in \mathbb{R}^n, \quad \beta \in \mathbb{N}_0^n
\]

(15)
and

\[ \Omega^\beta(x) = \sum_{m \in \mathbb{Z}^n} (\omega^\beta) (m) e^{-imx}, \quad x \in \mathbb{R}^n. \]

Let \( \varphi_0 \) and \( \varphi_1 \) as in connection with (12). Then for \( \beta \in \mathbb{N}_0^n \) and \( m \in \mathbb{Z}^n \),

\[
(\Phi^\beta_F)(\xi) = \varphi_0(\xi) \omega^\beta(\xi), \quad \xi \in \mathbb{R}^n,
\]

\[
(\Phi^\beta_M)(\xi) = \varphi_1(\xi) \omega^\beta(\xi), \quad \xi \in \mathbb{R}^n,
\] (16)

and

\[
\Phi^\beta_{jm}(x) = \begin{cases} 
\Phi^\beta_F(x - m) & \text{if } j = 0, \\
\Phi^\beta_M(2^j x - m) & \text{if } j \in \mathbb{N}.
\end{cases}
\] (17)

Remark 7. We followed [31] in the slight modification according to [35, Sections 3.2.1, 3.2.2]. There one finds also further explanations. In particular,

\[
\omega^\beta(x) = (2\pi)^{-n/2} \Omega^\beta(x) \quad \text{if} \quad x \in (-\pi, \pi)^n.
\] (18)

Hence, \( \Phi^\beta_F, \Phi^\beta_M \) and \( \Phi^\beta_{jm} \) are analytic functions belonging to \( S(\mathbb{R}^n) \). Furthermore, according to [35, Remark 3.22] there is a constant \( c \geq 0 \) such that for all \( K \in \mathbb{N} \), all \( \alpha \in \mathbb{N}_0^n \) with \( |\alpha| \leq 2K \), all \( \beta \in \mathbb{N}_0^n \), all \( \xi \in \mathbb{R}^n \) and some constant \( c_K \),

\[
|D^\alpha \omega^\beta(\xi)| \leq c_K (1 + |\beta|)^{2K} 2^{c|\beta|}.
\] (19)

Both (18) and (19) will be of some service for us later on. They can also be derived by elementary arguments. Let \( \sigma_p \) and \( \sigma_{pq} \) be the same numbers as in (9) and (10). Let

\[
\lambda^\beta_{jm}(f) = 2^{jn} (f, \Phi^\beta_{jm}) \quad \text{for} \quad f \in S'(\mathbb{R}^n),
\] (20)

according to the usual dual pairing in \( S(\mathbb{R}^n), S'(\mathbb{R}^n) \). Let \( p = 1 \) if \( 1 \leq p \leq \infty \) and \( p = 1 \) if \( 0 < p < 1 \). Let \( L_\infty(\mathbb{R}^n, w_\sigma) \) be the obviously normed space of all functions \( f \) such that \( w_\sigma f \in L_\infty(\mathbb{R}^n) \) where \( w_\sigma(x) = (1 + |x|^2)^{\sigma/2} \). The next theorem deals with representations of type (8) where \( \beta \in \mathbb{N}_0^n \), \( j \in \mathbb{N}_0 \), \( m \in \mathbb{Z}^n \). In any case the corresponding series converge absolutely (and hence unconditionally) in \( L_p(\mathbb{R}^n) \) if \( p < \infty \) and in \( L_\infty(\mathbb{R}^n, w_\sigma) \) with \( \sigma < 0 \) if \( p = \infty \). This justifies the writing in (8) and in what follows.

Theorem 8. Let \( k^\beta_{jm} \) as in (1)-(4) and let \( \Phi^\beta_{jm} \) according to Definition 6 be two fixed systems. Let \( \rho \geq 0 \) and \( 0 < q \leq \infty \).
Let \( 0 < p \leq \infty \) and \( s > \sigma_p \). Then \( f \in S'(\mathbb{R}^n) \) is an element of \( B^s_{pq}(\mathbb{R}^n) \) if, and only if, it can be represented as

\[
    f = \sum_{\beta,j,m} \lambda_{jm}^{\beta} k_{jm}^{\beta}, \quad \lambda \in b_{pq}^{s,\varrho},
\]

absolute convergence being in \( L_p(\mathbb{R}^n) \) if \( p < \infty \) and in \( L_{\infty}(\mathbb{R}^n, w_\sigma) \) where \( \sigma < 0 \) if \( p = \infty \). Furthermore,

\[
    \| f \|_{B^s_{pq}(\mathbb{R}^n)} \sim \inf \| \lambda \|_{b_{pq}^{s,\varrho}}
\]

are equivalent quasi-norms where the infimum is taken over all admissible representations (21).

(ii) Let \( 0 < p < \infty \) and \( s > \sigma_{pq} \). Then \( f \in S'(\mathbb{R}^n) \) is an element of \( F^s_{pq}(\mathbb{R}^n) \) if, and only if, it can be represented as

\[
    f = \sum_{\beta,j,m} \lambda_{jm}^{\beta} k_{jm}^{\beta}, \quad \lambda \in f_{pq}^{s,\varrho},
\]

absolute convergence being in \( L_p(\mathbb{R}^n) \). Furthermore,

\[
    \| f \|_{F^s_{pq}(\mathbb{R}^n)} \sim \inf \| \lambda \|_{f_{pq}^{s,\varrho}}
\]

are equivalent quasi-norms where the infimum is taken over all admissible representations (22).

(iii) Let \( \lambda_{jm}(f) \) be given by (20). Let \( 0 < p \leq \infty \) and \( s > \sigma_p \). Then \( f \in B^s_{pp}(\mathbb{R}^n) \) can be represented as

\[
    f = \sum_{\beta,j,m} \lambda_{jm}(f) k_{jm}^{\beta},
\]

absolute convergence being in \( L_p(\mathbb{R}^n) \) if \( p < \infty \) and in \( L_{\infty}(\mathbb{R}^n, w_\sigma) \) where \( \sigma < 0 \) if \( p = \infty \). Furthermore, \( \lambda(f) \in b_{pp}^{s,\varrho} \) and

\[
    \| f \|_{B^s_{pp}(\mathbb{R}^n)} \sim \| \lambda(f) \|_{b_{pp}^{s,\varrho}}
\]

(equivalent quasi-norms).

Remark 9. Parts (i) and (ii) are special cases of more general quarkonial decomposition theorems according to [30, Section 2] which may also be found (without proofs) in [35, Section 1.6]. Part (iii) is covered by by [31] and [35,
Section 3.2.2]. We add two comments. First one may ask whether one has for all spaces $B_{pq}^{\ast}(\mathbb{R}^n)$ covered by (i) and all spaces $F_{pq}^{\ast}(\mathbb{R}^n)$ covered by (ii) optimal frame representations as in (23) with (20), (24). This is possible but requires some extra efforts. We refer to [30, Corollary 2.12]. Secondly, one can replace in the above convergence assertions $L_{\bar{p}}$ by any

$$L_r \text{ with } p \leq r \leq \infty \text{ and } \frac{1}{r} > \frac{1}{p} - \frac{s}{n}.$$  

This follows from well-known embeddings. The choice $r = \bar{p} \geq 1$ makes clear that everything happens within the framework of $S'(\mathbb{R}^n)$. For our later purposes the choice $r = p$, hence absolute convergence in $L_p(\mathbb{R}^n)$ (with the indicated modification if $p = \infty$) is better adapted to what follows.

