Some Variational Aspect of Fractal Sets

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1 Introduction

In this talk we are concerned with transmission problems involving highly conductive layers of fractal type imbedded in Euclidean domains.

There is a huge literature dealing with transmission problems as these problems arise naturally in various fields: electrostatic, magnetostatics, hydraulic fracturing and studies of absorption or irrigation techniques. For instance, in electrostatics and magnetostatics, the model problem which describes the heat transfer through an infinitely conductive layer is a transmission problem (see the paper of H. Pham Huy and E. Sanchez-Palencia [35]). Another engineering application is the model problem of the flow of oil in a fractured medium in order to increase the flow of oil from a "reservoir" into a producing oil well (see the paper of J. R. Cannon and G. H. Meyer, [4]). Variety of applications leading to transmission conditions, not necessarily of order two, can be found in the book of R. Dautray and J. L. Lions [5].

In many applications one is indeed interested in enhancing the layer absorption and diffusion, for a given conductivity of the layer material, this could be also achieved by raising as much as possible the "surface" of the layer with respect to the surrounding volume. In this respect, suitable layers of fractal type may provide a new interesting setting adequate to the preceding goal. In all these applications the mathematical model is an elliptic or parabolic boundary value problem with a transmission condition on the interface (layer) that involves at the same time traces of functions from classical spaces as Sobolev spaces and intrinsic Laplaceans within the layer. The matching in the same problem of Euclidean and fractal analytic notions provide a significant crossing of Euclidean and fractal theories and tools. We refer to the paper of U. Mosco (see [30]) for an exhaustive

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discussion of this topic and we limit ourselves to a stationary version of the model problem in a simple geometry. Let $Q$ denote a (bounded open) domain in $\mathbb{R}^3$ e.g. $Q = \Omega \times ]0, 1[$

Figure 1: The Koch roof

where $\Omega$ is the (open) parallelogram with vertices $(0, 0), (1, 0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$ and let $S$ denote the layer: the Koch roof located in a median position inside $Q$. $S$ divides $Q$ in two subdomains $Q^1$ and $Q^2$.

To our knowledge, second order transmission problems with fractals or pre-fractals layers are new, the first examples having been given by M. R. Lancia [20], for the stationary case and by M. R. Lancia and P. Vernole [22] for the evolution case.

As already mentioned highly conductive layers are characterized with respect to the surrounding space for having much greater conductivity or permeability: heat or flow in the space is absorbed by the layer and starts diffusing within it much more efficiently than in the surrounding volume and the normal derivative from each side of the layer has a jump across the layer which acts as a source term for the Laplace operator generating the layer diffusion. The resulting boundary transmission condition is thus of second order what is, in some sense, unusual for second order elliptic boundary value problems. Moreover, the condition has an implicit character since the source term of the layer equation - the jump of the normal derivatives - is not among the data of the problem, but depends on the
solution itself. In the stationary case formally the equations are:

\begin{align}
-\Delta u^i &= f \quad \text{in } Q^i \quad \tag{1.1} \\
u &= 0 \quad \text{in } \partial Q \quad \tag{1.2} \\
u^1 &= u^2 \quad \text{in } S \quad \tag{1.3} \\
u_{|S} &= 0 \quad \text{in } \partial S \quad \tag{1.4} \\
\frac{\partial u^1}{\partial n_1} + \frac{\partial u^2}{\partial n_2} &= \Delta_S^\prime u \quad \text{in } S \quad \tag{1.5}
\end{align}

where \( u^i = u_{|Q^i}, \ i = 1, 2; \ \frac{\partial}{\partial n_i} \) denotes the (formal) exterior derivative to the boundary of \( Q^i \) and \( \Delta_S^\prime \) the (formal) Laplace operator on \( S \).

The rigorous definition of the operators and of the functional spaces in the previous equations is one of the main technical difficulties of this type of problem. The previous conditions can be seen as the Euler conditions satisfied by the minimizer of a suitable energy functional. Hence a natural approach is to prove existence and uniqueness of the weak solution by variational principles and then to establish regularity results in order to state rigorously the strong formulation.

In section 2 we introduce the pre-fractal problem \( (P_n) \) and we present the main regularity results, in section 3 we are concerned with the fractal problem \( (P) \), in section 4 we deal with the asymptotic convergence and in section 5 we briefly discuss some numerical results.

