Fakultät für Mathematik und Informatik
Friedrich-Schiller-Universität Jena
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Numerik Partieller Differentialgleichungen I (B.Sc.), and Theory and Numerical Analysis of Partial Differential Equations I (M.Sc.)

Exercise Set 2

Submission:
If you wish to submit one of the marked (highlighted with $*$) exercises from Exercise Sheet 2, it must be submitted in class on **06.05.2026** or sent via email before the class starts. Please write your full name and matriculation number at the top right of your submission.

Contact:
For any questions regarding the course or exercises, please send an email to Christos Pervolianakis (christos.pervolianakis@uni-jena.de).

Exercise 1(*): Let Ω be an open, bounded Lipschitz domain on $\mathbb{R}^2$. Then, in view of Proposition 1.5.1(Inequality Poincaré-Friedrichs) of the lecture notes, we know that exists a constant C_F that depends only on the Ω, such that,

$$\|u\|_{L^2(\Omega)} \leq C_F \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega). \tag{1}$$

Let $\lambda > 0$ and $\widetilde{\Omega} = \lambda^{-1}\Omega$. Show that,

$$\|u\|_{L^2(\widetilde{\Omega})} \leq \lambda^{-1} C_F \|\nabla u\|_{L^2(\widetilde{\Omega})}, \quad \forall u \in H_0^1(\widetilde{\Omega}), \tag{2}$$

where the constant C_F is the same in both (1) and (2).

Hint. *To solve this exercise, one needs to use the so–called scaling argument. Define the appropriate affine mapping between Ω and $\widetilde{\Omega}$ and use (1).*

Exercise 2(*): Let $K \subset \mathbb{R}^2$ a triangle and $v \in H^2(K)$ with norm $\|v\|_{H^2(K)}$.

1. Let T a sub-triangle of K defined $T := \operatorname{conv}\{A, B, C\}$ with $F = \operatorname{conv}\{A, B\}$ and τ the tangent vector. Prove that
$$|v(B) - v(A)| \leq 2|F|^{1/2}\rho^{-1/2}(1 + \operatorname{diam}(T)^2)^{1/2}\|v\|_{H^2(T)},$$
with $\rho := \frac{2|T|}{|F|}$.
2. For any two points A, B in K, there exists a $C \in K$, such that with $F = \operatorname{conv}\{A, B\}$, $T := \operatorname{conv}\{A, B, C\}$, the ρ^{-1} is uniformly bounded by some constant $C(K)$ that depends only on K, but not on A, B or T.
3. Using (1),(2), conclude that v is Hölder continuous with exponent $1/2$, i.e., the following Sobolev embedding $H^2(K) \hookrightarrow \mathcal{C}^{0,1/2}(K)$. In other words, prove that there exists a constant $C > 0$ such that
$$\|v\|_{C^{0,1/2}(K)} \leq C\|v\|_{H^2(K)} \quad \text{for all } v \in H^2(K).$$

Hint. *Apply the trace inequality, e.g., Theorem 1.4.1 for the function* $\nabla v \cdot \tau$.

Exercise 3: Let a, $b \in \mathbb{R}$, $a < b$, and $p \in [1, \infty)$.

1. Show that

$$\max_{a \le x \le b} |f(x)| \le (b-a)^{-1/p} \|f\|_{L^p(a,b)} + (b-a)^{1-1/p} \|f'\|_{L^q(a,b)}, \quad \forall\, f \in W^{1,p}([a,b]).$$

2. Prove that $W^{1,p}([a,b])$ embeds continuously in $\mathcal{C}([a,b])$.

Exercise 4: Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, partitioned into N Lipschitz subdomains

$$\Omega = \bigcup_{n=1}^{N} \Omega_n.$$

Let $k \ge 0$ be an integer and $p \in [1, \infty]$. Suppose a function v satisfies: (a) $v|_{\Omega_n} \in W^{k+1,p}(\Omega_n)$ for each $n = 1, \dots, N$, and (b) $v \in \mathcal{C}^k(\Omega)$. Show that $v \in W^{k+1,p}(\Omega)$.

Exercise 5: (Poincaré Inequality with Vanishing Mean) Let Ω be an open, bounded Lipschitz domain on $\mathbb{R}^2$. Then, exists a constant $\widetilde{C}_F$ that depends only on the Ω, such that,

$$\|u\|_{L^2(\Omega)} \le \widetilde{C}_F \|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H^1(\Omega), \text{ with } \int_\Omega u \, dx = 0.$$

Exercise 6: Prove Lemma 1.7.1