3 Proof of Theorem 1, further comments

3.1 Proof of Theorem 1

Step 1. First we prove (5), (6) with $r = 0$ and $l = 0$. Recall that $K^{\gamma}(x) = (2^{-j}x)^{\gamma}K(x)$ with $\gamma \in \mathbb{N}_0^n$. By (23) and (20), (17) we have

$$K^{\gamma}(x) = \sum_{\beta,j,m} \lambda^{\beta}_{jm}(K^{\gamma}) k^{\beta}_{jm}(x),$$

$$\lambda^{\beta}_{jm}(K^{\gamma}) = 2^{jn} \int_{\mathbb{R}^n} K^{\gamma}(y) \Phi^{\beta}_M(2^{j}y - m) \, dy$$

if $j \in \mathbb{N}$. In case of $j = 0$ one has to modify here and in what follows appropriately. Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq 2A$ for some $A \in \mathbb{N}$. Then it follows from (16), based on (18), (15), (19) and Stirling’s formula for $N!$ that for any $\varkappa > 0$ and any $\beta \in \mathbb{N}_0^n$,

$$\left| D^\alpha(\Phi^{\beta}_M)^{\gamma}(\xi) \right| \leq c_{A,\varkappa} 2^{-\varkappa|\beta|}.$$

By the support properties of $\varphi_1$ according to (12) and standard assertions of the Fourier transform it follows on the one hand that for any $b > 0$, any $\varkappa > 0$ and any $\beta \in \mathbb{N}_0^n$,

$$\left| \Phi^{\beta}_M(y) \right| \leq c_{b,\varkappa} 2^{-\varkappa|\beta|}(1 + |y|)^{-b}, \quad y \in \mathbb{R}^n, \quad (27)$$
and on the other hand that for any \( L \in \mathbb{N} \),
\[
(\Phi_M^\beta)^\vee (\xi) = |\xi|^{2L} \frac{\varphi_1(\xi)}{|\xi|^{2L}} \Omega^\beta(\xi) = |\xi|^{2L} \tilde{\Omega}^\beta(\xi)
\]
and hence
\[
\Phi_M^\beta(y) = \Delta^L \tilde{\Phi}_M^\beta(y), \quad y \in \mathbb{R}^n,
\]
where \( \Delta^L \) is the \( L \)th power of the Laplacian \( \Delta \). Here \( \tilde{\Phi}_M^\beta \) has the same properties as \( \Phi_M^\beta \). In particular we have (27) with \( \tilde{\Phi}_M^\beta \) in place of \( \Phi_M^\beta \). Then it follows by partial integration that
\[
\int_{\mathbb{R}^n} (\Delta^L K^\gamma)(y) \tilde{\Phi}_M^\beta(2^j y - m) \, dy = 2^{2jL} \int_{\mathbb{R}^n} K^\gamma(y) \Phi_M^\beta(2^j y - m) \, dy.
\]
By (26), (27) with \( \tilde{\Phi} \) in place of \( \Phi \), and the properties of \( K^\gamma \) one obtains that
\[
|\lambda^\gamma_{jm}(K^\gamma)| \leq c 2^{(n-2jL)2-|\gamma|2-\varepsilon|\beta|} \int_{|y| \leq 2^j} (1 + 2^j|y - m|)^{-b} \, dy \leq c' 2^{j(n-b-2L)2^{-|\gamma|2-\varepsilon|\beta|} (1 + |m|)^{-b}}.
\] (28)
Since first \( b > 0 \), then \( L \in \mathbb{N} \) and \( \varepsilon > 0 \) are at our disposal, (25), (28) justify (5), (6) with \( r = 0 \) and \( l = 0 \), hence
\[
|\lambda^\gamma_{jm}(K^\gamma)| \leq c 2^{-|\gamma|2-\varepsilon|\beta|} 2^{-jM} (1 + |m|)^{-b}.
\] (29)

**Step 2.** Let \( r = 0 \) and \( l \in \mathbb{Z}^n \). Then it follows from (25) that
\[
K^\gamma_{0,l}(x) = K^\gamma(x - l) = \sum_{\beta,j,m} \lambda^\beta_{jm}(K^\gamma) k^\beta_{j,m+2l}(x) = \sum_{\beta,j,m} \nu^\beta_{jm}(K^\gamma_{0,l}) k^\beta_{jm}(x)
\] (30)
with \( \nu^\beta_{jm}(K^\gamma_{0,l}) = \lambda^\beta_{jm-2l}(K^\gamma) \). By (29) one gets
\[
|\nu^\beta_{jm}(K^\gamma_{0,l})| \leq c 2^{-|\gamma|2-\varepsilon|\beta|} 2^{-jM} (1 + |m - 2jl|)^{-b},
\] (31)
hence (6) with \( r = 0 \). Let \( r \in \mathbb{N} \) and \( l \in \mathbb{Z}^n \). Then it follows from (30) that
\[
K^\gamma_{rl}(x) = K^\gamma(2^r x - l) = \sum_{\beta,j,m} \nu^\beta_{jm}(K^\gamma_{0,l}) k^\beta_{j+r,m}(x) = \sum_{\beta,j \geq r,m} \mu^\beta_{jm}(K^\gamma_{rl}) k^\beta_{jm}(x)
\] (32)
with \( \mu_{j,m}^{\beta}(K_{rl}^\gamma) = \nu_{j-r,m}^{\beta}(K_0^\gamma) \). By (31) one gets (6).

**Step 3.** By (20) and the above index-shifting in \( \mathbb{Z}^n \) and \( \mathbb{N}_0 \) the coefficients \( \mu_{j,m}^{\beta}(K_{rl}^\gamma) \) in (32) can be written as in (5) with some \( \Psi_{j,r,m,l}^{\beta,\gamma} \in S(\mathbb{R}^n) \) which can be calculated explicitly. By (5), (6) and the properties of \( k_{j,m}^{\beta} \) it follows that (5) converges in any norm

\[
\sup_{x \in \mathbb{R}^n, |\alpha| \leq L} (1 + |x|^2)^{\sigma/2} |D^\alpha g(x)|
\]

with \( \sigma > 0 \) and \( L \in \mathbb{N} \). Hence (5) converges unconditionally in \( S(\mathbb{R}^n) \).

### 3.2 Multiresolution property

**Remark 10.** The question arises which properties for the functions \( k \) and \( K \) in Theorem 1 are really needed to get an assertion of type (5), (6). As for \( k \) both the above arguments and also the frame representation in Theorem 8(iii) work for any resolution of unity according to (2) based on a compactly supported \( C^\infty \) function \( k \) in \( \mathbb{R}^n \). (Some technical modifications are necessary.) As for \( K \) we used only that \( K \) is a compactly supported \( C^\infty \) function. But one can surely weaken \( C^\infty \) by \( C^N \) and the assumption that \( K \) has a compact support by a corresponding assumption that \( K \) decays rapidly enough (in dependence on the applications having in mind).

**Remark 11.** The recent theory of orthogonal wavelet bases in, say, \( L_2(\mathbb{R}) \) is usually based on the so-called multiresolution analysis with the typical assertion

\[
\Phi(x) = \sum_{m=-\infty}^{\infty} a_m \Phi(2x - m), \quad x \in \mathbb{R},
\]

for the scaling function \( \Phi \), where \( a_m \) are suitable complex numbers. One may consult, for example, [37]. By Theorem 1 and the above comments one has the following counterpart.