## 2 Pre-fractal problems

We consider a 3–dimensional Euclidean domain \( Q \) containing a fractal subset \( S \), the layer. Our basic model refers to the geometry illustrated in Fig 1. Here the layer is of the type

\begin{equation}
S_h = K_h \times I, \quad \tag{2.1}
\end{equation}

where \( K_h \) is the pre-fractal Lipschitz curve occurring in the construction of the Koch curve in the plane, whose endpoints are \( A \) and \( B \) and \( I = [0, L] \) is a real interval (for simplicity we take \( L = 1 \), (see e.g. [12] and [6]).

The layer is embedded in a 3–dimensional box:

\[ Q = \Omega \times (0, 1), \]

where \( \Omega \) is the parallelogram with vertices \( (0, 0), (1, 0), (\frac{1}{2}, \frac{\sqrt{3}}{2}) \) and \( (\frac{1}{2}, -\frac{\sqrt{3}}{2}) \).
Any point of $Q$ has coordinates $(x_1, x_2, y)$ and the boundary of $S$ belongs to the boundary of $Q$.

In order to state the variational principle, we need to define an energy functional $E_h$ of the type

$$E_h = E_Q + E_{S_h}$$

(2.2)

where $E_Q$ is the volume energy and $E_{S_h}$ the layer energy.

We assume that the energy $E_Q$ is simply the usual Dirichlet integral

$$E_Q[u] = \int_Q |\nabla u|^2 dQ$$

(2.3)

where $dQ = dx_1 dx_2 dy$ is the Lebesgue volume measure on $\mathbb{R}^3$.

The space of functions of finite energy on $Q$, vanishing on $\partial Q$, is the usual Sobolev space $H^1_0(Q)$. It is well known that these functions have a well defined trace on the Lipschitz surface $S_h$ (see e.g. J. Necas [33]).

We now describe the construction of the layer energy $E_{S_h}$, by first considering the case where $K_h$ is the pre-fractal Lipschitz curve occurring in the construction of the Koch curve, see Fig.2.

We set

$$E_{S_h}[u] = \sigma_h^1 \int_I dy \int_{K_h} |D_\ell u|^2 d\ell + \sigma_h^2 \int_I d\ell \int_I |D_y u|^2 dy.$$  

(2.4)

The domain of the total energy form $E_h$ is the space

$$D_0(E_h) = \{ u \in H^1_0(Q) : u_{|S_h} \in H^1_0(S_h) \}.$$  

(2.5)
In (2.5), $H_0^1(Q)$ denotes the usual Sobolev space in $Q$ and $H_0^1(S_h)$ the Sobolev space on $S_h$ according to J. Necas, [33] (see also P. Grisvard [10]). We note that integrals in the right-hand side of (2.4) turn out to be the sum of integrals over the "faces" $S_j$

$$\sum_j \left( \int_{S_j} \sigma_1^2 |D_\ell u|^2 + \sigma_2^2 |D_y u|^2 \right) dS$$

where $D_\ell$ denotes the tangential derivative along the pre-fractal $K_h$ and $D_y$ the usual partial derivative in the $y$ direction; $d\ell$ denotes the one-dimensional measure on $K_h$ relative to the arc-length $\ell$ and $dS$ the surface measure on $S$, that is, $dS = d\ell dy$.

$\sigma_1^2$ and $\sigma_2^2$ are positive constants that, here as $h$ is fixed, do not play any role; a good choice of these constants will be essential in the asymptotic theory, where we see the prefractal layers converging to the fractal layer.

By $E_h$, in the following, we shall denote both the quadratic functional and the associated (symmetric) bilinear form.

We can easily prove (see [31]).

**Proposition 2.1** The space $D_0(E_h)$ given by (2.5) is a Hilbert space under the norm

$$\|u\|_{D_0(E_h)} = (E[u])^{1/2}$$

and $E_h$, with domain $D_0(E_h)$, is a regular, strongly local Dirichlet form in $L^2(Q)$.

For definitions and main properties of Dirichlet forms see e.g. [7], [28] and [9]. As consequence we have:

**Corollary 2.2** Let $f$ be a given function in $L^2(Q)$. Then, there exists a unique $u_h \in D_0(E_h)$ that minimizes the functional

$$\frac{1}{2} E_h[u] - \int_Q f u \, dQ$$

in $D_0(E_h)$.

