Fakultät für Mathematik und Informatik
Friedrich-Schiller-Universität Jena
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Numerik Partieller Differentialgleichungen I (B.Sc.), and Theory and Numerical Analysis of Partial Differential Equations I (M.Sc.)

Exercise Set 1

Submission:

If you wish to submit one of the marked (highlighted with $*$) exercises from Exercise Sheet 1, it must be submitted in class on **22.04.2026** or sent via email before the class starts. Please write your full name and matriculation number at the top right of your submission.

Contact:

For any questions regarding the course or exercises, please send an email to Christos Pervolianakis (christos.pervolianakis@uni-jena.de).

Exercise 1(*): Let $\Omega = (-1, 1)$ and let the function $u : \Omega \to \mathbb{R}$ defined such that $u(x) := |x|^{3/2} - 1$.

1. Decide whether $u \in H^1(\Omega)$. If yes, find its weak derivative.
2. We assume that exists functions $u_1 \in \mathcal{C}^1([-1, 0])$, $u_2 \in \mathcal{C}^1([0, 1])$ with $u_1(0) = u_2(0)$. We define the function u as $u|_{(-1,0)} := u_1$ and $u|_{(0,-1)} := u_2$. Decide whether $u \in H^1(\Omega)$.
3. What happens in case where $u_1(0) \neq u_2(0)$? Does $u \in H^1(\Omega)$? Explain your answer.

Hint. *Use the definition of weak derivative and similar arguments to Examples 1.2.1 and 1.2.2.*

Exercise 2(*): Let $k \in \mathbb{N}$, $k \geq 1$, and $v \in \mathcal{C}^k(\Omega)$. Prove that, up to the order k, the weak derivatives and the classical derivatives of v coincide.

Hint. *Use the fundamental lemma of calculus of variations that holds also for $L^1_{loc}(\Omega)$ functions, i.e., if $g \in L^1_{loc}(\Omega)$ and*

$$\int_\Omega g\psi \, dx = 0, \quad \forall\, \psi \in \mathcal{C}_c^\infty(\Omega),$$

then $g = 0$ almost everywhere in Ω.

Exercise 3: Let $(H, (\cdot, \cdot))$ a real normed space with inner product and $\|\cdot\|$ the norm that derived from that inner product. Let $v, w \in H$, prove that

1. $\|v + w\|^2 = \|v\|^2 + \|w\|^2 + 2(v, w)$.
2. If $(v, w) = 0$, then $\|v + w\|^2 = \|v\|^2 + \|w\|^2$.
3. $\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$.

Exercise 4: Let $\widetilde{H}$ be a finite–dimensional subspace of an inner product space H, with $\dim \widetilde{H} = d$, and let $\{x_1, \ldots, x_d\}$ be a basis of $\widetilde{H}$. Show that the condition

$$(y - x, z) = 0 \quad \forall z \in \widetilde{H}$$

is equivalent to

$$(x_i, y) = (x, x_i), \quad i = 1, \dots, d.$$

If $y = \alpha_1 x_1 + \cdots + \alpha_n x_d$, prove that the coefficients $\alpha_1, \dots, \alpha_d$ satisfy the linear system

$$\sum_{j=1}^{d} (x_i, x_j)\alpha_j = (x, x_i), \quad i = 1, \dots, d.$$

The matrix

$$G(x_1, \dots, x_d) := \big((x_i, x_j)\big)_{i,j=1}^{d}$$

is called the Gram matrix. Show that G is symmetric and positive definite.

Exercise 5: Let $\Omega = (0, 1)$ and $H = L^2(0, 1)$ (real-valued). Let $f \in L^2(0, 1)$ and let $G \subset H$ be the space of real polynomials of degree $\leq n - 1$. Find the best approximation to f in G.

Exercise 6: Let $v, w \in H^1(\Omega)$. Prove that the product vw is weakly differentiable and satisfies $\partial_j(vw) = (\partial_j v)w + v(\partial_j w)$. Can we say that vw belongs to $H^1(\Omega)$?

Exercise 7: For $a, b \geq 0$, and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, verify that

$$\log\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geq \frac{1}{p}\log a^p + \frac{1}{q}\log b^q.$$

Then, using the latter, prove the Young inequality.

Exercise 8: Let $\Omega = (-1, 1)$. Prove that if $v \in L^1_{loc}(\Omega)$ has a second order weak derivative, then it has also first order weak derivative.