Numerical Analysis of Ordinary Differential Equations

Fall 2023

Description of the problem

We seek function $y : [a,b] \to \mathbb{R}$ to be the solution of initial value problem

$$y'(t) = f(t, y(t)), \qquad t \in [a,b], \tag{0.1}$$
$$y(a) = y_0, \tag{0.2}$$

where the function $f : [a,b] \times \mathbb{R} \to \mathbb{R}$, the real numbers a, b with $a < b$ and the $y_0 \in \mathbb{R}$, are given.

Explicit Euler method

Given a natural number M, we recall a uniform partition of $[a,b]$ with mesh size $h = \frac{b-a}{M}$ and nodes the points $t^n = a + nh$, for $n = 0, \ldots, M$. The explicit Euler, produces the approximation $Y^n \in \mathbb{R}$ of $y(t^n)$ from

$$Y^n = Y^{n-1} + h\, f(t^{n-1}, Y^{n-1}), \quad n = 1, \ldots, M, \tag{0.3}$$
$$Y^0 = y_0. \tag{0.4}$$

Exercise

Write a program that computes the approximations of explicit Euler $\{Y^n\}_{n=0}^{M}$ for the initial value problem,

$$y'(t) = \frac{1}{10} y(t), \qquad t \in [0,1], \tag{0.5}$$
$$y(0) = 1. \tag{0.6}$$

The exact solution of (0.5)-(0.6) is $y(t) = e^{\frac{t}{10}},\ t \in [0,1]$. In order to check your program, compute the order of convergence for a given M, which is given from

$$\mathcal{E}(M) := \max_{0 \le n \le M} |Y^n - y(t^n)|\,. \tag{0.7}$$

Compute the error (0.7) for two different natural numbers $M_1 < M_2$, and compute the experimental order of convergence with M_1, M_2, which is given by

$$p(M_1, M_2) = \frac{\ln\left(\frac{\mathcal{E}(M_2)}{\mathcal{E}(M_1)}\right)}{\ln\left(\frac{M_1}{M_2}\right)}. \tag{0.8}$$

See that $p(M_1, M_2) \approx 1$.