**Corollary 12.** Let \( k \) be a compactly supported \( C^\infty \) function in \( \mathbb{R}^n \) generating the resolution of unity (2). Let \( k_\beta(x) = x^\beta k(x) \) where \( \beta \in \mathbb{N}_0^n \). Let \( N \in \mathbb{N} \). Then there are complex numbers \( \lambda_{j,m}^{\beta} \) such that

\[
k(x) = \sum_{\beta \in \mathbb{N}_0^n} \sum_{j=N}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{j,m}^{\beta} k_\beta(2^j x - m), \quad x \in \mathbb{R}^n,
\]
unconditional convergence being in $S(\mathbb{R}^n)$. For any $\varkappa > 0$, $M > 0$, $b > 0$, there is a positive constant $c = c_{\varkappa, M, b}$ (depending on $k$ and $N$) such that

$$|\lambda^\beta_{jm}| \leq c 2^{-\varkappa|\beta|} 2^{-jM} (1 + |m|)^{-b}.$$\hspace{1cm} \square

Proof. This follows from the arguments in Step 1 in Section 3.1, the comments in Remark 10 and (29).

Remark 13. One may speak about the multiresolution property if coarse grids such as $k(x - m)$ or $k^\beta(x - m)$ can be resolved by finer grids $k^\beta_{jm}$, $j \in \mathbb{N}$, generated by the same function $k$.

4 Spaces on $\mathbb{R}^n$

4.1 Definition and basic assertions

Theorem 1 is a cornerstone of what follows. The sequence spaces $b^{s,\varepsilon}_{pq}$ and $f^{s,\varepsilon}_{pq}$ have the same meaning as in Definition 4 and Remark 5. Furthermore, $L_p(\mathbb{R}^n)$ are the usual Lebesgue spaces, quasi-normed by (11), and $L_\infty(\mathbb{R}^n, w_\sigma)$ has the same meaning as in Remark 7.

Definition 14. Let $k$ be a non-negative $C^\infty$ function in $\mathbb{R}^n$ with (1) for some $J \in \mathbb{N}$ (one may fix $J = n$) and (2). Let $k^\beta_{jm}$ as in (4). Let $\varepsilon \geq 0$, $0 < q \leq \infty$, and $s > 0$.

(i) Let $0 < p \leq \infty$. Then $\mathfrak{B}^{s,\varepsilon}_{pq}(\mathbb{R}^n)$ is the collection of all $f \in L_p(\mathbb{R}^n)$ which can be represented as

$$f = \sum_{\beta,j,m} \lambda^\beta_{jm} k^\beta_{jm}, \quad \lambda \in b^{s,\varepsilon}_{pq},$$

absolute convergence being in $L_p(\mathbb{R}^n)$ if $p < \infty$ and in $L_\infty(\mathbb{R}^n, w_\sigma)$ with $\sigma < 0$ if $p = \infty$. Let

$$\|f\| \mathfrak{B}^{s,\varepsilon}_{pq}(\mathbb{R}^n) = \inf \|\lambda\| b^{s,\varepsilon}_{pq}$$

where the infimum is taken over all admissible representations (33).

(ii) Let $0 < p < \infty$. Then $\mathfrak{F}^{s,\varepsilon}_{pq}(\mathbb{R}^n)$ is the collection of all $f \in L_p(\mathbb{R}^n)$ which can be represented as

$$f = \sum_{\beta,j,m} \lambda^\beta_{jm} k^\beta_{jm}, \quad \lambda \in f^{s,\varepsilon}_{pq},$$

11
absolute convergence being in $L_p(\mathbb{R}^n)$. Let
\[ \| f \|_{\mathcal{F}^s_{pq}(\mathbb{R}^n)} = \inf \| \lambda \|_{F^s_{pq}(\mathbb{R}^n)}, \]
where the infimum is taken over all admissible representations (34).

**Remark 15.** To avoid any misunderstanding we remark that the absolute convergence of (33) is a consequence of $\lambda \in \mathcal{b}^s_{pq}$ and not an extra requirement. Similarly for (34).

**Theorem 16.** The above spaces $\mathcal{B}^s_{pq}(\mathbb{R}^n)$ and $\mathcal{F}^s_{pq}(\mathbb{R}^n)$ are quasi-Banach spaces. They are independent of $\varrho$ and $k$ (equivalent quasi-norms). Furthermore, for all admitted $s, p, q$,
\[ \mathcal{B}^s_{pq}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n), \quad \mathcal{F}^s_{pq}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n), \] (35)
and
\[ \mathcal{B}^s_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow \mathcal{F}^s_{pq}(\mathbb{R}^n) \hookrightarrow \mathcal{B}^s_{p,\max(p,q)}(\mathbb{R}^n). \] (36)

**Proof. Step 1.** Let $\varrho \geq 0$ be fixed. The continuous embedding (36) follows from a corresponding inclusion for the underlying sequence spaces, [29, Proposition 13.6, p.75]. We prove (35). By (36) it is sufficient to deal with the first inclusion. Let $0 < p \leq 1$ (otherwise one has to modify appropriately). Then it follows from (1), (3), and (14) that
\begin{align*}
\left\| \sum_{\beta,j,m} \lambda_{jk}^\beta k_{jm}^\beta \right\|_{L_p(\mathbb{R}^n)}^p & \leq \sum_{\beta,j} \int_{\mathbb{R}^n} \left\| \sum_m \lambda_{jm}^\beta (2^j x - m) \right\|_{L_p(\mathbb{R}^n)}^p d x \\
& \leq c \sum_{\beta,j} 2^{-jn} 2^{-\varepsilon |\beta| p} \sum_m |\lambda_{jm}^\beta|^p \\
& \leq c' \sum_{\beta,j} 2^{-\varepsilon |\beta| p - js_p}.
\end{align*}
(37)
This proves (35), where we used $\varepsilon > 0$ and $s > 0$. Then it follows by standard arguments that $\mathcal{B}^s_{pq}(\mathbb{R}^n)$ and $\mathcal{F}^s_{pq}(\mathbb{R}^n)$ are quasi-Banach spaces.

**Step 2.** It remains to prove the independence of $\varrho$ and $k$. First deal with the $\mathcal{B}$-spaces. We assume that $f$ is given by the counterpart of (33) with $K$ in place of $k$, where $K$ has the same meaning as in Theorem 1, hence
\[ f(x) = \sum_{\gamma,r,l} \nu_{\gamma}^r K_{jm}(x), \quad \nu \in \mathcal{b}^{s,\varrho}_{pq}, \] (38)

12
We insert (5), in the reformulation (32), in (38) and get

\[ f(x) = \sum_{\beta,j,m} \lambda^\beta_{jm} k^\beta_{jm}(x) \]

with

\[ \lambda^\beta_{jm} = \sum_{\gamma \in \mathbb{N}^n_0} \sum_{t=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \nu^\gamma_{t,l} \cdot \mu^\beta_{jm}(K^\gamma_{rl}). \]  

(39)

Let \( \nu^\gamma_{t,l} = 0 \) if \( r < 0 \) and \( t = j - r \). Then it follows from (6) (with the \( \mu \)-coefficients on the left-hand side) that

\[ 2^{j(s-n/p)} 2^{\nu|\gamma|} |\lambda^\beta_{jm}| \]

\[ \leq c \sum_{\gamma} 2^{-\varepsilon|\gamma|} \sum_{t=0}^{\infty} \sum_{l \in \mathbb{Z}^n} 2^{(j-t)(s-n/p)} |\nu^\gamma_{t-j,l}| 2^{-tM'} (1 + |m - 2^t l|)^{-b} \]

\[ \leq c' \sum_{\gamma} 2^{-\varepsilon|\gamma|} \left( \sum_{t,l} 2^{(j-t)(s-n/p)q} |\nu^\gamma_{t-j,l}|^q 2^{-tM''q} (1 + |m - 2^t l|)^{-b'q} \right)^{1/q} \]

