Numerical Analysis of Ordinary Differential Equations

Exercise set 2
Fall 2023

Exercise 1 *Let $M \in \mathbb{R}^{m,m}$ a non-positive definite matrix, i.e., $(Mx, x) \leq 0$ for all $x \in \mathbb{R}^m$. Let the initial value problem*

$$\begin{cases} y'(t) = My(t), \quad t \geq 0, \\ y(0) = y_0, \end{cases}$$

where $y_0 \in \mathbb{R}^m$. We discretize the initial value problem with implicit Euler, using a uniform mesh of $[0,1]$ with step $h = 1/M$, $M \in \mathbb{N}$. Let that $\{Y^n\}_{n=0}^M$ are the grid function of implicit Euler. Prove that

$$\|Y^n\| \leq \|Y^{n-1}\|, \;\; n = 1, \dots, M,$$

where $\|\cdot\|$ the Euclidean norm.

Hint *Write the global formulation of the numerical method applied to the above IVP and then, take the inner product with Y^n.*

Exercise 2 *Let the initial value problem*

$$\begin{cases} x'(t) = -2x(t) + y(t), \quad t \in [0,1], \\ y'(t) = 2x(t) - 2y(t), \quad t \in [0,1], \\ x(0) = x_0, \\ y(0) = y_0, \end{cases}$$

where x_0, $y_0 \in \mathbb{R}$. We discretize the initial value problem with implicit Euler, using a uniform mesh of $[0,1]$ with step $h = 1/M$, $M \in \mathbb{N}$. Let that $\{(X^n, Y^n)\}_{n=0}^M$ are the grid function of implicit Euler. Prove that

$$(X^n)^2 + (Y^n)^2 \leq (X^{n-1})^2 + (Y^{n-1})^2, \;\; n = 1, \dots, M.$$

Compare the result with Exercise 9 of first exercises set.

Exercise 3 *Let the initial value problem*

$$\begin{cases} y'(t) = -e^{y(t)}, \quad t \in [0,1], \\ y(0) = 1, \end{cases}$$

where $y_0 \in \mathbb{R}$. We discretize the initial value problem with implicit Euler, using a uniform mesh of $[0,1]$ with step $h = 1/M$, $M \in \mathbb{N}$. Prove that the approximations are well defined.

Hint *Use the fact that the $-e^y$ is decreasing.*

Exercise 4 *Let the initial value problem with $f : [a,b] \times \mathbb{R} \to \mathbb{R}$,*

$$\begin{cases} y'(t) = f(t, y(t)), \quad t \in [a,b], \\ y(0) = y_0, \end{cases}$$

where $y_0 \in \mathbb{R}$. We discretize the initial value problem with implicit Euler, using a uniform mesh of $[0,1]$ with step $h = 1/M$, $M \in \mathbb{N}$. The grid function $\{Y^n\}_{n=0}^M$ is given by

$$\begin{cases} Y^0 = y_0, \\ Y^n = Y^{n-1} + h\, f(t^n, Y^n), \;\; n = 1, \dots, M. \end{cases}$$

Prove that the approximations are well defined if f a) satisfies the Lipchitz condition and b) the one-side Lipchitz condition.

Exercise 5 *Let the differential equation $y'(t) = f(t), \;\; f \in \mathbb{P}_1$. Prove that any explicit two-stage Runge-Kutta method of order $p = 2$, integrates exactly the above differential equation.*

Hint *Show that $-hT(t, y; h)$ is the remainder term $E(f)$ of the two-point quadrature formula.*

Exercise 6 *Prove that a Runge-Kutta method is consistent if and only if* $\sum_{s=1}^{r} a_s = 1$.

Hint *Use the definition of consistency.*

Exercise 7 *Let the initial value problem*

$$\begin{cases} y'(t) = 1, & t \in [0,1], \\ y(0) = 0, \end{cases}$$

Let $M \in \mathbb{N}$ *and* $h = 1/M$ *and* Y^M *the approximation of* $y(1)$ *that gives a Runge-Kutta method with step* h*. If* $Y^M \to 1 = y(1), \ \ M \to \infty$*, prove that the Runge-Kutta method is consistent.*

Hint *Use the definition of consistency.*

Exercise 8 *Let the initial value problem*

$$\begin{cases} y'(t) = 1, & t \in [0,1], \\ y(0) = 0, \end{cases}$$

Let $M \in \mathbb{N}$ *and* $h = 1/M$ *and* Y^M *the approximation of* $y(1)$ *that gives a Runge-Kutta method with step* h*. If* $Y^M \to 1 = y(1), \ \ M \to \infty$*, prove that the Runge-Kutta method is consistent.*