The variational solution $u_h$ satisfies a second order transmission condition which is obtained via integration by parts and Green formulas in each subdomain $Q_h^i$. More precisely, we have
Theorem 2.3 Let $u_h$ be the variational solution (in Corollary 2.2) then we have that

\begin{align}
(2.6) \quad & u_h \in C(Q) \\
(2.7) \quad & u^1_h \in H^{\frac{5}{2}-\varepsilon}(Q^1_h), \; u^2_h \in H^{\frac{5}{2}-\varepsilon}(Q^2_h) \\
(2.8) \quad & \frac{\partial u^i_h}{\partial n^i} \in L^2(S_h), \; i = 1, 2
\end{align}

in particular conditions (1.2), (1.3) and (1.4) are satisfied pointwise; (1.1) and (1.5) almost everywhere and

\begin{equation}
(2.9) \quad \Delta_S = \Delta_{S_h} = \sigma^1_h D^2_x + \sigma^2_h D^2_y.
\end{equation}

Here $u^i_h$ is the restriction of $u_h$ to $Q^i_h$, $\frac{\partial u^i_h}{\partial n^i}$ the outward "normal derivatives", $D^2_x$ the "piecewise" second order tangential derivative along the sides of $K_h$ and $D^2_y$ the "usual" second order partial derivative in $y$.

In the proof of Theorem 2.3 we use Kondrat’ev type results.


For similar results and detailed proofs, in the case in which $K_h$ is the pre-fractal curve approximating the Von Koch snowflake, we refer to [21] (Theorem 4.3, Proposition 4.4 and Proposition 4.5).

Remark 2.1 Let me only note that the discrepancy between the Sobolev regularity exponents for $u^1_h$ and $u^2_h$ is due to the geometry of the polyhedra $Q^1_h$ and $Q^2_h$ which have different (largest) dihedral angles $\left(\frac{5}{3}\right)\pi$ and $\left(\frac{4}{3}\right)\pi$ respectively. As it is known from the regularity theory, the regularity of the solutions improves if the opening of the inner dihedral angles becomes smaller. This effect holds on, despite the implicit character of the equations, and the dependence of the regularity exponent on the angle remains unperturbed. Let me incidentally remark that here the presence of ”conical” points does not affect the required regularity and we find the same regularity of the 2-dimensional case. To our knowledge,
there are not in the literature regularity results for this type of transmission problems, neither for the classical "flat" layer considered by E. Sanchez-Palencia, the difficulty is due to the implicit character of the problem (see (1.5)). We have proved for the "classical" case that both the restrictions $u_1^h$ and $u_2^h$ belong to the Sobolev space $H^2(Q_h^i)$. (see [25], Theorem 3.2)

3 Fractal problem

We now consider the case of the layer $S = K \times I$, whose section is the fractal Koch curve $K$. We introduce the coordinates described in Fig. 2, where every $P \in S$ is uniquely described by its projection $x = (x_1, x_2)$ on the plane $(x_1, x_2)$ and by its projection $y$ on the interval $[0, 1]$, hence, $P = (x, y)$. It is well known that on the Koch curve $K$ there exists an invariant measure $\mu$, that is a regular positive Borel measure $\mu$, that, after normalization, coincides with the restriction to $K$ of the $d_f$–dimensional Hausdorff measure $\mathcal{H}^{d_f}$ of $\mathbb{R}^2$:

$$\mu = (\mathcal{H}^{d_f}(K))^{-1}\mathcal{H}^{d_f}|_K,$$

where

$$d_f = \frac{\ln 4}{\ln 3}$$

is the Hausdorff dimension of $K$. The measure $\mu$ has the property that there exist two positive constants $c_1, c_2$

$$c_1 r^{d_f} \leq \mu(B_e(x, r) \cap K) \leq c_2 r^{d_f}, \quad \forall x \in K,$$

(see J. E. Hutchinson [12] and K. Falconer [6]), where $B_e(x, r)$ denote the Euclidean ball in $\mathbb{R}^2$. According to A. Jonsson and H. Wallin [17], we say that $K$ is a $d_f$–set.