(40)

where \( \varepsilon > 0 \), \( M'' > 0 \), \( b' > 0 \) are at our disposal. We take the \( \ell_q \)-quasinorm with respect to \( m \in \mathbb{Z}^n \). The related factors at the right-hand side of (40) can be estimated independently of \( 2^t l \) (choosing \( b' > 0 \) sufficiently large). Then the summation over \( l \in \mathbb{Z}^n \) gives the desired \( \ell_q \)-blocks for the \( \nu_{j-t,l} \)-coefficients. Afterwards one can do the same with respect to \( \ell_p \) and the summation over \( j \) at the expense of \( M'' > 0 \). Since \( \varepsilon > 0 \) one gets

\[ \| \lambda \|_{\ell_q^{s,\nu}} \leq c \| \nu \|_{\ell_p^{s,\nu}} \leq c' \| \nu \|_{\ell_p^{s,\nu}} \]

(41)

for any \( \nu \geq 0 \) and any \( \nu \geq 0 \). This proves the independence of the spaces \( \mathfrak{B}^{s,q}_{pq}(\mathbb{R}^n) \) both of \( \nu \) and of \( k \).

Step 3. We prove that the \( \mathfrak{F} \)-spaces are independent of \( \nu \) and \( k \). We need some preparations. Recall that for locally integrable functions \( g \) in \( \mathbb{R}^n \),

\[ (Mg)(x) = \sup |Q|^{-1} \int_Q |g(y)| \, dy, \quad x \in \mathbb{R}^n, \]

(42)

is the Hardy-Littlewood maximal function, where the supremum is taken over all cubes \( Q \) centred at \( x \). Let \( 0 < p < \infty \), \( 0 < q \leq \infty \), and \( 0 < w < \min(p,q) \).
Then there is a constant $c$ such that for all such functions $g_{jm}$ with $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$,
\[
\left\| \left( \sum_{j,m} (M|g_{jm}|^w) (\cdot)^{q/w} \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)} \leq c \left\| \left( \sum_{j,m} |g_{jm}|^q \right)^{1/q} \right\|_{L_p(\mathbb{R}^n)}.
\]  
(43)

This vector-valued maximal inequality goes back to [11]. A short proof may be found in [26, pp. 303-305]. As for the use of (43) in the context of $F$-spaces and further references one may consult [28, p. 89] and [29, p. 79]. We need a second preparation. Let $\chi_r$ be the characteristic function of the cube of side-length $2^{-r}$ with $r \in \mathbb{N}_0$ centred at the origin. Then it follows from (42) that for some $c > 0$ (which is independent of $r$)
\[
(M \chi_r)(x) \geq c \min \left[ 1, 2^{-rn} |x|^{-n} \right], \quad x \in \mathbb{R}^n.
\]  
(44)

We compare $\chi_{jm}$ with $M \chi_{rl}$ as needed in (39) in connection with $f_{pq,\varrho}$ according to Definition 4. In particular $r \leq j$. Then it follows for $x = 2^{-j}m$ from (44) that
\[
(M \chi_{rl})(x)^{1/w} \geq c \min \left[ 1, (2^{-rn} |2^{-j}m - 2^{-r}l|^{-n})^{1/w} \right] \\
\geq c \min \left[ 1, |m - 2^{-r}l|^{-n/w} \right].
\]  
(45)

This estimate remains valid for all $x \in \mathbb{R}^n$ with $\chi_{jm}(x) = 1$ on the left-hand side of (45) and the same right-hand side. With $t = j - r$ one gets
\[
\chi_{jm}(x) \leq c' \left( 1 + |m - 2^t l|^{n/w} \right) (M \chi_{j-t,l})(x)^{1/w}, \quad x \in \mathbb{R}^n.
\]  
(46)

We multiply (39) with $\chi_{jm}(x)$ and use (46). Since $b > 0$ in (40) is at our disposal we arrive at a counterpart of (40). Using (43) one gets the $f$-counterpart of (41). This completes the proof.

**Remark 17.** That $\varrho$ in Definition 14 and Theorem 16 can be chosen arbitrarily large (equivalent quasi-norms) looks a little bit like black magic. It follows from (40) and, as a consequence, (41) with $\varphi = \varrho$. To get a better understanding one can ask for the dependence of $c$ in (6) on $\varphi = \varrho$, what, in turn, directly influences the constants in equivalent quasi-norms for different values of $\varrho$. In a slightly different but near-by context we estimated corresponding constants in [30, (2.81), p. 23]. This suggests that $c$ in (6) can be replaced by $c_1 2^{q\varphi}$ with $c_1 > 0$, $c_2 > 0$, independent of $\varphi$. Hence the price to pay for rapid decay $2^{-\varphi|\beta|}$ are exponentially exploding constants $c_1 2^{c_2\varphi}$. 

14
4.2 Properties

Let $\sigma_p$, $\sigma_{pq}$ as in (9), (10).

**Theorem 18.** Let $B_{pq}^s(\mathbb{R}^n)$, $F_{pq}^s(\mathbb{R}^n)$ be as in Definition 2 and let $\mathfrak{B}_{pq}^s(\mathbb{R}^n)$, $\mathfrak{F}_{pq}^s(\mathbb{R}^n)$ be as in Definition 14. Then

$$B_{pq}^s(\mathbb{R}^n) = \mathfrak{B}_{pq}^s(\mathbb{R}^n) \quad \text{if} \quad s > \sigma_p$$

and

$$F_{pq}^s(\mathbb{R}^n) = \mathfrak{F}_{pq}^s(\mathbb{R}^n) \quad \text{if} \quad s > \sigma_{pq},$$

interpreted as subspaces of $S'(\mathbb{R}^n)$.

**Proof.** This follows immediately from Theorem 8 and Remark 9. \hfill \Box

**Remark 19.** By Theorem 8 and Remark 9 we have for all spaces in the above theorem the frame representation (23), (20). But it is not clear whether there is a corresponding counterpart of those $\mathfrak{B}$-spaces and $\mathfrak{F}$-spaces which are not covered by the above theorem. If $0 < s < \sigma_p$ (hence $p < 1$) then nothing like (47), (48) can be expected. In this case $B_{pq}^s(\mathbb{R}^n)$ and $F_{pq}^s(\mathbb{R}^n)$ contain singular distributions, for example the $\delta$-distribution whereas $\mathfrak{B}_{pq}^s(\mathbb{R}^n)$ and $\mathfrak{F}_{pq}^s(\mathbb{R}^n)$ are subsets of $L_p(\mathbb{R}^n)$. But there might be an other possibility. Let

$$(\Delta^l_h f)(x) = f(x + h) - f(x), \quad \Delta^{l+1}_h = \Delta^l_h \Delta^l_h,$$

where $x \in \mathbb{R}^n$, $h \in \mathbb{R}^n$, $l \in \mathbb{N}$, and let

$$\omega_m(f, t)_p = \sup_{|h| \leq t} \|\Delta^m_h f \|_{L_p(\mathbb{R}^n)}, \quad 0 < t < \infty,$$

be the usual modulus of continuity. Let $s > 0$, $0 < p \leq \infty$, $0 < q \leq \infty$. Let $s < m \in \mathbb{N}$. Then, by definition, $B_{pq}^s(\mathbb{R}^n)$ is the collection of all $f \in L_p(\mathbb{R}^n)$ such that

$$\|f \|_{B_{pq}^s(\mathbb{R}^n)} = \|f \|_{L_p(\mathbb{R}^n)} + \left( \int_0^t t^{-sq} \omega_m(f, t)^q \frac{dt}{t} \right)^{1/q} < \infty \quad (49)$$

