Numerical Analysis of Ordinary Differential Equations

Exercise set 1
Fall 2023

Exercise 1 *Let $p : [a, b] \to \mathbb{R}$ a continuous function. Prove that every solution of homogeneous linear differential equation $y'(t) = p(t)y(t),\ t \in [a, b]$ is of form*

$$y(t) = Ce^{\int_a^t p(s)\,ds},$$

where C a constant.

Hint *Multiply with $\int_a^t p(s)\,ds$ and integrate by parts.*

Exercise 2 *Let the initial value problem*

$$\begin{cases} y' = \sqrt{|y|}, & 0 \le t \le 2, \\ y(0) = 1. \end{cases}$$

Prove that for this initial value problem, the Theorem 1.2 of lecture notes, holds for sufficient c and L. Find the solution y.

Hint *To compute y use the method of separation of variables.*

Exercise 3 *Let the initial value problem*

$$\begin{cases} y' = \sqrt{|1 - y^2|}, & t \ge 0, \\ y(0) = 1. \end{cases}$$

1. *Why we should expect that this problem will not have a unique solution?*
2. *Prove that $y(t) = 1$ and $y(t) = \cosh t$ are solutions of the problem in every interval $[0, b]$, $b > 0$.*
3. *Which is the longest interval of form $[b, 0]$ in which $y(t) = \cos t$ is solution?*

Exercise 4 *Prove Theorem 1.3.*

Exercise 5 *(Gronwall inequality in integral form)*
Let $\phi(t) \ge 0$ a continuous function on $[0, T]$ and $\alpha, \beta \in \mathbb{R}$ with $\beta \ge 0$. If holds that

$$\phi(t) \le \alpha + \beta \int_0^t \phi(s)\,ds, \quad \forall\, t \in [0, T],$$

prove that

$$\phi(t) \le \alpha\, e^{\beta\, t}, \ \ \forall\, t \in [0, T].$$

Hint *Let the function*

$$\psi(t) = \alpha + \beta \int_0^t \phi(s)\,ds, \ \ t \in [0, T].$$

Then, show that $\mathrm{e}^{-\beta\, t}\psi(t)$ is decreasing function.

Exercise 6 *(Gronwall inequality in differential form)*
Let $\phi(t) \ge 0$ a continuous differentiable function on $[0, T]$ and $\beta \in \mathbb{R}$ with $\beta \ge 0$. If holds that

$$\phi'(t) \le \beta\phi(t), \quad \forall\, t \in [0, T],$$

prove that

$$\phi(t) \le \phi(0)\, e^{\beta\, t}, \ \ \forall\, t \in [0, T].$$

Hint *One way is to use previous exercise.*

Exercise 7 *Let $M \in \mathbb{R}^{m,m}$ a matrix. Let the initial value problem*

$$\begin{cases} y'(t) = My(t), & t \geq 0, \\ y(0) = y_0, \end{cases}$$

where $y_0 \in \mathbb{R}^m$.

1. *In accordance to the definition of e^x for a real number, define the matrix e^M as*

$$e^M := \sum_{l=0}^{\infty} \frac{1}{l!} M^l.$$

Let $\| \cdot \|$ a matrix norm, prove that

$$\forall \epsilon > 0,\ \exists n \in \mathbb{N},\ \forall k \in \mathbb{N} \quad \left\| \sum_{l=n}^{n+k} \frac{1}{l!} M^l \right\| \leq \epsilon,$$

and therefore the matrix e^M is well-defined.

2. *Prove that the exact solution is given by*

$$y(t) = e^{t\,M} y_0, \quad t \geq 0.$$

Exercise 8 *Let $M \in \mathbb{R}^{m,m}$ a non-positive definite matrix, i.e., $(Mx, x) \leq 0$ for all $x \in \mathbb{R}^m$. Let the initial value problem*

$$\begin{cases} y'(t) = My(t), & t \geq 0, \\ y(0) = y_0, \end{cases}$$

where $y_0 \in \mathbb{R}$. Prove that the euclidean norm $\|y(\cdot)\|$ is a decreasing function.

Exercise 9 *Let the initial value problem*

$$\begin{cases} x'(t) = -2x(t) + y(t), & t \geq 0, \\ y'(t) = 2x(t) - 2y(t), & t \geq 0, \\ x(0) = x_0, \\ y(0) = y_0, \end{cases}$$

where x_0, $y_0 \in \mathbb{R}$. Prove that $x(\cdot)^2 + y(\cdot)^2$ is a decreasing function.

Hint *Use previous exercise.*

Exercise 10 *Let $M \in \mathbb{R}^{m,m}$ an anti-symmetric matrix, i.e., $M = -M^T$. Let the initial value problem*

$$\begin{cases} y'(t) = My(t), & t \geq 0, \\ y(0) = y_0, \end{cases}$$

where $y_0 \in \mathbb{R}$. Prove that the euclidean norm $\|y(t)\| = \|y_0\|$.

Hint *Prove and use the property that $(Mx, y) = -(x, My)$, $M \in \mathbb{R}^{m,m}$ anti-symmetric matrix.*

Exercise 11 *Let y_0, $x_0 \in \mathbb{R}$ and the initial value problems,*

$$\begin{cases} y'(t) = f(t, y(t)), & t \geq 0, \\ y(0) = y_0, \end{cases} \quad \text{and} \quad \begin{cases} x'(t) = f(t, x(t)), & t \geq 0, \\ x(0) = x_0, \end{cases}$$

where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies the one-side Lipschitz condition with respect to y, uniformly on t, i.e.,

$$\forall\, t \in [a, b],\ \ \forall\, y_1,\, y_2 \in \mathbb{R},\ \ (f(t, y_1) - f(t, y_2))(y_1 - y_2) \leq \mu (y_1 - y_2)^2,$$

where μ is constant. Prove that for all $t \in [a, b]$, it holds that

$$|y(t) - x(t)| \leq e^{\mu(t-a)} |y_0 - x_0|.$$

Exercise 12 *Let the initial value problem,*

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [a, b], \\ y(a) = y_0, \end{cases}$$

where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function that satisfies the one-side Lipschitz condition with respect to* y*, uniformly on* t*, i.e.,*

$$\forall\, t \in [a, b],\ \ \forall\, y_1,\, y_2 \in \mathbb{R},\ \ (f(t, y_1) - f(t, y_2))(y_1 - y_2) \le \mu(y_1 - y_2)^2,$$

where μ *is constant.*

1. *Let* $u(t) := e^{-\nu(t-a)} y(t)$*. Show that* u *satisfies the initial value problem*

$$\begin{cases} u'(t) = F(t, u(t)), & t \in [a, b], \\ u(a) = y_0, \end{cases}$$

with

$$F(t, v) := e^{-\mu(t-a)} f(t, e^{\mu(t-a)} v) - \mu v.$$

2. *Prove that* F *satisfies the one-side satisfies the one-side Lipschitz condition with respect to* v*, uniformly on* t*.*