Numerical Analysis of Ordinary Differential Equations

Fall 2023

Description of the problem

We seek function $y : [a, b] \to \mathbb{R}$ to be the solution of initial value problem

$$y'(t) = f(t, y(t)), \qquad t \in [a, b], \tag{0.1}$$
$$y(a) = y_0, \tag{0.2}$$

where the function $f : [a, b] \times \mathbb{R} \to \mathbb{R}$, the real numbers a, b with $a < b$ and the $y_0 \in \mathbb{R}$, are given.

Implicit Euler method

Given a natural number M, we recall a uniform partition of $[a, b]$ with mesh size $h = \frac{b-a}{M}$ and nodes the points $t^n = a + nh$, for $n = 0, \dots, M$. The implicit Euler, produces the approximation $Y^n \in \mathbb{R}$ of $y(t^n)$ from

$$Y^n = Y^{n-1} + h\, f(t^n, Y^n), \quad n = 1, \dots, M, \tag{0.3}$$
$$Y^0 = y_0. \tag{0.4}$$

Exercise

Write a program that computes the approximations of implicit Euler $\{Y^n\}_{n=0}^{M}$ for the initial value problem,

$$y'(t) = e^{-y(t)}, \qquad t \in [0, 5], \tag{0.5}$$
$$y(0) = 1. \tag{0.6}$$

The exact solution of (0.5)-(0.6) is $y(t) = \ln(t + e)\,,\ t \in [0, 5],\ t \in [0, 1]$. In order to check your program, compute the order of convergence for a given M, which is given from

$$\mathcal{E}(M) := \max_{0 \le n \le M} |Y^n - y(t^n)|\,. \tag{0.7}$$

Compute the error (0.7) for two different natural numbers $M_1 < M_2$, and compute the experimental order of convergence with M_1, M_2, which is given by

$$p(M_1, M_2) = \frac{\ln\left(\frac{\mathcal{E}(M_2)}{\mathcal{E}(M_1)}\right)}{\ln\left(\frac{M_1}{M_2}\right)}. \tag{0.8}$$

See that $p(M_1, M_2) \approx 1$.

Hint

For every $n = 1, \dots, M$, the Y^n is the solution of a non-linear equation. Thus, one way to approximate it, is the Newton method.

Newton Method

The Newton method gives approximations of the real root of a function $g(x)$, $x \in \mathbb{R}$, for which the derivative $g'(x)$ is known. The approximations are

$$\begin{aligned} x^{(l)} &= x^{(l-1)} - \frac{g(x^{(l-1)})}{g'(x^{(l-1)})}, \quad l \ge 0, \\ x^{(0)} &= x_0, \end{aligned} \tag{0.9}$$

with $x_0 \in \mathbb{R}$ an initial approximation of the root.

Implementation of Newton method to (0.3)-(0.4)

We want to approximate the solution of (0.3). Define $x := Y^n$. Then,

$$x - Y^{n-1} - h\, e^{-x} = 0. \tag{0.10}$$

Thus, $g(x) := x - Y^{n-1} - h\, e^{-x},\ x \in \mathbb{R}$. For initial condition, we get $x_0 = Y^{n-1}$, where Y^{n-1} the approximation on the previous time step t^{n-1}. In every step of implicit Euler method, Y^n is approximated (Newton method approximate Y^n) with L iterations of Newton method in view of (0.9). Then we set $Y^n := x^L$. Compactly,

For $n = 1, \ldots, M$,
Step 1: Set $x_0 := Y^{n-1}$
Step 2: For $l = 1, \ldots, L$, we compute

$$\begin{aligned} x^{(l)} &= x^{(l-1)} - \frac{g(x^{(l-1)})}{g'(x^{(l-1)})} \\ x^{(0)} &= Y^{n-1}, \end{aligned}$$

Step 3: Set $Y^n := x^{(L)}$.