(with the usual modification if $q = \infty$). These spaces are independent of $m$ (equivalent quasi-norms). We refer to [8, Ch. 2, §10]. The interest in these
spaces comes from approximation theory and numerics. We refer to [25] and [9]. Furthermore one can complement (47) by

$$B^{s}_{pq}(\mathbb{R}^n) = \mathcal{B}^{s}_{pq}(\mathbb{R}^n) \text{ if } 0 < p \leq \infty, 0 < q \leq \infty, s > \sigma_p,$$

[28, Section 2.6.1]. In any case, $B^{s}_{pq}(\mathbb{R}^n)$ are subspaces of $L_p(\mathbb{R}^n)$ and one may ask whether

$$B^{s}_{pq}(\mathbb{R}^n) = \mathcal{B}^{s}_{pq}(\mathbb{R}^n) \text{ if } 0 < p \leq \infty, 0 < q \leq \infty, s > 0.$$  \hspace{1cm} (50)

For the $F$-spaces there is also a good candidate for an $F$-counterpart of (50). Yu.V. Netrusov complemented in [22] the scale $F^{s}_{pq}(\mathbb{R}^n)$ by spaces $F^{s}_{pq}(\mathbb{R}^n) = FL^{s}_{pq}(\mathbb{R}^n)$ with $s > 0, 0 < p < \infty, 0 < q \leq \infty$, as subspaces of $L_p(\mathbb{R}^n)$. We refer in this context also to the substantial recent paper [18] in continuation of [22] and [1, Section 10], especially to [18, Theorem 3.14]. There one finds characterisations of $F^{s}_{pq}(\mathbb{R}^n)$ both in terms of atoms and by means of differences, the $F$-counterpart of (49). Although some details must be checked it is quite clear that one has both (50) and

$$\mathcal{F}^{s}_{pq}(\mathbb{R}^n) = F^{s}_{pq}(\mathbb{R}^n) \text{ if } 0 < p < \infty, 0 < q \leq \infty, s > 0.$$  \hspace{1cm} (51)

One direction follows from the observation that quarkonial representations according to Definition 14 can be converted into atomic decompositions according to [22, 18]. Conversely one can apply the reproducing formula (5) to any atom (where now $\gamma = 0$) and argue as in the proof of Theorem 16. Together with the indicated characterisations in terms of differences one gets both (51) and (50).

**Remark 20.** For the spaces $F^{s}_{pq}(\mathbb{R}^n)$ with $s > \sigma_{pq}$ one has (48) and also characterisations in terms of means of differences, [28, Theorem 3.5.3, p. 194]. The recent paper [7] indicates that such a characterisation in terms of means of differences cannot be expected if $\sigma_p < s < \sigma_{pq}$ (hence $0 < q < p$).

One may also consult the formulations in [35, Section 1.11.8, Theorem 1.116, Remark 1.117]. This suggests, combined with [18, Theorem 3.14], that

$$F^{s}_{pq}(\mathbb{R}^n) \neq F^{s}_{pq}(\mathbb{R}^n) \text{ if } 0 < p < \infty, 0 < q \leq \infty, 0 < s < \sigma_{pq},$$  \hspace{1cm} (52)

as subspaces of $L_p(\mathbb{R}^n)$, or, if, in addition, $s > \sigma_p$, as subspaces of $S'(\mathbb{R}^n)$. Of course, $0 < q < p < \infty, \sigma_p < s < \sigma_{pq}$, is the most surprising case in (52). Comparing atomic decompositions for $F^{s}_{pq}(\mathbb{R}^n)$, [22, 12, 13], with corresponding decompositions for $F^{s}_{pq}(\mathbb{R}^n)$, [22], [18, Theorem 3.14], it comes out as a
by-product that the usual cancelation conditions for atoms in \( F_{pq}^s(\mathbb{R}^n) \) with \( s < \sigma_{pq} \) are indispensable. One may consult in this context also [35, Section 1.5, Theorem 1.19, Remark 1.20]. This settles a long standing question.

**Definition 21.** A space \( A(\mathbb{R}^n) \) according to Definition 2 or to Definition 14 is said to have the positivity property if any \( f \in A(\mathbb{R}^n) \) can be decomposed as

\[
f = f_1 - f_2 + i f_3 - i f_4 \quad \text{with} \quad f_i \geq 0, \quad f_i \in A(\mathbb{R}^n),
\]

and

\[
\| f \|_{A(\mathbb{R}^n)} \sim \sum_{l=1}^{4} \| f_l \|_{A(\mathbb{R}^n)}
\]

where the equivalence constants are independent of \( f \).

**Remark 22.** Recall that \( f \geq 0 \) for \( f \in S'(\mathbb{R}^n) \) means that \( f(\varphi) \geq 0 \) for any real non-negative \( \varphi \in S(\mathbb{R}^n) \).

**Theorem 23.** All spaces \( B_{pq}^s(\mathbb{R}^n) \) and \( F_{pq}^s(\mathbb{R}^n) \) according to Definition 14 have the positivity property.

**Proof.** Decomposing \( \lambda^\beta_{jm} \in \mathbb{C} \) in (33) naturally one gets this assertion from \( k^\beta_{jm} \geq 0 \) according to (3), (4). \( \square \)

**Remark 24.** In particular the spaces \( B_{pq}^s(\mathbb{R}^n) \) in (47) and \( F_{pq}^s(\mathbb{R}^n) \) in (48) have the positivity property. This is known, [33] and [35, Section 3.3.2]. There one finds also further assertions of this type. In particular, the spaces \( B_{pq}^s(\mathbb{R}^n) \) and \( F_{pq}^s(\mathbb{R}^n) \) with \( 0 < p \leq \infty \) (\( p < \infty \) for the \( F \)-spaces) with \( 0 < q \leq \infty \) and \( -\infty < s < \sigma_p \) do not have the positivity property. As a consequence, these spaces cannot be represented as in Theorem 8(i),(ii).