We recall that a Lagrangean $L_x(u, v)$ is defined on $K$ and

$$E_K(u, v) = \int_K L_x(u, v)(dx)$$

is a regular strongly local Dirichlet form with domain $D_K$ dense in $L^2(K, \mu)$. $D_K$ is a Hilbert space with respect to the norm

$$\|u\|_{D_K} = (E_K(u, u) + \|u\|^2_{L^2(K, \mu)})^{1/2}.$$

The regularity implies that $D_K$ is not trivial (i.e. not made only by constant functions). Moreover, the functions in $D_K$ possess a continuous representative, which is actually Hölder continuous on $K$. For all the previous properties we refer e.g. to [23].
We now consider the subspace

\begin{equation}
D_0(K) = \{ u \in D_K : u = 0 \text{ on } A \text{ and } B \}.
\end{equation}

\(D_0(K)\) is a closed subspace of \(D_K\), hence a Hilbert space with respect to the norm \((3.5)\).

From the Poincaré inequality:

\begin{equation}
\|u\|_{L^2(K,\mu)} \leq c (E_K(u,u))^{\frac{1}{2}}, \ u \in D_0(K)
\end{equation}

(see e.g. [20]), we deduce that

\begin{equation}
\|u\|_{D_0(K)} = (E_K(u,u))^{\frac{1}{2}}
\end{equation}

is an equivalent norm in \(D_0(K)\).

As the form \(E_K\) is closed in \(L^2(K,\mu)\), there exists a non-positive self-adjoint operator \(\Delta_K\) in \(L^2(K,\mu)\), with dense domain in \(L^2(K,\mu)\).

We define the product Lagrangean \(L_{x,y} \cdot \cdot \) on the fractal \(S = K \times I\) by setting

\begin{equation}
L_{x,y}(u,v) = L_x(u,v)(dx)dy + D_yuD_yvdy\mu(dx)
\end{equation}

on the set

\begin{equation}
D_{L_{x,y}} = C_0(S) \cap L^2(I;D_0(K)) \cap L^2(K;H^1_0(I))
\end{equation}

where

\begin{equation}
L^2(K) = L^2(K,\mu).
\end{equation}

Here \(L_x(\cdot,\cdot)(dx)\) denotes the measure-valued Lagrangian of the energy for \(E_K\) of \(K\) (see \((3.4)\)), now acting on \(u(x,y)\) and \(v(x,y)\) as a functions of \(x \in K\) for a.e. \(y \in I\) and \(\mu(dx)\) is the measure defined in \((3.1)\) acting on each section \(K\) of \(S\) for a.e. \(y \in I\).

We use the notation

\(L_x[u] = L_x(u,u), u \in D_{L_{x,y}}.\)

The following functional is well defined for every \(u \in D_{L_{x,y}}\)

\begin{equation}
E_S[u] = \sigma^1 \int_I dy \int_K L_x[u](dx) + \sigma^2 \int_K d\mu(x) \int_I |D_yu|^2 dy.
\end{equation}

with \(\sigma^1\) and \(\sigma^2\) positive constants. By \(D_0(S)\) we denote the completion of \(D_{L_{x,y}}\) in the intrinsic norm:

\begin{equation}
\|u\|_S = (E_S[u])^{\frac{1}{2}}
\end{equation}
We remark that functional in (3.12) is also well defined on the larger domain $D_0(S)$. By $m$ we denote the product measure on $S$,

$$dm = d\mu(x) dy.$$ 

The measure $m$ has the property that there exist two positive constants $c_1, c_2$

$$c_1 r^d \leq m(B_e(P, r) \cap S) \leq c_2 r^d, \quad \forall P \in S,$$ 

where

$$d = d_f + 1 = \frac{\log 12}{\log 3},$$

and where $B_e(P, r)$ denote the Euclidean ball in $\mathbb{R}^3$. As before, $S$ is a $d$–set where now $d = d_f + 1$. From the Poincaré inequality on $K$, see (3.7), we derive easily the Poincaré inequality on $S$:

$$\|u\|_{L^2(S, m)} \leq c(E_S[u])^{\frac{1}{2}}, \quad u \in D_0(S).$$

**Proposition 3.1** $D_0(S)$ is a Hilbert space under the intrinsic norm (3.13) and the form $E_S$, with domain $D_0(S)$, is a regular Dirichlet form in $L^2(S, m)$.

