Numerical Analysis of Ordinary Differential Equations

Fall 2023

Description of the problem

We seek function $y : [a,b] \to \mathbb{R}$ to be the solution of initial value problem

$$y'(t) = f(t, y(t)), \qquad t \in [a,b], \tag{0.1}$$
$$y(a) = y_0, \tag{0.2}$$

where the function $f : [a,b] \times \mathbb{R} \to \mathbb{R}$, the real numbers a, b with $a < b$ and the $y_0 \in \mathbb{R}$, are given.

Implicit Midpoint Method

Given a natural number M, we recall a uniform partition of $[a,b]$ with mesh size $h = \frac{b-a}{M}$ and nodes the points $t^n = a + nh$, for $n = 0, \ldots, M$. The implicit midpoint method, produces the approximation $Y^n \in \mathbb{R}$ of $y(t^n)$ from

$$Y^n = Y^{n-1} + h\, f(t^{n+\frac{1}{2}}, Y^{n+\frac{1}{2}}), \quad n = 1, \ldots, M, \tag{0.3}$$
$$Y^0 = y_0, \tag{0.4}$$

where $t^{n+\frac{1}{2}} := \frac{t^n + t^{n-1}}{2}$ and $Y^{n+\frac{1}{2}} := \frac{Y^n + Y^{n-1}}{2}$, $n = 1, \ldots, M$.

Notice that every term of the grid function $\{Y^n\}_{n=0}^{N}$ that produces from implicit midpoint method, is a vector of $d-$dimension. Moreover, the solution Y^n of (0.3) is a solution of a non-linear system. Thus, we need to approximate it, for example with Newton Method.

Newton Method

The Newton's method produces approximations of the root of $g(x)$, $x \in \mathbb{R}$, where the derivative $g'(x)$ is known. The approximations are

$$\begin{aligned} x^{(m+1)} &= x^{(m)} - \frac{g(x^{(m)})}{g'(x^{(m)})}, \quad m \geq 0, \\ x^{(0)} &= x_0, \end{aligned} \tag{0.5}$$

with $x_0 \in \mathbb{R}$ an initial approximation of the root that we want to approximate such that $g'(x_0) \neq 0$.

The Newton's method can be generalized for system of equation on $\mathbb{R}^d$. Let the function $G : \mathbb{R}^d \to \mathbb{R}^d$ with $G(x) = (g_1(x), \ldots, g_d(x))^T$. The aim is to approximate the root of G. Then,

$$\begin{aligned} J(x^{(m)}) \left(x^{(m+1)} - x^{(m)} \right) &= -G(x^{(m)}), \quad m \geq 0, \\ x^{(0)} &= x_0, \end{aligned} \tag{0.6}$$

where $J(x) \in \mathbb{R}^{d \times d}$ the Jacobian matrix G, which is defined as

$$J_{ij}(x) = \frac{\partial G_i}{\partial x_j}(x), \quad i,j = 1, \ldots, d, \tag{0.7}$$

and $x_0 \in \mathbb{R}^d$ an initial approximation of the root, such that the matrix $J(x_0)$ is invertible.

Application of Newton's method to implicit midpoint method

Every vector $\left\{ Y^{n+\frac{1}{2}} \right\}_{n=0,\ldots,M}$ will approximated by (0.6) where

$$\begin{aligned} x^{(0)} &= Y^{n-1}, \\ G(x) &= x - Y^{n-1} - h\, f(t^{n+\frac{1}{2}}, x), \quad x \in \mathbb{R}^d, \\ J_{ij}(x) &= \delta_{ij} - h \frac{\partial f_i}{\partial x_j}(t^{n+\frac{1}{2}}, x), \quad x \in \mathbb{R}^d, \quad \text{for } \ i,j = 1, \ldots, d. \end{aligned} \tag{0.8}$$

Then, after the approximation of $Y^{n+\frac{1}{2}}$, we set $Y^n := 2Y^{n+\frac{1}{2}} - Y^{n-1}$.

Lotka-Volterra

Let a closed habitat with two species, where one is the predator and the other is the prey, e.g. foxes and rabbits, respectively. We are interested on the evolution of the initial population of each species. If the function y_1 represents the population of the prey and y_2 the population of the predator, the model that used is Lotka-Volterra which is describing by the following equations

$$\begin{aligned}
y_1'(t) &= \alpha y_1(t) - \beta y_1(t) y_2(t), \qquad t \in [a,b], && (0.9)\\
y_2'(t) &= -\gamma y_2(t) + \delta y_1(t) y_2(t), \quad t \in [a,b], && (0.10)\\
y_1(a) &= \overline{y}_0,\\
y_2(a) &= \tilde{y}_0.
\end{aligned}$$

with real numbers $\alpha, \beta, \gamma, \delta > 0$, which describes the interaction between two species and $\overline{y}_0$, $\tilde{y}_0 \in \mathbb{R}^2$ the initial population.

Assume the initial data

$$[a,b] = [0,20], \quad Y^0 = (\overline{y}_0, \tilde{y}_0)^T = (1,\, 0.5)^T, \;\; \alpha = 2,\, \beta = 1.1,\, \gamma = 1 \;\text{ and }\; \delta = 0.5, \quad N = [500, 10^4, 2 \cdot 10^4].$$

Print the pairs of approximations $\{Y_1^n, Y_2^n\}_{n=0}^N$, and the graph of $\{t^n, Y_1^n\}_{n=0}^N$ and $\{t^n, Y_2^n\}_{n=0}^N$. Compare the results with explicit Euler.