## 5 Spaces on sets

### 5.1 Preliminaries, sequence spaces

The Definitions 4 for sequence spaces and 14 for function spaces can be extended to sets \( M \) in \( \mathbb{R}^n \). Two cases seem to be of special interest, first \( M = \Omega \) is a domain (= open set) in \( \mathbb{R}^n \) and secondly, \( M = \Gamma \) is a compact set (= fractal) in \( \mathbb{R}^n \). As for spaces \( B_{pq}^s(M) \) and \( F_{pq}^s(M) \) one always needs a
substitute of (35). In case of \( M = \Omega \) one has \( L_p(\Omega) \), naturally quasi-normed by
\[
\|f|L_p(\Omega)\| = \left( \int_{\Omega} |f(x)|^p \, dx \right)^{1/p}, \quad 0 < p \leq \infty,
\]
onobviously modified if \( p = \infty \). If \( \Gamma \) is a compact set in \( \mathbb{R}^n \) then there is
the temptation to rely on the remarkable observation in [36] that there is a
Radon measure \( \mu \) in \( \mathbb{R}^n \) satisfying the doubling condition with
\[
supp \, \mu = \Gamma \text{ compact, } \quad 0 < \mu(\Gamma) < \infty. \tag{53}
\]
However our point of view is different. We give preference to finite Radon
measures in \( \mathbb{R}^n \) (with or without the doubling condition) such that its support \( \Gamma \) according to (53) is compact. Such measures can be asked to which
spaces \( B^{s}_{pq}(\mathbb{R}^n) \) they belong. This results in a rather sharp fingerprint, called
Besov characteristics, of \( \mu \). We studied this in detail in [35] and the under-
lying papers. Although it might be useful in the context of this paper we
restrict ourselves to a coarser case, leaving more refined considerations to
later occasions.
Again let \( Q_{jm} \) with \( j \in \mathbb{N}_0 \) and \( m \in \mathbb{Z}^n \) be the above cubes with sides parallel
to the axes of coordinates, centred at \( 2^{-j}m \) and having side-length \( 2^{-j+1} \).
Recall that \( C^\sigma(\mathbb{R}^n) \) are the Hölder-Zygmund spaces according to (13). Then
\( \mu \in C^{-n}(\mathbb{R}^n) \) for any Radon measure \( \mu \) with (53). This assertion can be
sharpened as follows. Let
\[
0 < \sigma \leq n \quad \text{and} \quad \mu_j = \sup_{m \in \mathbb{Z}^n} \mu(Q_{jm}) \quad \text{with} \quad j \in \mathbb{N}_0. \tag{54}
\]
Then
\[
\mu \in C^{-\sigma}(\mathbb{R}^n) \iff \sup_j 2^{j(n-\sigma)} \mu_j < \infty. \tag{55}
\]
We refer to [32] or [35, Section 1.12]. For \( M \subset \mathbb{R}^n \) (either a domain \( \Omega \) or a
compact set \( \Gamma \)) we abbreviate
\[
\sum_{m}^{M,j} = \sum_{m \in \mathbb{Z}^n, Q_{jm} \cap M \neq \emptyset} \quad \text{where} \quad j \in \mathbb{N}_0.
\]
As above \( \chi_{jm} \) is the characteristic function of \( Q_{jm} \).
Definition 25. Let $\varrho \geq 0$, $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$. Let $M \subset \mathbb{R}^n$ and

$$\lambda = \left\{ \lambda_{jm}^\beta \in \mathbb{C} : \beta \in \mathbb{N}_0^n, j \in \mathbb{N}_0, m \in \mathbb{Z}^n \text{ with } Q_{jm} \cap M \neq \emptyset \right\}.$$ 

Then

$$\| \lambda \|_{b^{s, \varrho}_{pq}(M)} = \sup_{\beta \in \mathbb{N}_0^n} 2^{q|\beta|} \left( \sum_{j=0}^{\infty} 2^{j(s-n/p)q} \left( \sum_m |\lambda_{jm}^\beta|^p \right)^{q/p} \right)^{1/q}$$

(56)

and

$$\| \lambda \|_{f^{s, \varrho}_{pq}(M)} = \sup_{\beta \in \mathbb{N}_0^n} 2^{q|\beta|} \left( \left( \sum_{j=0}^{\infty} \sum_m M_{jm} 2^{jsq} |\lambda_{jm}^\beta|^q \chi_{jm}(\cdot) \right)^{1/q} \right)$$

(57)

with the usual modification if $p = \infty$ and/or $q = \infty$.

Remark 26. This is the counterpart of Definition 4. We denote the corresponding sequence spaces by $b^{s, \varrho}_{pq}(M)$ and $f^{s, \varrho}_{pq}(M)$. There arise some questions, especially in connection with (57). First we remark that

$$b^{s, \varrho}_{pp}(M) = f^{s, \varrho}_{pp}(M), \quad 0 < p \leq \infty.$$ 

One may ask what happens if one replaces $L_p(\mathbb{R}^n)$ by $L_p(\Gamma, \mu)$, quasi-normed by

$$\| f \|_{L_p(\Gamma, \mu)} = \left( \int_{\Gamma} |f(\gamma)|^p \mu(d\gamma) \right)^{1/p}$$

(58)

in case of $M = \Gamma$ with (53). In case of $M = \Omega$ we agree that (58) stands for $L_p(\Omega)$ with the Lebesgue measure $\mu(dx) = dx$ in place of $\mu$. Recall that a Radon measure $\mu$ in $\mathbb{R}^n$ with (53) is called isotropic if there is a continuous strictly increasing function $h$ on the interval $[0, 1]$ with $h(0) = 0$, $h(1) = 1$, and

$$\mu(B(\gamma, r)) \sim h(r) \quad \text{with } \gamma \in \Gamma \text{ and } 0 < r < 1,$$

(59)

where $B(\gamma, r)$ is a ball centred at $\gamma$ and of radius $r$. Then $\Gamma$ is called a $h$-set.

The special case $h(r) = r^d$ with $0 \leq d \leq n$ results in $d$-sets, and $\mu = \mathcal{H}^d|\Gamma$ is the restriction of the Hausdorff measure $\mathcal{H}^d$ in $\mathbb{R}^n$ to $\Gamma$. A detailed study of $h$-sets may be found in [4, 5]. Let $f^{s, \varrho, \mu}_{pp}$ be the spaces originating from (57) with $L_p(\Gamma, \mu)$ in place of $L_p(\mathbb{R}^n)$. Let $\Gamma$ be a $d$-set with $0 < d < n$. Then

$$f^{s, \varrho, \mu}_{pp} = f^{s, \varrho}_{pp}(\Gamma) = b^{s, \varrho}_{pp}(\Gamma), \quad 0 < p < \infty, \quad \sigma = s - \frac{n}{p} + \frac{d}{p}.$$ 

(60)
We indicate a proof of this assertion. It follows by (43), (44) that one can replace the characteristic function $\chi_{jm}$ of the cubes $Q_{jm}$ with $Q_{jm} \cap \Gamma \neq \emptyset$ in (57) by the characteristic function of balls centred at $\Gamma$ of radius $\sim 2^{-j}$, covering $\Gamma$, having pairwise distance $\geq 2^{-j}$. But then (60) follows from (59) with $h(r) = r^d$. If $p \neq q$ then the spaces $f_{pq}^{s,\varrho}(\Gamma)$ and $f_{pq}^{s,\varrho,\mu}$ might be unrelated. Nevertheless in case of isotropic measures $\mu$ the intrinsically defined spaces $f_{pq}^{s,\varrho,\mu}$ may be the better choice when it comes to the definition of related $\mathcal{F}$-spaces. There may well be a related counterpart of the vector-valued maximal inequality (43) paving the way to a corresponding theory for $\mathcal{F}$-spaces parallel to the above considerations in $\mathbb{R}^n$. In case of $d$-sets such a maximal inequality had been used in [16, pp. 80/81]. If $\mu$ is non-isotropic and non-doubling then the spaces $f_{pq}^{s,\varrho}(\Gamma)$ and $f_{pq}^{s,\varrho,\mu}$ and even $b_{pq}^{s,\varrho}(\Gamma)$ and (appropriately defined) $b_{pq}^{s,\varrho,\mu}$ seem to be unrelated. We stick here at the spaces from Definition 25 adding some assertions originating from the geometry of $M$.

**Definition 27.** (i) A domain $\Omega$ in $\mathbb{R}^n$ is said to be interior regular if there is a positive number $c$ such that $|\Omega \cap B| \geq c|B|$ for any ball centred at $\partial \Omega$ with radius less than 1.

(ii) A compact set $\Gamma$ in $\mathbb{R}^n$ is said to be porous if there is a number $\eta$ with $0 < \eta < 1$ such that one finds, for any ball $B(x, r)$, centred at $x \in \mathbb{R}^n$ and of radius $0 < r < 1$, a ball $B(y, \eta r)$ with

$$B(y, \eta r) \subset B(x, r) \quad \text{and} \quad B(y, r\eta) \cap \Gamma = \emptyset.$$ (61)

**Remark 28.** Interior regularity of domains plays a role in connection with atomic decompositions of function spaces in non-smooth domains, [10, p. 59]. Porosity comes from fractal geometry, [21, pp. 156-158]. We used this property, also called ball condition, in [30, pp. 138-141] in connection with function spaces on fractals. There one finds a characterisation of $h$-sets $\Gamma$ to be porous. In particular, any $d$-set with $d < n$ is porous. Furthermore if $\Gamma$ is porous then $|\Gamma| = 0$.