We now come back to the total energy functional $E = E_Q + E_S$ (see (2.2) and (2.3)), $S$ being the fractal layer. The domain of the form $E$ is the space

$$D_0(E) = \left\{ u \in H^1_0(Q) : u|_S \in D_0(S) \right\}.$$ 

and we have

**Theorem 3.2** The space $D_0(E)$ given by (3.17) is a Hilbert space under the intrinsic norm

$$\|u\|_{D_0(E)} = (E[u])^{1/2}$$

and the form $E$, with domain $D_0(E)$, is a regular Dirichlet form in $L^2(Q)$.

As consequence:

**Corollary 3.3** Given $f \in L^2(Q)$, there exists a unique $u \in D_0(E)$ that minimizes the functional

$$\frac{1}{2} E[u] - \int_Q f u dQ.$$
We refer to [31] (Proposition 3.1, Theorem 3.2 and Theorem 3.3) for the proof of all the previous results.

The variational solution $u$ satisfies a second order transmission condition which is obtained via integration by parts and Green formulas in each subdomains $Q^i$. In the fractal case the normal derivatives have to be intended in a suitable sense, that is, a dual sense, namely they belong to the dual of the subspace $B^{2,2}_{\beta,0}$ of the Besov space $B^{2,2}_\beta$ where $\beta$ is equal to $\frac{d_f}{2}$; we recall that $d_f$ is the Hausdorff dimension of $K$. Roughly speaking $B^{2,2}_{\beta,0}$ is the fractal analogous of the Lions-Magenes space $H^{\frac{3}{2}}_{0,0}(\Gamma)$ that is defined when $\Gamma$ is a Lipschitz surface. Then, the variational solution $u$ satisfies the transmission condition in a dual sense: that is the sense of the dual of the domain $D_0(S)$. More precisely, denoting by $u^i$ the restriction to $Q^i$, $\frac{\partial u^i}{\partial n^i}, i = 1, 2$ the outward "normal derivative" and $\left[\frac{\partial u}{\partial n}\right] = \frac{\partial u_1}{\partial n_1} + \frac{\partial u_2}{\partial n_2}$ the jump of the normal derivative, we can state the following:

**Theorem 3.4** Let $u$ be the variational solution of Corollary 3.3 then we have that

\begin{align}
(3.20) & \quad u^i \in H^2_{loc}(Q^i) \\
(3.21) & \quad \frac{\partial u^i}{\partial n^i} \in \left(B^{2,2}_{\beta,0}(S)\right)', \beta = \frac{d_f}{2}, i = 1, 2
\end{align}

and the transmission condition (1.5) holds in $(D_0(S))'$ that is

\begin{align}
(3.22) & \quad \left<\Delta_S u|_S, z\right>_{(D_0(S))',D_0(S)} = \left<\left[\frac{\partial u}{\partial n}\right]_S, z\right>_{(D_0(S))',D_0(S)}
\end{align}

where $\left(B^{2,2}_{\beta,0}(S)\right)'$ is the dual of $B^{2,2}_{\beta,0}(S)$, $(D_0(S))'$ is the dual of $D_0(S)$, $\Delta_S$ is the variational operator from $D_0(S) \rightarrow (D_0(S))'$ and $<\cdot,\cdot>_{(D_0(S))',D_0(S)}$ is the duality pairing between $(D_0(S))'$ and $D_0(S)$.

The proof is based on A. Jonsson and H. Wallin results for Besov spaces (see [17] and also H. Triebel [36]). Moreover the characterization of the domain $D(K)$ in terms of the Lipschitz spaces stated by M. R. Lancia and myself plays an important role (see [23]).

A. Jonsson introduced these Lipschitz spaces ([15]) to characterize the domain of the Dirichlet form of the Brownian motion on the Sierpinski gasket; he also established their location in the Besov scale and several interesting properties. Later on K. P. Paluba ([34]) and T. Kumagai ([18]) gave further applications.
4 Asymptotic theory

An asymptotic "constructive" theory for the pre-fractal approximation is an important step toward the numerical analysis of the problem.

We address the question whether the pre-fractal solutions $u_h$ do converge to the fractal solution $u$, as the pre-fractal layers $S_h$ converge to the fractal layer $S$ for $h \to \infty$.

As $h$ increases the pre-fractal layers develop increasing surface up to reach the infinite 2-dimensional area of the limit fractal surface. The norm in the fractional Sobolev spaces (with smoothness index greater than 1) (see (2.7)) blows up as $h \to +\infty$. So we are forced to choose a different notion of convergence, actually an energy convergence. The

![Figure 3: Asymptotics](image)

Euclidean pre-fractal energies must be re-normalized in order to keep the energy finite in the limit. This amounts to choose the constants $\sigma_1^h$ and $\sigma_2^h$ in (2.4) conveniently.

