Fakultät für Mathematik und Informatik
Friedrich-Schiller-Universität Jena
Dr. Christos Pervolianakis

FRIEDRICH-SCHILLER-
UNIVERSITÄT
JENA

Numerik Partieller Differentialgleichungen I (B.Sc.), and Theory and Numerical Analysis of Partial Differential Equations I (M.Sc.)

Exercise Set 3

Submission:
If you wish to submit one of the marked (highlighted with $*$) exercises from Exercise Sheet 3, it must be submitted in class on **20.05.2026** or sent via email before the class starts. Please write your full name and matriculation number at the top right of your submission.

Contact:
For any questions regarding the course or exercises, please send an email to Christos Pervolianakis (christos.pervolianakis@uni-jena.de).

Exercise 1(*): Show that the Neumann problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} &= g \quad \text{on } \partial\Omega \end{aligned}$$

has a unique weak solution $u \in H^1(\Omega)$ satisfying

$$\int_\Omega u \, dx = 0$$

if and only if the compatibility condition

$$\int_\Omega f \, dx + \int_{\partial\Omega} g \, ds = 0$$

holds.

Exercise 2(*): Prove Theorem 2.4.3.

Exercise 3: Let $u \in C^3(\Omega)$. Show that

$$|\Delta u|^2 - |D^2 u|^2 = \operatorname{div}\left(\nabla u \, \Delta u - \frac{1}{2}\nabla|\nabla u|^2\right).$$

Exercise 4: Prove that the Laplacian is represented in polar coordinates (r, ϕ) as follows

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}.$$

Exercise 5: Let the function given by

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x|, & \text{if } d = 2, \\ -\frac{1}{d(d-2)\alpha(d)}\frac{1}{|x|^{d-2}}, & \text{if } d > 2, \end{cases}$$

where $\alpha(d) \neq 0$ a real number. Show that $\Delta\,\Phi(x) = 0, \ \forall\, x \in \mathbb{R}^d \backslash \{0\}$.

Exercise 6: Let $\alpha \in (0, 2\pi)$ and define the domain

$$\Omega = \{(r\cos\theta, r\sin\theta) : 0 < r < 1,\ 0 < \theta < \alpha\}.$$

Let $\Gamma_D = \partial\Omega$ (Dirichlet boundary) and $\Gamma_N = \emptyset$, and define $f = 0$ in Ω, and boundary data

$$u_D(r,\theta) = \begin{cases} 0 & \text{for } \theta = 0 \text{ or } \theta = \alpha, \\ \sin\left(\dfrac{\pi\theta}{\alpha}\right) & \text{for } r = 1. \end{cases}$$

Show that

$$u(r,\theta) = r^{\pi/\alpha} \sin\left(\frac{\pi\theta}{\alpha}\right)$$

is a weak solution of the Poisson problem.