**Proposition 29.** (i) Let $\Omega$ be an interior regular domain in $\mathbb{R}^n$ and let $\tilde{\chi}_{jm}$ be the characteristic function of $\tilde{Q}_{jm} \cap \Omega$, where $\tilde{Q}_{jm}$ is a cube centred at $2^{-j}m$ and of side-length $2^{-j+2}$. Let $p < \infty$. Then the right-hand side of (57) with $\tilde{\chi}_{jm}$ in place of $\chi_{jm}$ is equivalent to the quasi-norm in (57) (intrinsic characterisation).

(ii) Let $p < \infty$ and let $\Gamma$ be a compact porous set in $\mathbb{R}^n$. Then $f_{pq}^{s,\varrho}(\Gamma)$ is
independent of \(q\), in particular
\[
f_{pq}^{s,\rho}(\Gamma) = b_{pp}^{s,\rho}(\Gamma).
\] (62)

Proof. Part (i) follows from (43), (44). As for part (ii) we first remark that one can replace, again by (43), (44), the cubes \(Q_{jm}\) by balls of radius \(2^{-j}\) centred at \(\Gamma\) having pairwise distance \(\geq 2^{-j}\). For each such ball one can choose a sub-ball according to (61) of radius \(\eta 2^{-j}\) where one may assume in addition that each sub-ball has a distance to \(\Gamma\) of at least \(c 2^{-j}\) for some \(c > 0\). Then all these sub-balls have a controlled overlapping, one may even assume that all these sub-balls have pairwise disjoint supports. This proves (62).

\[\square\]

Remark 30. In particular, if \(\Omega\) is interior regular then one can replace \(\chi_{jm}\) in (57) by \(\tilde{\chi}_{jm}\) (characteristic function of \(\tilde{Q}_{jm}\)) and \(L^p(\mathbb{R}^n)\) by \(L^p(\Omega)\) (equivalent quasi-norms).

5.2 Function spaces

Again \(M\) is either a domain \(\Omega\) in \(\mathbb{R}^n\) or a compact set \(\Gamma\) originating from a measure \(\mu\) with (53)-(55). To unify our notation we put \(L_p(\Omega) = L_p(M, \mu)\) if \(M = \Omega\) is a domain. If the domain \(\Omega\) is unbounded then \(L_{\infty}(\Omega, w_\sigma)\) stands for all \(f\) with \(w_\sigma f \in L_{\infty}(\Omega)\) where \(w_\sigma(x) = (1 + |x|^2)^{\sigma/2}\). Let \(k_{jm}^\beta\) be the same functions as in Definition 14. Furthermore \(b_{pq}^{s,\rho}(M)\) and \(f_{pq}^{s,\rho}(M)\) have the same meaning as in Definition 25 and Remark 26.

Definition 31. Let \(0 < p \leq \infty\) (\(p < \infty\) for the \(\mathcal{F}\)-spaces), \(0 < q \leq \infty\) and \(\rho \geq 0\). Let either
\[
M = \Omega \quad \text{domain in } \mathbb{R}^n \quad \text{and} \quad s > 0,
\]
or
\[
M = \Gamma \quad \text{with} \quad \mu \in C^{-\sigma}(\mathbb{R}^n), \quad 0 < \sigma \leq n, \quad s > \sigma/p,
\] (63)
according to (53)-(55). Then \(\mathcal{B}_{pq}^s(M, \mu)\) is the collection of all \(f \in L_p(M, \mu)\) which can be represented as
\[
f = \sum_{\beta, j} \sum_m M_{j}^{M, \beta} \lambda_{jm}^\beta k_{jm}^\beta, \quad \lambda \in b_{pq}^{s,\rho}(M),
\] (64)
absolute convergence being in \( L_p(M, \mu) \) (with the modification \( L_\infty(\Omega, w_\infty) \), \( \kappa < 0 \), if \( p = \infty \) and \( \Omega \) is an unbounded domain). Let

\[
\|f|_B^{s,pq}(M, \mu)\| = \inf \|\lambda|_b^{s,\varrho}(M)\|
\]

where the infimum is taken over all admissible representations (64). Similarly \( \mathcal{F}^{s,pq}_\mu(M, \mu) \) is the collection of all \( f \in L_p(M, \mu) \) which can be represented as

\[
f = \sum_{\beta,j} \sum_m^{M,j} \lambda_\beta^j k_{jm}^\beta, \quad \lambda \in f^{s,\varrho}_p(M),
\]

absolute convergence being in \( L_p(M, \mu) \). Let

\[
\|f|_{\mathcal{F}^{s,pq}_\mu}(M, \mu)\| = \inf \|\lambda|_f^{s,\varrho}(M)\|
\]

where the infimum is taken over all admissible representations (65).

**Remark 32.** This is the direct counterpart of Definition 14, including the comment in Remark 15. Obviously in case of domains \( M = \Omega \) one would prefer to write \( B^{s,pq}_\mu(\Omega) \) and \( \mathcal{F}^{s,pq}_\mu(\Omega) \).

**Theorem 33.** (i) The above spaces \( \mathcal{B}^{s,pq}_\mu(M, \mu) \) and \( \mathcal{F}^{s,pq}_\mu(M, \mu) \) are quasi-Banach spaces. They are independent of \( \varrho \) and \( k \) (equivalent quasi-norms). Furthermore for all admitted \( s, p, q \),

\[
\mathcal{B}^{s,pq}_\mu(M, \mu) \hookrightarrow L_p(M, \mu), \quad \mathcal{F}^{s,pq}_\mu(M, \mu) \hookrightarrow L_p(M, \mu),
\]

and

\[
\mathcal{B}^{s,\min(p,q)}_\mu(M, \mu) \hookrightarrow \mathcal{F}^{s,pq}_\mu(M, \mu) \hookrightarrow \mathcal{B}^{s,\max(p,q)}_\mu(M, \mu).
\]

(ii) Let, in addition, \( \Gamma \) in (53) be porous according to Definition 27(ii). Then \( \mathcal{F}^{s,pq}_\mu(\Gamma, \mu) \) is independent of \( q \), in particular,

\[
\mathcal{F}^{s,pq}_\mu(\Gamma, \mu) = \mathcal{B}^{s,pp}_\mu(\Gamma, \mu).
\]

**Proof.** We prove the first inclusion in (66) in case of \( M = \Gamma \) with (53)-(55). Let \( p < \infty \) (if \( p = \infty \) then one has to modify appropriately). One gets in analogy to (37),

\[
\left\| \sum_m^{\Gamma,j} \lambda_{jm}^\beta k_{jm}^\beta |L_p(\Gamma, \mu)|^p \right\| \leq c 2^{-\varepsilon|\beta|} \sum_m^{\Gamma,j} |\lambda_{jm}^\beta|^{p} \mu_j
\]

\[
\leq c' 2^{-\varepsilon|\beta|} 2^{-(n-\sigma)j} 2^{-jsp+jn}
\]

\[
\leq c' 2^{-\varepsilon|\beta|} 2^{-j(sp-\sigma)}.
\]

22
By (63) one gets the first inclusion in (66). All other assertions in part (i) can
be obtained as in the proof of Theorem 16. Part (ii) follows from Proposition
29(ii). □

5.3 Properties, comments, snowflaked transforms

We collect a few comments and references to related papers.