The good choice of the re-normalization factors is obtained by taking into account the effect of the $d_f$-dimensional length intrinsic to the fractal curve $K$.

**Theorem 4.1** Let $u$ be the variational solution of the fractal transmission problem in the domain $Q$ of $\mathbb{R}^3$, with layer $S = K \times I$ with $K$ the fractal Koch curve. For every integer $h \geq 1$, let $u_h$ be the variational solution of the transmission problems in $Q$ with pre-fractal layer $S_h = K_h \times I$. If we scale the energy functionals (2.4), by taking $\sigma_1^h = \sigma_1^1(3^{d_f-1})^h$ and $\sigma_2^h = \sigma_2^1(3^{1-d_f})^h$ then as $h \to \infty$ we find:

\begin{equation}
(4.1) \quad u_h \to u \text{ strongly in } H^1_0(Q)
\end{equation}
\[
\int_{S_h} \frac{\partial u_h}{\partial n_i} \phi \, d\ell \to \langle \frac{\partial u_h}{\partial n_i}, \phi \rangle_{(B^2_{p,0}(S))^\prime, B^2_{p,q}(S)}, \quad \forall \phi \in H^1_0(Q)
\]

where \( \beta = df/2, \ i = 1, 2 \).

\[
\int_{S_h} \triangle_{S_h} u_h \phi \, d\ell \to \langle \triangle_S u, \phi \rangle_{(D_0(S))^\prime, D_0(S)}, \quad \forall \phi \in D_0(E).
\]

Remark 4.1 From the probabilistic point of view, Brownian motions penetrating fractal sets – a probabilistic counterpart of the analytic variational approach adopted here – have been constructed by T. Lindstrøm [26] and T. Kumagai [18], however without reference to transmission problems and related transmission conditions.

5 Numerical results

Let me conclude this talk with some numerical results obtained by E. Vacca in her PHD Thesis.

Consider the prefractal two-dimensional transmission problems: hence \( Q = \Omega \) is the parallelogram and \( S_h = K_h \) the prefractal Koch curve that divides \( \Omega \) in two adjacent subdomains \( \Omega_1 \) and \( \Omega_2 \) (see Figure 4). The variational solution \( u_h \) enjoys some regularity properties (see [24] for details and proofs). In particular we have:

**Theorem 5.1** Let \( u_h \) be the variational solution (as in Corollary 2.2). Then

\[
u_h \in C_0(\overline{\Omega})
\]

\[
r_i^\mu D^\alpha u_h \in L^2(\Omega_i), \quad |\alpha| = 2, \ i = 1, 2, \ \mu_1 > \frac{2}{5}, \ \mu_2 > \frac{1}{4}
\]

\[
u_h \in H^2(K_h)
\]

Where \( r_i = r_i(x) \) denotes the distance of the point \( x \) from the vertex of the nearest corner "reentrant" in \( \Omega_i \) and

\[H^2(K_h) = \{ u \in H^1(K_h) : u_{\mid M} \in H^2(M) \quad \forall M \ "\text{segment}\" \ of \ K_h \}.\]
Remark 5.1 There is a strict relation between the weights in Theorem 5.1 and the smoothness "exponent" in the fractional Sobolev spaces (see Theorem 2.3) here let me stress the fact that the value of weights $\mu_i$ plays an important role in the error estimate as we will see in Theorem 5.2 (following).

Problems $(P_h)$ are approximated by the "s-version" of Galerkin's finite element method. Let me recall the main definitions:

Definition 5.1 A triangulation $T_i = T_i(s)$ of $\Omega_i$ is regular and conformal if

- $\bar{\Omega}_i = \bigcup_{T \in T_i} T$
- $\hat{T} \neq \emptyset, \forall \ T \in T_i$
- $\hat{T}_1 \cap \hat{T}_2 = \emptyset, \ \forall \ T_1, T_2 \in T_i: \ T_1 \neq T_2$
- $T_1 \cap T_2 \neq \emptyset, \ T_1 \neq T_2 \Rightarrow T_1 \cap T_2 = \text{edge or vertex}$
- $\exists \sigma > 0 \text{ such that } \max_{T \in T_i} \left( \frac{s_T}{\eta_T} \right) \leq \sigma$

where $s_T = \text{diam}(T)$ and $\eta_T = \sup \{ \text{diam}(B) : B \text{ ball} \subset T \}$.

