Numerical Analysis of Ordinary Differential Equations

Exercise set 3
Fall 2023

Exercise 1 *Let the initial value problem with $M \in \mathbb{R}^{m,m}$ a constant matrix,*

$$\begin{cases} y'(t) = My(t), & t \geq 0, \\ y(0) = y_0, \end{cases}$$

where $y_0 \in \mathbb{R}^m$. Assume that all the eigenvalues $\lambda_j \in \mathbb{C}$ of M have the property that $Re\, \lambda_j \leq 0$, i.e., they have non-positive real part.

Discretize the initial value problem with the semi-implicit family of methods

$$\begin{array}{c|cc} \mu & \mu & 0 \\ 1-\mu & 1-2\mu & \mu \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array},$$

where $\mu \in \mathbb{R}$. In every time step we should solve two linear systems. What do we notice about these systems?

Exercise 2 *Prove that the family of the semi-implicit methods of form*

$$\begin{array}{c|cc} \mu & \mu & 0 \\ 1-\mu & 1-2\mu & \mu \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array},$$

where $\mu \in \mathbb{R}$. Find $\mu \in \mathbb{R}$ such that the method is $A-$stable.

Exercise 3 *Prove that the method with Butcher tableau*

$$\begin{array}{c|cc} \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \\ \frac{3}{4} & \frac{3}{8} & \frac{3}{8} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array},$$

is $A-$stable but not $B-$stable and algebraic stable.

Exercise 4 *Prove that the family of the semi-implicit methods of form*

$$\begin{array}{c|cc} \mu & \mu & 0 \\ 1-\mu & 1-2\mu & \mu \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array},$$

where $\mu \in \mathbb{R}$ is $B-$stable for $\mu \geq \frac{1}{4}$.

Exercise 5 *Prove that the Gauss-Legendre method of two stages*

$$\begin{array}{c|cc} \frac{1}{2}-\mu & \frac{1}{4} & \frac{1}{4}-\mu \\ \frac{1}{2}+\mu & \frac{1}{4}+\mu & \frac{1}{4} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array},$$

where $\mu = \frac{\sqrt{3}}{6}$ is $B-$stable.

Hint *Prove that is algebraic stable with $m_{ij} = 0,\ 1 \leq i, j \leq 2$.*

Exercise 6 *Prove that the trapezoidal rule*

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array},$$

is not $B-$stable.

Hint *Find an initial value problem of form $y' = f(t, y)$ where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function that*

$$\forall\, t \in [a, b], \ \ \forall\, y_1,\, y_2 \in \mathbb{R}, \ \ (f(t, y_1) - f(t, y_2))(y_1 - y_2) \leq 0.$$

For example take $f(t, y) := g(t)y$, where g a continuous function with non-positive values. Then, find the function g such that $Y^n = -2Y^{n-1}$.

Exercise 7 *Prove that the method with Butcher tableau*

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \tau & \frac{\tau}{2} & \frac{\tau}{2} \\ \hline & 1-\frac{1}{2\tau} & \frac{1}{2\tau} \end{array},$$

is A−stable if and only if $\tau = 1$. Which method is for $\tau = 1$?

Hint *Prove that for these method, we have the following rational approximation of the exponential $\phi : \mathbb{C} \to \mathbb{C}$,*

$$\phi(z) = \frac{2 + 2(2-\tau)z + (1-\tau)z^2}{2 - \tau z}.$$

Exercise 8 *Let the initial value problem*

$$\begin{cases} y'(t) = f(t, y(t)), \quad t \in [a, b], \\ y(0) = y_0, \end{cases}$$

where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function that

$$\forall\, t \in [a, b], \ \ \forall\, y_1,\, y_2 \in \mathbb{R}, \ \ (f(t, y_1) - f(t, y_2))(y_1 - y_2) \leq 0.$$

We discretize the initial value problem with an algebraic stable method Runge-Kutta of order p, with Butcher tableau

$$\begin{array}{c|c} \mu & \Lambda \\ \hline & \alpha^T \end{array},$$

with uniform partition of $[a, b]$ with step $h = \frac{b-a}{M}$. Assume that the exact solution is sufficiently smooth, prove the following error estimation

$$\max_{0 \leq n \leq M} |y(t^n) - Y^n| \leq Ch^p,$$

where C is independent of h.

Hint *In this exercise our aim is to prove an error estimate without using the Lipschitz constant. We need to prove that*

$$\left(Y^n - y(t^n) - T(t^{n-1}, y(t^{n-1}); h)\right)^2 \leq \left(Y^{n-1} - y(t^{n-1})\right)^2,$$

which means that

$$|Y^n - y(t^n)| \leq |Y^{n-1} - y(t^{n-1})| + |T(t^{n-1}, y(t^{n-1}); h)|.$$

Then, use that the method has order p.