Remark 34. Let $M = \Omega$ be a domain in $\mathbb{R}^n$. Let $0 < p \leq \infty$ ($p < \infty$ for
the $\mathcal{F}$-spaces), $0 < q \leq \infty$, $s > 0$. Then it follows from the Definitions 14
and 31 that the spaces on $\Omega$ are restrictions of the corresponding spaces on
$\mathbb{R}^n$ in the usual interpretation,

$$\| f \|_{\mathfrak{B}^s_{pq}(\Omega)} = \inf \| g \|_{\mathfrak{B}^s_{pq}(\mathbb{R}^n)}, \quad g|\Omega = f,$$

as subspaces of $L_p(\Omega)$. Similarly for the $\mathcal{F}$-spaces. Since the Fourier-analytical
spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ are also defined by restrictions of the corresponding
spaces on $\mathbb{R}^n$ one gets in particular

$$\mathfrak{B}^s_{pq}(\Omega) = B^s_{pq}(\Omega) \quad \text{if} \quad s > \sigma_p, \quad \mathcal{F}^s_{pq}(\Omega) = F^s_{pq}(\Omega) \quad \text{if} \quad s > \sigma_{pq},$$

as a consequence of Theorem 18.

Remark 35. Let $\mu$ and $\Gamma$ as in (53)-(55). We dealt in [29, 30] and recently
in [35, Sections 1.17.2,7] with spaces on $\Gamma = \text{supp} \mu$. In particular if $0 < 1/p = t < 1, 0 < q \leq \infty$, and $s > 0$ then we defined quite naturally

$$B^s_{pq}(\Gamma, \mu) = \text{tr}_\mu B^{s+|s_\mu(1-t)|}_{pq}(\mathbb{R}^n)$$

as trace spaces, where $s_\mu(1-t)$ is the Besov characteristics of the Radon
measure $\mu$. We do not go into detail here. We only mention that in case of
d-sets, which we briefly mentioned after (59), one has $|s_\mu(1-t)| = t(n-d)$. If
$s + |s_\mu(1-t)| > \sigma t$ with $\mu$ and $\sigma$ as in (63), then $s > 0$ and

$$B^s_{pq}(\Gamma, \mu) = \mathfrak{B}^{s+|s_\mu(1-t)|}_{pq}(\Gamma, \mu). \quad (67)$$

The approach via traces is more subtle (so far) if $1 < p < \infty$. On the other
hand, the $\mathfrak{B}$-spaces as introduced in Definition 31 apply to all $0 < p \leq \infty$. 23
Remark 36. There are several proposals in the literature to introduce function spaces on abstract spaces \((X, \varrho, \mu)\) where \(X\) is a set, \(\varrho\) a (quasi-)metric and \(\mu\) a related Borel measure. In case of (first order) Sobolev spaces we refer to [14]. Another approach to spaces of Besov type on compact sets \(\Gamma\) in \(\mathbb{R}^n\) is based on the observation that \(\Gamma\) can be equipped with a doubling measure, [36], briefly mentioned at the beginning of Section 5.1. Refinements of this approach and related Besov spaces, based on (first) differences and atoms may be found in [20] and [6]. Further more detailed references can be found in [35, Section 1.17.5]. Nearer to us are spaces of type \(B^{s}_{pq}(X)\) and \(F^{s}_{pq}(X)\) on spaces \((X, \varrho, \mu)\) of homogeneous type mostly restricted to \(1 \leq p \leq \infty\) (but there are also some extensions to \(p < 1\)), \(1 \leq q \leq \infty\), \(|s| < 1\), where \(\mu\) is a Borel measure satisfying the doubling condition. Detailed references may be found in [35, Section 1.17.5]. We mention here the surveys [15] and [16]. In case of the spaces \(F^{s}_{pq}(X)\) the underlying technique might be nearer to the sequence spaces \(f^{s,\varrho,\mu}_{pq}\) mentioned briefly between (59) and (60) than to spaces of \(f^{s,\varrho}_{pq}\)-type as used above, hoping that there is a counterpart of the mighty instrument of vector-valued maximal inequalities according to (43) as used in [16]. But this requires apparently some regularity of \(\mu\), such as to be isotropic. For general measures a related counterpart of (43) is not available. Nevertheless there is a way to circumvent this difficulty, [17]. It should be mentioned that the theory of the spaces \(B^{s}_{pq}(X)\) and \(F^{s}_{pq}(X)\) as developed in [15, 16, 17] (with or without the vector-valued Hardy-Littlewood maximal inequalities) on the one hand and the approach presented in this paper on the other hand have one point in common. Both rely decisively on reproducing formulas of type (5) and their abstract counterparts in \((X, \varrho, \mu)\).

Remark 37. The above homogeneous space \((X, \varrho, \mu)\) is called a \(d\)-space for some \(d > 0\), if the doubling measure \(\mu\) satisfies

\[
\mu(B(x, r)) \sim r^d \quad \text{for } x \in X \text{ and } 0 < r \leq \text{diam}X < \infty,
\]

where \(B(x, r)\) is a ball centred at \(x\) and of radius \(r\). There is a number \(0 < \varepsilon_0 \leq 1\) such that one finds for any \(\varepsilon\) with \(0 < \varepsilon \leq \varepsilon_0\) a bi-Lipschitzian map \(H : \mathbb{R}^n \to \mathbb{R}^n\),

\[
H : (X, \varrho^\varepsilon, \mu) \iff (\Gamma, \varrho_n, \mathcal{H}^{d/\varepsilon}|\Gamma),
\]

of the so-called snowflaked version \((X, \varrho^\varepsilon, \mu)\) of \((X, \varrho, \mu)\) onto the indicated \(d/\varepsilon\)-set \(\Gamma\) in some (high-dimensional) \(\mathbb{R}^n\) with \(d/\varepsilon < n\), where \(\varrho_n\) is the corresponding Euclidean metric and \(\mathcal{H}^{d/\varepsilon}\) is the related Hausdorff measure.
This is the so-called **snowflaked transform** of \((X, \varrho, \mu)\). It gives the possibility to transfer

\[
B^s_{pq}(\Gamma, \mu) = \mathcal{B}^s_{pq} \frac{u^{d/\varepsilon}}{p} (\Gamma, \mu)
\]

according to (67) to the \(d\)-space \((X, \varrho, \mu)\), hence

\[
f = \sum_{\beta,j} \sum_{m}^{M,j} \lambda^j m H^{-1} k_{jm}^\beta
\]

as the transferred representation (65). In particular \(H^{-1} k_{jm}^\beta \geq 0\). We did this in detail in [34] restricted to \(s > 0\) and \(1 < p = q < \infty\). Now it can be extended to all related spaces covered by Definition 31 and Theorem 33. The snowflaked transform goes back to [2]. We refer also to [19] for a detailed account. A description and further references may also be found in [34] and [35, Sections 1.17,9]. But the main point in connection with the above context is the observation that this snowflaked transform works on the much larger scale of quasi-metric spaces \((X, \varrho, \mu)\) with doubling measures, [19]. More detailed references are given in [34] and [35, Section 1.17]. Based on Definition 31 one can transfer in this way \(\mathcal{B}^s\)-spaces to huge classes of homogeneous spaces \((X, \varrho, \mu)\) for all \(s > 0\), \(0 < p \leq \infty\), \(0 < q \leq \infty\). But this has not yet been done.

**References**


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