Figure 4: The Koch curve
Here \( s = \sup \{ s_T : \ T \in \mathcal{T} \} \) denotes the size of the triangulation \( \mathcal{T} \).

The choice of an appropriate triangulation is a crucial point in order to obtain more precise discrete solutions and better error estimates. From now on \( h \) denotes the step of iteration in the prefractal curve \( K_h \) and \( n \) the index of the discretization; hence as \( n \) goes to infinity the "size" of the triangulation \( s = s(n) \) goes to zero.

**Definition 5.2** The family of triangulations \( \mathcal{T}_{n,i}^h \), \( h \in \mathbb{N} \), \( n \in \mathbb{N} \), \( i = 1, 2 \), is "adapted" to Problem \((P_h)\) if

- the vertices of the prefractal curves \( K_h \) are nodes of the triangulations
- the meshes are conformal and regular according to Definition (5.1)
- \( \exists \ \sigma > 0 \) such that:
  
  \[
  \left\{ \begin{array}{ll}
  s_T \leq \sigma s \frac{1}{r_i} & \forall T \in \mathcal{T}_{n,i}^h : T \cap K_h \neq \emptyset \\
  s_T \leq \sigma s \cdot \inf_T r_i & \forall T \in \mathcal{T}_{n,i}^h : T \cap K_h = \emptyset 
  \end{array} \right.
  \]

where \( \mu_1 = \frac{2}{5} + \varepsilon \), \( \mu_2 = \frac{1}{4} + \varepsilon \), \( 0 < \varepsilon \) "small".

Here \( s = s(n) = \sup \{ \text{diam}(T), T \in \mathcal{T}_{n,i}^h \}, \ i = 1, 2 \} \) is the size of the triangulation and \( r_i = r_i(x) \) denotes the distance of the point \( x \) from the vertex of the nearest corner "reentrant" in \( \Omega_i \).

Let \( V_{1,h}^n \) denote the "discrete" space:

\[
V_{1,h}^n = \left\{ v \in C^0 \left( \overline{\Omega} \right), \ v = 0 \text{ on } \partial \Omega, \ v|_{\partial r} \text{ polynomial of degree 1} \right\}
\]

that is a subspace of the domain \( D_0(E) \), (see (2.5)), hence there exist a unique "discrete" solution in \( V_{1,h}^n \) that minimizes the total energy:

\[
(5.4)
\]

\[
(5.4) \quad u_{h,n} = \arg \min_{V_{1,h}^n} \left\{ \frac{1}{2} E[u] - \int_Q f u \ dQ \right\}
\]

(see also Corollary 2.2).

The following estimates hold:

**Theorem 5.2** Let \( u_h \) be the variational solution of \((P_h)\) (see Corollary 2.2) and \( u_{h,n} \) the "discrete solution" in \( V_{1,h}^n \) (see (5.4)).
Then we have:

\[
\|u_h - u_{h,n}\|_{D^0(E)} \leq C s \left\{ \sum_{i=1}^{2} \sum_{|\alpha| = 2} \| r_i^{\alpha} D^\alpha u_h \|_{C^2(\Omega_i)} + \| u_h \|_{H^2(K_h)} + \| u_h \|_{H^0(\Omega)} \right\}
\]

where \( C \) is independent from \( h \) and \( n \), \( s = s(n) = \sup \{ \text{diam}(T), \ T \in T^h \} \) and \( r_i = r_i(x) \) denotes the distance of the point \( x \) from the vertex of the nearest corner “reentrant” in \( \Omega_i \).

**Remark 5.2** In the previous assumptions and notations, using (ordinary) fractional Sobolev spaces we would obtain a worse estimate: i.e.:

\[
\|u_h - u_{h,n}\|_{D^0(E)} \leq C s^{3/5 - \varepsilon} \left\{ \| u_{h1} \|_{H^{3/5 - \varepsilon}(\Omega_1)} + \| u_{h2} \|_{H^{3/5 - \varepsilon}(\Omega_2)} + \| u_h \|_{H^2(K_h)} \right\}.
\]

**References**


